

ERGODIC BEHAVIOUR OF NONSTATIONARY REGENERATIVE PROCESSES¹

BY

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ABSTRACT. Let V_t be a regenerative process whose successive generations are not necessarily identically distributed and let A be a measurable set in the range of V_t . Let μ_n be the mean length of the n th generation and α_n be the mean time V_t is in A during the n th generation. We give conditions ensuring $\lim_{t \rightarrow \infty} \text{Prob}\{V_t \in A\} = \alpha/\mu$ where $\lim_{n \rightarrow \infty} (1/n) \sum_{j=1}^n \mu_j = \mu$ and $\lim_{n \rightarrow \infty} (1/n) \sum_{j=1}^n \alpha_j = \alpha$.

Introduction and main results. A nonstationary regenerative process V_t may be viewed as a succession of independent generations or cycles whose cycles are not necessarily identically distributed (a good example is a classical regenerative process (see [8]) whose stochastic mechanism is nonuniformly perturbed). The successive cycles are labelled 1, 2, 3, . . . and we could start off the process at the n th cycle, say, in which case we denote the process by $V_t^{(n)}$ ($V_t^{(1)}$ is denoted by V_t). However the process is started, it is defined on a measure space $\{\Omega, \mathcal{F}\}$. Starting off at the n th cycle merely induces a different probability measure $P^{(n)}$ on $\{\Omega, \mathcal{F}\}$ ($P^{(1)}$ is denoted by P).

Let $(T_n)_{n=1}^\infty$ denote the lengths of the cycles starting at cycle 1. Hence $(T_n)_{n=1}^\infty$ is a sequence of independent random variables defined on $\{\Omega, \mathcal{F}, P\}$. Also we assume $T_n > 0$ for all n . Define $S_n = \sum_{m=1}^n T_m$.

If the T_m all take values on a lattice (for simplicity the integers) we call it the lattice case; otherwise we call it the continuous case. As in [2] and [3] we maintain a dichotomy between the two cases. In the former case $R [R_+]$ represents the integers [nonnegative integers]; B_+ is the σ -field of subsets of R_+ and m is counting measure. In the latter case $R [R_+]$ represents $(-\infty, \infty)$ $[[0, \infty)]$; B_+ is the σ -field of Borel sets on R_+ and m is Lebesgue measure.

Let K_n be a partition of $\{1, 2, \dots\}$ of the form $K_n = \{i | i_n < i \leq i_{n+1}\}$. Given K_n define $Y_n = \sum_{i \in K_n} T_i$. For $d > 0$, and $\varepsilon = 1/2r$, r an integer, set

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$$\begin{aligned}
 B_k(\varepsilon) &= \{x \mid -\varepsilon < x - 2k\varepsilon \leq \varepsilon\}, \\
 q_{nk}(\varepsilon, d) &= \min[\text{Prob}\{Y_n \in B_k(\varepsilon)\}, \text{Prob}\{Y_n - d \in B_k(\varepsilon)\}], \\
 q_n(\varepsilon, d) &= \sum_{k=-\infty}^{\infty} q_{nk}(\varepsilon, d), \quad Q_m = \sum_{n=1}^m q_n\left(\frac{1}{2}, 1\right).
 \end{aligned}$$

DEFINITION 1. The sequence $\{T_n\}_{n=1}^{\infty}$ is called strongly d -mixing if $\forall \varepsilon$ there exists a sequence K_n such that $\sum_{n=0}^{\infty} q_n(\varepsilon, d) = \infty$. Furthermore the sequence $\{T_n\}_{n=1}^{\infty}$ is called strongly mixing if the closure of the smallest subgroup containing $\{d\}\{T_n\}_{n=1}^{\infty}$ is strongly d -mixing is R .

CONDITION C(a). $(T_n)_{n=1}^{\infty}$ is a strongly mixing sequence (see also [2, Definition 3] and also Mineka [4]).

Let F^n be the distribution of T_n .

CONDITION C(b). There exists a distribution G with finite mean such that $F^n(s) \geq G(s)$ for all n and all $s \in R_+$. Let $\bar{\mu}$ be the mean of G .

CONDITION C(c). If $\mu_n = ET_n$ is the mean length of the n th cycle then $\inf \mu_n \geq \bar{\mu} > 0$. This condition is vacuous in the lattice case.

It is clear that V_t is a very special semiregenerative process (see [2] and [3]) with embedded semi-Markov chain $(n + 1, T_n)_{n=0}^{\infty}$ ($T_0 = 0$). We may therefore apply the ergodic results ([2, Theorem 4] and [3, Theorem 4]) to our special case:

THEOREM 1. If A is a measurable set in the range of V_t , if Conditions C(a)–C(c) hold and if in the continuous case the functions $P^{(n)}\{V_s \in A, X_1 > s\}$ (X_1 denotes the length of the starting cycle) are uniformly directly Riemann integrable (see [3, Definition 4]) then

$$\lim_{\substack{t \rightarrow \infty \\ t \in R_+}} \sum_{n=1}^{\infty} \left| P\{V_t \in A, S_{n-1} \leq t < S_n\} - \frac{\alpha_n}{\mu_n} P\{S_{n-1} \leq t < S_n\} \right| = 0 \quad (1)$$

where $\alpha_n = \int dP^{(n)} \int_0^{X_1} \chi_{\{V_t^{(n)} \in A\}}(s) \cdot m(ds)$; that is α_n is the mean time V_t is in A during the n th cycle.

The goal of this paper is to evaluate the weighting $P\{S_{n-1} \leq t < S_n\}$ as $t \rightarrow \infty$. We define the following supplementary conditions:

Let the variance of F^n be σ_n^2 and set $A_n = \sum_{j=1}^n \mu_j$, $B_n^2 = \sum_{j=1}^n \sigma_j^2$.

CONDITION S(a). There exist constants $\underline{\sigma}$ and $\bar{\sigma}$ such that $\sigma_n^2 \leq (\bar{\sigma})^2$ and $n(\underline{\sigma})^2 \leq B_n^2$.

CONDITION S(b). $\lim_{b \rightarrow \infty} (1/\sigma_k^2) \int_{|x| < b} (x - \mu_k)^2 F^k(dx) = 1$ uniformly in k .

CONDITION S(c). There exists a $\mu > 0$ such that $\lim_{n \rightarrow \infty} \sqrt{n} (A_n/n - \mu) = 0$.

The chief results of this paper (the proofs are given in the next section) are as follows.

THEOREM 2. *If Conditions C(a)–C(c) and S(a)–S(c) hold, then*

$$\lim_{\substack{t \rightarrow \infty \\ t \in R_+}} \sum_{n=1}^{\infty} \left| P \{ S_{n-1} \leq t < S_n \} - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - n\mu)^2}{2B_n^2} \right\} \right| = 0. \quad (2)$$

THEOREM 3. *If Conditions C(a)–C(c) and S(a)–S(c) hold, if (1) holds and if there exists an α such that*

$$\lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{1}{n} \sum_{j=1}^n \alpha_j - \alpha \right) = 0$$

then

$$\lim_{\substack{t \rightarrow \infty \\ t \in R_+}} P \{ V_t \in A \} = \frac{\alpha}{\mu}.$$

This generalizes the classical ergodic result for regenerative processes (see [8]).

Lemmas and proofs. Conditions C(a)–C(c) are verified throughout.

LEMMA 1. *If Conditions S(a) are satisfied then for any $\epsilon > 0$ there exists a λ such that*

$$\lim_{t \rightarrow \infty} \sum_{|(t - A_n)/\sqrt{t}| > \lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - A_n)^2}{2B_n^2} \right\} < \epsilon.$$

PROOF.

$$\begin{aligned} & \sum_{(t - A_n)/\sqrt{t} > \lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - A_n)^2}{2B_n^2} \right\} \\ & < \sum_{k=1}^{t/\gamma\mu} \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{(\bar{\sigma})^2((t - \lambda\sqrt{t})/\bar{\mu} - k)}} \exp \left\{ \frac{-(\lambda\sqrt{t} + k\bar{\mu})^2}{2(\bar{\sigma})^2((t - \lambda\sqrt{t})/\bar{\mu} - k)} \right\} \\ & \quad + \sum_{k=t/\gamma\bar{\mu} + 1}^{(t - \lambda\sqrt{t})/\bar{\mu}} \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\bar{\sigma}} \exp \left\{ \frac{-(\lambda\sqrt{t} + k\bar{\mu})^2}{2(\bar{\sigma})^2((t - \lambda\sqrt{t})/\bar{\mu} - k)} \right\} \end{aligned}$$

(where $0 < (1/\gamma)(\bar{\mu}/\mu) < 1$ and $t > \lambda^2(1 - 1/\gamma)^{-2}$)

$$\begin{aligned} & < \frac{(\bar{\mu})^{3/2}}{\bar{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{t/\gamma\mu} \frac{1}{\sqrt{t - \lambda\sqrt{t} - x\bar{\mu}}} \exp \left\{ \frac{-\bar{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t} + x\bar{\mu})^2}{(t - \lambda\sqrt{t} - x\bar{\mu})} \right\} dx \\ & \quad + \frac{t}{\bar{\mu}} \left(1 - \frac{1}{\gamma} \right) \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\bar{\sigma}} \exp \left\{ \frac{-\bar{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(t/\gamma)^2}{t} \right\}. \end{aligned}$$

We now choose $0 < \alpha < 1$ such that

$$t - \lambda\sqrt{t} - x\bar{\mu} > \alpha(t - \lambda\sqrt{t} - x\bar{\mu})$$

for $0 < x < t/\gamma\bar{\mu}$. It suffices to find a suitable α for $x = t/\gamma\bar{\mu}$. For $x = t/\gamma\bar{\mu}$ the above inequality yields $t(1 - \alpha - \bar{\mu}/\gamma\bar{\mu} + \alpha/\gamma) > \lambda\sqrt{t}(1 - \alpha)$. Pick $\bar{\alpha}$ so that $\beta = (1 - \alpha - \bar{\mu}/\gamma\bar{\mu} + \alpha/\gamma) > 0$; that is $0 < \alpha < (1 - 1/\gamma)^{-1}(1 - \bar{\mu}/\gamma\bar{\mu})$ and $\beta t > \lambda\sqrt{t}(1 - \alpha)$; that is $t > (\lambda(1 - \alpha)/\beta)^2$. Therefore for α, γ and β as above and $t > \max\{\lambda^2(1 - 1/\gamma)^{-2}, (\lambda(1 - \alpha)/\beta)^2\}$

$$\begin{aligned} & \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{t/\gamma\bar{\mu}} \frac{1}{\sqrt{t - \lambda\sqrt{t} - x\bar{\mu}}} \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t} + x\bar{\mu})^2}{(t - \lambda\sqrt{t} - x\bar{\mu})}\right\} dx \\ & < \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_0^{t/\gamma\bar{\mu}} \frac{1}{\sqrt{t - \lambda\sqrt{t} - x\bar{\mu}}} \\ & \qquad \qquad \qquad \cdot \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t} + x\bar{\mu})^2}{(t - \lambda\sqrt{t} - x\bar{\mu})}\right\} dx \\ & = \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\bar{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_{t(1-1/\gamma)-\lambda\sqrt{t}}^{t-\lambda\sqrt{t}} \frac{1}{\sqrt{s}} \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(t-s)^2}{s}\right\} ds \end{aligned}$$

(where $s = t - \lambda\sqrt{t} - x\bar{\mu}$)

$$\begin{aligned} & = \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\bar{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_{\lambda\sqrt{t}(t-\lambda\sqrt{t})^{-1/2}}^{(t/\gamma+\lambda\sqrt{t})(t(1-1/\gamma)-\lambda\sqrt{t})^{-1/2}} (1 - x(x^2 + 4t)^{-1/2}) \\ & \qquad \qquad \qquad \cdot \exp\left\{\frac{-\underline{\mu}}{(\bar{\sigma})^2} \cdot x^2\right\} dx \end{aligned}$$

(where $x = (t - s)/\sqrt{s}$ or $\sqrt{s} = (\sqrt{x^2 + 4t} - x)/2$)

$$< \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\bar{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_{\lambda}^{\infty} \exp\left\{\frac{-\underline{\mu}}{(\bar{\sigma})^2} x^2\right\} dx.$$

Therefore,

$$\begin{aligned} & \sum_{(t-A_n)/\sqrt{t} > \lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{\frac{-(t - A_n)^t}{2B_n^2}\right\} \\ & < \frac{(\bar{\mu})^{3/2}}{\underline{\sigma}\bar{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{\alpha}} \int_{\lambda}^{\infty} \exp\left\{\frac{-\underline{\mu}}{(\bar{\sigma})^2} x^2\right\} dx \\ & \quad + \frac{t}{\bar{\mu}} \left(1 - \frac{1}{\gamma}\right) \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\underline{\sigma}} \exp\left\{\frac{-\underline{\mu}}{2(\bar{\sigma})^2\gamma^2} t\right\} < \frac{\varepsilon}{2} \end{aligned}$$

for λ sufficiently large and $t > \max\{\lambda^2(1 - 1/\gamma)^{-2}, (\lambda(1 - \alpha)^2/\beta)\}$ where α , λ and β are defined as above.

Similarly

$$\begin{aligned} & \sum_{(t-A_n)/\sqrt{t} < -\lambda} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t - A_n)^2}{2B_n^2} \right\} \\ & < \sum_{k=1}^{\infty} \frac{\bar{\mu}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{(\bar{\sigma})^2((t + \lambda\sqrt{t})/\bar{\mu} + k)}} \exp\left\{ \frac{-(\lambda\sqrt{t} + k\bar{\mu})^2}{2(\bar{\sigma})^2((t + \lambda\sqrt{t})/\bar{\mu} + k)} \right\} \\ & < \frac{(\bar{\mu})^{3/2}}{\bar{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{t + \lambda\sqrt{t} + k\bar{\mu}}} \exp\left\{ \frac{-\bar{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t} + k\bar{\mu})^2}{(t + \lambda\sqrt{t} + k\bar{\mu})} \right\} \\ & < \frac{(\bar{\mu})^{3/2}}{\bar{\sigma}} \cdot \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\sqrt{t + \lambda\sqrt{t} + x\bar{\mu}}} \exp\left\{ \frac{-\bar{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(\lambda\sqrt{t} + x\bar{\mu})^2}{(t + \lambda\sqrt{t} + x\bar{\mu})} \right\} \\ & = \frac{(\bar{\mu})^{3/2}}{\bar{\sigma}\bar{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \int_{t+\lambda\sqrt{t}}^{\infty} \frac{1}{\sqrt{s}} \exp\left\{ \frac{-\bar{\mu}}{2(\bar{\sigma})^2} \cdot \frac{(s-t)^2}{s} \right\} ds \\ & \text{(where } s = t + \lambda\sqrt{t} + x\bar{\mu}\text{)} \\ & = \frac{(\bar{\mu})^{3/2}}{\bar{\sigma}\bar{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\lambda\sqrt{t}/\sqrt{t+\lambda\sqrt{t}}}^{\infty} \left(1 + \frac{x}{\sqrt{x^2 + 4t}}\right) \exp\left\{ \frac{-\bar{\mu}}{2(\bar{\sigma})^2} \cdot x^2 \right\} dx \\ & \text{(where } x = (t - s)/\sqrt{s} \text{ or } \sqrt{s} = (\sqrt{x^2 + 4t} + x)/2\text{)} \\ & < \frac{(\bar{\mu})^{3/2}}{\bar{\sigma}\bar{\mu}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\lambda}^{\infty} 2 \exp\left\{ \frac{-\bar{\mu}}{2(\bar{\sigma})^2} x^2 \right\} dx < \frac{\varepsilon}{2} \end{aligned}$$

for λ sufficiently large.

The result follows. \square

LEMMA 2. *If Conditions S(a) are verified, then for any $\varepsilon > 0$ there exist λ and $T(\lambda(\varepsilon))$ such that*

$$\sum_{|(t-A_n)/\sqrt{t}| > \lambda} P\{S_{n-1} < t < S_n\} < \varepsilon \quad \text{for all } t > T.$$

PROOF.

$$\sum_{|(t-A_n)/\sqrt{t}| > \lambda} P(S_{n-1} < t < S_n) \leq P(S_k > t) + P(S_l < t)$$

where

$$k = \sup\{n: A_n < t - \lambda\sqrt{t}\}$$

and

$$l = \inf \{ n: A_n > t + \lambda\sqrt{t} \},$$

$$P(S_k > t) \leq P(S_k - A_k > \lambda\sqrt{t}) \leq \frac{B_k^2}{t\lambda^2} \leq \frac{k(\bar{\sigma})^2}{t\lambda^2}.$$

However $\underline{\mu}k \leq A_k < t - \lambda\sqrt{t}$. Hence

$$\frac{k}{t} \leq \frac{1}{\underline{\mu}} \left(1 - \frac{\lambda}{\sqrt{t}} \right) \leq \frac{1}{\underline{\mu}} \quad \text{for } t > \lambda^2.$$

Therefore $P(S_k > t) \leq (1/\underline{\mu})(\bar{\sigma})^2/\lambda^2$.

$$P(S_l < t) \leq P(S_l - A_l < -\lambda\sqrt{t}) \leq \frac{B_l^2}{t\lambda^2} \leq \frac{l(\bar{\sigma})^2}{t\lambda^2}.$$

However $\underline{\mu}(l-1) \leq A_{l-1} < t + \lambda\sqrt{t}$. Hence

$$\frac{l}{t} \leq \frac{1}{\underline{\mu}} \left(1 + \frac{\lambda}{\sqrt{t}} \right) + \underline{\mu}.$$

Therefore,

$$P(S_k < t) \leq \left[\left(\frac{1}{\underline{\mu}} + \underline{\mu} \right) + \frac{1}{\underline{\mu}} \cdot \frac{\lambda}{\sqrt{t}} \right] \frac{(\bar{\sigma})^2}{\lambda^2}.$$

Hence,

$$\lim_{t \rightarrow \infty} [P(S_k > t) + P(S_l < t)] \leq \frac{(\bar{\sigma})^2}{\lambda^2} \left(\frac{2}{\underline{\mu}} + \underline{\mu} \right).$$

This can be made arbitrarily small by taking λ large.

LEMMA 3. *If Conditions S(a) are satisfied then for $L > 0$, $\varepsilon > 0$, $\lambda > 0$ there exists a $T > 0$ such that for all $t > T$ and for all x ($0 < x < L$) we have*

$$\left| \exp \left\{ \frac{-(t-x-A_n)^2}{2B_n^2} \right\} - \exp \left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < \varepsilon$$

for all n such that $|(t-A_n)/\sqrt{t}| < \lambda$.

PROOF.

$$\left| \exp \left\{ \frac{-(t-x-A_n)^2}{2B_n^2} \right\} - \exp \left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right|$$

$$< \left| 1 - \exp \left\{ \frac{(t-A_{n+1})^2}{2B_{n+1}^2} - \frac{(t-x-A_n)^2}{2B_n^2} \right\} \right|.$$

Next

$$\frac{(t - A_{n+1})^2}{2B_{n+1}^2} - \frac{(t - x - A_n)^2}{2B_n^2} = \frac{B_n^2(t - A_{n+1})^2 - B_{n+1}^2(t - x - A_n)^2}{2B_{n+1}^2B_n^2}.$$

Also

$$\begin{aligned} B_n^2(t - A_n - \mu_{n+1})^2 - (B_n + \sigma_{n+1}^2)(t - x - A_n)^2 \\ = B_n^2(t - A_n)^2 - 2B_n^2\mu_{n+1}(t - A_n) + (\mu_{n+1})^2B_n^2 \\ - B_n^2(t - A_n)^2 - B_n^2x^2 + 2B_n^2x(t - A_n) \\ - \sigma_{n+1}^2(t - A_n)^2 + 2x\sigma_{n+1}^2(t - A_n) - \sigma_{n+1}^2x^2. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{(t - A_{n+1})^2}{2B_{n+1}^2} - \frac{(t - x - A_n)^2}{2B_n^2} \\ = \frac{-2\mu_{n+1}(t - A_n) + 2x(t - A_n)}{2B_{n+1}^2} + \frac{(\mu_{n+1})^2 - x^2}{2B_{n+1}^2} \\ + \frac{2x\sigma_{n+1}^2(t - A_n) - \sigma_{n+1}^2(t - A_n)^2}{2B_n^2B_{n+1}^2} - \frac{\sigma_{n+1}^2x^2}{2B_n^2B_{n+1}^2} \end{aligned}$$

as $t \rightarrow \infty, n \rightarrow \infty$. Moreover $B_n^2 \geq n(\sigma)^2$ hence

$$\{(\mu_{n+1})^2 - x^2\}/2B_{n+1}^2 \rightarrow 0, \quad t \rightarrow \infty,$$

and

$$-\sigma_{n+1}^2x^2/2B_n^2B_{n+1}^2 \rightarrow 0, \quad t \rightarrow \infty,$$

for n such that $|(t - A_n)/\sqrt{t}| < \lambda$. Next $A_n - \lambda\sqrt{t} < t < A_n + \lambda\sqrt{t}$ implies that $t - \lambda\sqrt{t} \leq n\bar{\mu}$. Hence

$$\frac{|t - A_n|}{B_n^2} \leq \frac{\lambda\sqrt{t}}{B_n^2} \leq \frac{\lambda\sqrt{t}}{n(\sigma)^2} \leq \frac{\lambda\sqrt{t}\bar{\mu}}{\sigma(t - \lambda\sqrt{t})}.$$

Therefore $|t - A_n|/B_n^2 \rightarrow 0$ and

$$\frac{(t - A_n)^2}{B_n^2B_{n+1}^2} = \frac{(t - A_n)}{B_n^2} \cdot \frac{(t - A_n)}{B_{n+1}^2} \rightarrow 0$$

as $t \rightarrow \infty$ by the same reasoning. Hence we have

$$\frac{(t - A_{n+1})^2}{2B_{n+1}^2} - \frac{(t - x - A_n)^2}{2B_n^2} \rightarrow 0$$

which gives the result. \square

LEMMA 4. If Conditions S(a) are verified then $\forall \varepsilon > 0, \lambda > 0$ there exists a $T > 0$ such that for $t > T$

$$\left| \int_0^t (1 - F^{n+1}(t-s)) \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} m(ds) \right. \\ \left. - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < \varepsilon$$

for n such that $|(t - A_n)/\sqrt{t}| < \lambda$.

PROOF.

$$\int_L^\infty (1 - F^n(x))m(dx) < \int_L^\infty (1 - G(x))m(dx) \rightarrow 0$$

as $L \rightarrow \infty$. Let L be such that $\int_L^\infty (1 - G(x))m(dx) < \varepsilon/4$. Next

$$\left| \int_0^t (1 - F^{n+1}(t-s)) \left[\frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} \right. \right. \\ \left. \left. - \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right] m(ds) \right| \\ < \int_0^t (1 - G(t-s)) \frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} \right. \\ \left. - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| m(ds).$$

Moreover

$$\frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < 1;$$

hence

$$\int_0^t (1 - G(t-s)) \frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| m(ds) \\ < \int_{t-L}^t (1 - G(t-s)) \frac{1}{\sqrt{2\pi}} \left| \exp\left\{ \frac{-(s-A_n)^2}{2B_n^2} \right\} \right. \\ \left. - \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| m(ds) + \frac{\varepsilon}{4}.$$

By Lemma 3 there exists a $T > L$ such that for $t > T$ and s such that $t - L \leq s < t$ we have

$$\left| \exp\left\{ \frac{-(s - A_n)^2}{2B_n^2} \right\} - \exp\left\{ \frac{-(t - A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < \frac{\varepsilon}{4\bar{\mu}}$$

for n such that $|(t - A_n)/\sqrt{t}| < \lambda$. Hence we have

$$\begin{aligned} & \left| \int_0^t (1 - F^{n+1}(t - s)) \left[\frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s - A_n)^2}{2B_n^2} \right\} \right. \right. \\ & \qquad \qquad \qquad \left. \left. - \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(t - A_{n+1})^2}{2B_{n+1}^2} \right\} \right] m(ds) \right| \\ & < \bar{\mu} \cdot \frac{\varepsilon}{4\bar{\mu}} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \end{aligned}$$

However

$$\begin{aligned} & \left| \int_0^t (1 - F^{n+1}(t - s))m(ds) - \mu_{n+1} \right| \cdot \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(t - A_{n+1})^2}{2B_{n+1}^2} \right\} \\ & < \left| \mu_{n+1} - \int_0^L (1 - F^{n+1}(s))m(ds) \right| < \frac{\varepsilon}{2}. \end{aligned}$$

This gives the result. \square

We now digress to establish certain useful results related to local limit theorems.

PROPOSITION 1. *In the continuous case suppose $\{T_n\}_{n=1}^\infty$ is strongly mixing (we need not assume that the T_n are positive valued) and suppose T_1 has a bounded density; then for all $s, 0 < s < l$, where $s \in R_+$ and $l \in R_+$,*

$$\lim_{n \rightarrow \infty} \sum_{k=-\infty}^\infty P \{kl < S_n < kl + s\} = \frac{s}{l}.$$

PROOF. First suppose $s = xp, l = xq$ where p and q are integers ($x \in R_+$). Consider the function $f(z) = \chi_{\cup_{k=-\infty}^\infty [kl, kl+x)}(z)$. Next let ${}^2S_n = \sum_{k=2}^n T_k$ so

$$Ef(S_n) = Ef({}^2S_n + T_1) = E \int_{-\infty}^\infty f({}^2S_n + y)F^1(dy) = Eg({}^2S_n)$$

where

$$g(x) = \int_{-\infty}^\infty f(x + y)F^1(dy).$$

Since F^1 has a bounded derivative, g is uniformly continuous and applying Theorem 2 in Orey [7] we have:

$$\lim_{n \rightarrow \infty} [Ef(S_n) - Ef(S_n + x)] = \lim_{n \rightarrow \infty} [Eg({}^2S_n) - Eg({}^2S_n + x)] = 0.$$

Similarly

$$\lim_{n \rightarrow \infty} [Ef(S_n) - Ef(S_n + kx)] = 0$$

so

$$\lim_{n \rightarrow \infty} \left[qEf(S_n) - E \sum_{k=0}^{q-1} f(S_n + kx) \right] = 0;$$

that is $\lim_{n \rightarrow \infty} [qEf(S_n) - 1] = 0$. Hence $\lim_{n \rightarrow \infty} E \sum_{k=0}^{p-1} f(S_n + kx) = p/q$ which gives:

$$\lim_{n \rightarrow \infty} \sum_{k=-\infty}^{\infty} P \{kl < S_n < kl + s\} = \frac{s}{l}$$

if s/l is rational. For arbitrary s, l we can find $S_L < s < S_U$ such that S_L/l and S_U/l are rational. Applying the above

$$\frac{S_L}{l} \leq \lim_{n \rightarrow \infty} \sum_{k=-\infty}^{\infty} P \{kl < S_n < kl + s\}$$

and

$$\overline{\lim}_{n \rightarrow \infty} \sum_{k=-\infty}^{\infty} P \{kl < S_n < kl + s\} \leq \frac{S_U}{l}.$$

The result follows by taking S_L and S_U arbitrarily close to s . \square

The analogous result in the lattice case is obtained similarly. We also have:

PROPOSITION 2. *In the continuous case under the hypotheses of Proposition 1 $\lim_{n \rightarrow \infty} \sup_{s \in [e, \infty)} |\prod_{k=1}^n f_k(t)| = 0$ where $\epsilon > 0$ and $f_k(t)$ is the characteristic function of T_k .*

PROOF. Let \bar{T}_1 be a random variable independent of the T_k for $k > 2$ with characteristic function

$$\bar{f}_1(s) = \begin{cases} 1 - |s|/\epsilon, & \text{for } |t| \leq \epsilon, \\ 0 & \text{for } |t| > \epsilon \end{cases}$$

(see Mineka [5]). Let $\bar{S}_n = \bar{T}_1 + T_2 + \dots + T_n$. Now $(n + 1, T_n)_{n=0}^{\infty}$ ($T_0 = 0$) is a semi-Markov chain with state space $\{1, 2, \dots\}$ and transitions from n to $n + 1$ for all n . It is clear that all bounded, harmonic functions on the underlying chain are constants. Moreover $(n + 1, T_n)_{n=0}^{\infty}$ is strongly mixing. Therefore by Lemma 1(a) in [3], for any harmonic function h on $(n + 1, S_n)_{n=0}^{\infty}$, $h(n, x) = C_h$, a constant, a.e.- m for each n . We may consider $(n + 1, \bar{S}_n)_{n=0}^{\infty}$ to be the same chain with a different initial measure. By hypothesis both S_n and \bar{S}_n are absolutely continuous w.r.t. m so, by Theorem 1 in [3],

$$\lim_{n \rightarrow \infty} \|P \{S_n \in dx\} - P \{\bar{S}_n \in dx\}\| = 0$$

($\| \cdot \|$ is the total variation on R).

Now $Ee^{is\bar{T}_1} = 0$ for $|s| > \epsilon$ and hence $Ee^{is\bar{S}_n} = 0$ for $|s| > \epsilon$. Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{s \in [\epsilon, \infty)} \left| \prod_{k=1}^n f_k(s) \right| &= \lim_{n \rightarrow \infty} \sup_{s \in [\epsilon, \infty)} |Ee^{isS_n} - Ee^{is\bar{S}_n}| \\ &< \lim_{n \rightarrow \infty} \|P \{S_n \in dx\} - P \{\bar{S}_n \in dx\}\| = 0 \end{aligned}$$

since $|e^{isx}| \leq 1$ for all s . \square

PROPOSITION 3. *In the lattice case if Condition C(a) is satisfied then*

$$\prod_{k=1}^{\infty} \left[\max_{0 < n < h} P \{T_k = n \pmod{h}\} \right] = 0$$

for any integer $h > 2$.

PROOF. Suppose there exists an integer \bar{h} such that

$$\prod_{k=1}^{\infty} \left[\max_{0 < m < \bar{h}} P \{T_k = m \pmod{\bar{h}}\} \right] > 0.$$

Then there exists a sequence of integers $\{m_k\}_{k=1}^{\infty}$ such that

$$\prod_{k=1}^{\infty} \left[P \{T_k - m_k = 0 \pmod{\bar{h}}\} \right] > \epsilon > 0.$$

Let $\bar{T}_k = T_k - m_k$; $\bar{S}_n = \bar{T}_1 + \bar{T}_2 + \dots + \bar{T}_n$. Hence $P\{\bar{T}_n = 0 \pmod{\bar{h}}\}$ for all $n\} > \epsilon > 0$. Hence $P\{\bar{S}_n = 0 \pmod{\bar{h}} \text{ ult}\} > \epsilon > 0$. However the tail field of $\{\bar{S}_n\}_{n=1}^{\infty}$ is clearly contained in the tail field of $\{S_n\}_{n=1}^{\infty}$ which is trivial by hypothesis C(a). Hence $P\{\bar{S}_n = 0 \pmod{\bar{h}} \text{ ult}\} = 1$. However

$$\begin{aligned} \sum_{i=-\infty}^{\infty} |P \{S_n = i\} - P \{S_n = i + 1\}| \\ > \left| P \left\{ S_n = \sum_{k=1}^n m_k \pmod{\bar{h}} \right\} - P \left\{ S_n = \sum_{k=1}^n m_k + 1 \pmod{\bar{h}} \right\} \right| \\ = |P \{ \bar{S}_n = 0 \pmod{\bar{h}} \} - P \{ \bar{S}_n = 1 \pmod{\bar{h}} \}| \\ \rightarrow 1 \text{ as } n \rightarrow \infty \end{aligned}$$

since $\bar{S}_n = 0 \pmod{\bar{h}}$ ultimately. But by the argument used in Proposition 2 (or by Theorem 2 in [7])

$$\lim_{n \rightarrow \infty} \|P \{S_n \in dx\} - P \{S_n + 1 \in dx\}\| = 0.$$

This gives a contradiction and hence completes the proof. \square

We now state:

LEMMA 5. *If Conditions C(a) and S(a)–S(b) are verified and T_1 has a bounded density then*

$$\lim_{n \rightarrow \infty} \left| B_n p_n(s) - \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{-(s - A_n)^2}{2B_n^2} \right\} \right| = 0$$

uniformly in $s \in R$ where $p_n(s) = P\{S_n = s\}$ in the lattice case and $p_n(s)$ is the density of S_n in the continuous case.

PROOF. It suffices to check the conditions for Theorem 1 in Mineka [5] in the lattice case and Theorem 1 in Muhin [6] in the continuous case. This is easy using Propositions 1–3. \square

In the discrete case Condition S(b) may be weakened to the Lindeberg condition: for each $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{1}{B_n^2} \sum_{k=1}^n \int_{|x| > \epsilon B_n} (x - \mu_k)^2 F^k(dx) = 0$$

plus a stronger mixing condition: $\inf_{n \rightarrow \infty} \sqrt{Q_n} / B_n > 0$. See [1].

LEMMA 6. *If Conditions S(a)–S(b) are verified and if $W_t(n) = P\{S_{n-1} < t < S_n\}$ then, for all $\lambda > 0$,*

$$\lim_{t \rightarrow \infty} \sum_{|(t - A_n)/\sqrt{t}| < \lambda} \left| W_t(n) - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - A_n)^2}{2B_n^2} \right\} \right| = 0.$$

PROOF. By Remark 1 in [3] if \tilde{T}_1 is a random variable independent of $\{T_n\}_{n=1}^\infty$ with bounded derivative then

$$\lim_{t \rightarrow \infty} \sum_{n=1}^\infty |W_t(n) - P\{\tilde{S}_{n-1} < t < \tilde{S}_n\}| = 0$$

where $\tilde{S}_n = \tilde{T}_1 + T_2 + \dots + T_n$. Hence we may assume that T_1 has a bounded derivative. Now take $\epsilon > 0$. Next $B_n^2/n > (\sigma)^2$ for all n as $n \rightarrow \infty$, so there is a K_2 such that

$$\sqrt{t/B_n^2 [t/\bar{\mu} - \lambda/\bar{\mu}\sqrt{t}]} < k \quad \forall t > K_2.$$

By Lemma 5 there exists an N such that for all $n > N$

$$\left| B_n p_n(s) - \frac{1}{\sqrt{2\pi}} \exp \left\{ \frac{-(s - A_n)^2}{2B_n^2} \right\} \right| < \frac{\epsilon}{\lambda k \bar{\mu}} \tag{3}$$

for all s . Hence there is a $K_3 > K_2$ such that the inequality $|(t - A_n)/\sqrt{t}| < \lambda$ implies $n > N$ for $t > K_3$. By Lemma 4 there exists a $T > K_3$ such that, for $t > T$,

$$\left| \int_0^t (1 - F^{n+1}(t-s)) \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s - A_n)^2}{2B_n^2} \right\} m(ds) - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t - A_{n+1})^2}{2B_n^2} \right\} \right| < \frac{\varepsilon}{\lambda k} \quad (4)$$

for n such that $|(t - A_n)/\sqrt{t}| < \lambda$. We remark that $W_i(n+1) = \int_0^t (1 - F^{n+1}(t-s)) p_n(s) m(ds)$, so, for $t > T$,

$$\begin{aligned} & \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \left| W_i(n+1) - \frac{1}{\sqrt{2\pi}} \cdot \frac{\mu_{n+1}}{B_n} \exp\left\{ \frac{-(t - A_{n+1})^2}{2B_{n+1}^2} \right\} \right| \\ &= \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{B_n} \left| \int_0^t (1 - F^{n+1}(t-s)) B_n p_n(s) m(ds) - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t - A_{n+1})^2}{2B_{n+1}^2} \right\} \right| \\ &< \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{B_n} \int_0^t (1 - F^{n+1}(t-s)) \cdot \left| B_n p_n(s) - \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-(s - A_n)^2}{2B_n^2} \right\} \right| m(ds) \\ &+ \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{B_n} \left| \int_0^t (1 - F^{n+1}(t-s)) \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{ \frac{-(s - A_n)^2}{2B_n^2} \right\} m(ds) - \frac{\mu_{n+1}}{\sqrt{2\pi}} \exp\left\{ \frac{-(t - A_{n+1})^2}{2B_{n+1}^2} \right\} \right| \\ &< \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{B_n} \bar{\mu} \cdot \frac{\varepsilon}{\lambda k \bar{\mu}} \quad (\text{by (3)}) \\ &+ \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{B_n} \frac{\varepsilon}{\lambda k} \quad (\text{by (4)}) \\ &< \frac{2\varepsilon}{\lambda k} \cdot \frac{1}{B_{[t/\bar{\mu} - \lambda\sqrt{t}/\bar{\mu}]}} \cdot \frac{2\lambda}{\underline{\mu}} \sqrt{t}, \end{aligned}$$

using $n\bar{\mu} > t - \lambda\sqrt{t}$. But $t > T > K_2$ so $\sqrt{t} / B_{\lfloor t/\bar{\mu} - \lambda\sqrt{t}/\bar{\mu} \rfloor} < k$. Therefore we have

$$\sum_{|(t-A_n)/\sqrt{t}| < \lambda} \left| W_t(n+1) - \frac{1}{\sqrt{2\pi}} \cdot \frac{\mu_{n+1}}{B_n} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \right| < \frac{4\epsilon}{\underline{\mu}}.$$

Finally

$$\begin{aligned} \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{\sqrt{2\pi}} \mu_{n+1} \exp\left\{ \frac{-(t-A_{n+1})^2}{2B_{n+1}^2} \right\} \left| \frac{1}{B_n} - \frac{1}{B_{n+1}} \right| \\ \leq 2 \frac{\lambda}{\underline{\mu}} \sqrt{t} \frac{\sigma_G}{B_{n-1}B_n} \end{aligned}$$

since $\sqrt{B_{n+1}^2 - B_n^2} > B_{n+1} - B_n$. This tends to 0 as $t \rightarrow \infty$. We remark that

$$W_t(\lfloor t/\bar{\mu} + \lambda\sqrt{t}/\bar{\mu} \rfloor) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

which implies

$$\sum_{|(t-A_n)/\sqrt{t}| < \lambda} \left| W_t(n) - \frac{1}{\sqrt{2\pi}} \cdot \frac{\mu_n}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \right| < 4\epsilon$$

which yields the result. \square

PROPOSITION 1. *If Conditions S(a)–S(b) are verified then*

$$\lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \left| W_t(n) - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \right| = 0.$$

PROOF. This is immediate from Lemmas 1, 2 and 6.

REMARKS. Proposition 1 gives

$$\lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} = \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} W_t(n) = 1.$$

LEMMA 7. *If Conditions S(a)–S(c) are satisfied,*

$$\begin{aligned} \lim_{t \rightarrow \infty} \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \left| \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \right. \\ \left. - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp\left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} \right| = 0. \end{aligned}$$

PROOF.

$$\begin{aligned} & \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \left| \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \right. \\ & \qquad \qquad \qquad \left. - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} \right| \\ & < \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \\ & \quad \cdot \left| 1 - \exp \left\{ \frac{-2tA_n + A_n^2 + 2tn\mu - n^2\mu^2}{2B_n^2} \right\} \right|. \end{aligned}$$

However

$$\frac{-2tA_n + A_n^2 + 2tn\mu - n^2\mu^2}{2B_n^2} = -\frac{[A_n - n\mu]}{2B_n} \cdot \left[\frac{2(t - A_n)}{B_n} + \frac{A_n - n\mu}{B_n} \right].$$

But

$$\frac{A_n - n\mu}{B_n} = \frac{A_n - n\mu}{\sqrt{n}} \cdot \frac{\sqrt{n}}{B_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Also $|(t - A_n)/\sqrt{t}| < \lambda$ implies $\sqrt{n} > ((t - \lambda\sqrt{t})/\bar{\mu})^{1/2}$, hence

$$\begin{aligned} \left| \frac{t - A_n}{\sqrt{n}} \right| & < \frac{\lambda\sqrt{t}}{\sqrt{n}} < \lambda\sqrt{t} \left(\frac{t - \lambda\sqrt{t}}{\bar{\mu}} \right)^{-1/2} \quad \text{if } \left| \frac{t - A_n}{\sqrt{t}} \right| < \lambda \\ & < k \quad \text{for all } t, \text{ for some constant } k. \end{aligned}$$

Therefore $(t - n\mu)/B_n = O(1)$. Hence for any $\epsilon > 0$, we may choose a $T > 0$ such that for $t > T$

$$\left| \exp \left\{ \frac{-2tA_n + A_n^2 + 2tn\mu - n^2\mu^2}{2B_n^2} \right\} - 1 \right| < \epsilon$$

for all n such that $|(t - A_n)/\sqrt{t}| < \lambda$. Therefore

$$\begin{aligned} & \overline{\lim} \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \left| \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\} \right. \\ & \qquad \qquad \qquad \left. - \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t-n\mu)^2}{2B_n^2} \right\} \right| \\ & < \epsilon \overline{\lim}_{t \rightarrow \infty} \sum_{|(t-A_n)/\sqrt{t}| < \lambda} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t-A_n)^2}{2B_n^2} \right\}. \end{aligned}$$

Now

$$\overline{\lim}_{t \rightarrow \infty} \sum_{|(t - A_n)/\sqrt{t}| < \lambda} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - A_n)^2}{2B_n^2} \right\} < 1.$$

The result follows. \square

LEMMA 8. *If Conditions S(a)–S(c) are satisfied then for any $\epsilon > 0$ there exists a λ_2 such that*

$$\lim_{t \rightarrow \infty} \sum_{|(t - n\mu)/\sqrt{t}| > \lambda_2} \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - n\mu)^2}{2B_n^2} \right\} < \epsilon.$$

PROOF. This follows from Lemma 1.

Note that

$$\frac{1}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} e^{-(\mu^2/2\sigma^2)s^2} ds = \frac{1}{\mu}.$$

Hence there is a $\bar{\lambda} > 0$ such that

$$\left| \frac{1}{\sqrt{2\pi} \sigma} \int_{-\bar{\lambda}/\sqrt{\mu}}^{\bar{\lambda}/\sqrt{\mu}} e^{-(\mu^2/2\sigma^2)s^2} ds - \frac{1}{\mu} \right| < \bar{\epsilon}.$$

If we break $[-\bar{\lambda}/\sqrt{\mu}, \bar{\lambda}/\sqrt{\mu}]$ at the following points:

$$\left\{ \dots, \frac{-n}{\sqrt{n-n}}, \dots, \frac{-1}{\sqrt{n-1}}, 0, \frac{1}{\sqrt{n+1}}, \dots, \frac{n}{\sqrt{n+n}}, \dots \right\},$$

we have

$$\begin{aligned} \frac{1}{\sqrt{2\pi} \sigma} \int_{-\bar{\lambda}/\sqrt{\mu}}^{\bar{\lambda}/\sqrt{\mu}} e^{-(\mu^2/2\sigma^2)s^2} ds &= \lim_{n \rightarrow \infty} \sum_{|n| < \bar{\lambda}\sqrt{n}/\mu} \frac{1}{\sqrt{2\pi}\sigma^2} \\ &\quad \cdot \frac{1}{\sqrt{n+n}} \exp \left\{ -\frac{1}{2} \frac{\mu^2}{\sigma^2} \left(\frac{n}{\sqrt{n+n}} \right)^2 \right\} \end{aligned}$$

since $(n + 1)/\sqrt{n + n + 1} - n/\sqrt{n + n} \approx 1/\sqrt{n + n}$. Let $\mu\bar{n} = t$:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}\sigma} \int_{-\bar{\lambda}/\sqrt{\mu}}^{\bar{\lambda}/\mu} e^{-(\mu^2/2\sigma^2)s^2} ds &= \lim_{t \rightarrow \infty} \sum_{|(t - n\mu)/\sqrt{t}| < \bar{\lambda}} \frac{1}{\sqrt{2\pi}} \\ &\quad \cdot \frac{1}{\sigma\sqrt{n}} \exp \left\{ \frac{-(t - n\mu)^2}{2\sigma^2 n} \right\}. \end{aligned}$$

This gives

$$\lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma\sqrt{n}} \exp \left\{ \frac{-(t - n\mu)^2}{2\sigma^2 n} \right\} = \frac{1}{\mu}.$$

PROOF OF THEOREM 2.

$$\begin{aligned} & \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \left| W_t(n) - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - n\mu)^2}{2B_n^2} \right\} \right| \\ & < \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \left| W_t(n) - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - A_n)^2}{2B_n^2} \right\} \right| \\ & \quad + \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \left| \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - A_n)^2}{2B_n^2} \right\} \right. \\ & \quad \quad \left. - \frac{\mu_n}{\sqrt{2\pi}} \cdot \frac{1}{B_n} \exp \left\{ \frac{-(t - n\mu)^2}{2B_n^2} \right\} \right|. \end{aligned}$$

The first expression goes to 0 as $n \rightarrow \infty$ by Proposition 1. The second goes to 0 as $n \rightarrow \infty$ using Lemma 7 and Lemmas 1 and 8.

PROOF OF THEOREM 3.

$$\lim_{\substack{t \rightarrow \infty \\ t \in R_+}} \left| P \{ V_t \in A \} - \sum_{m=1}^{\infty} \frac{\alpha_m}{\mu_m} W_t(m) \right| = 0$$

by Theorem 1. By Theorem 2,

$$\lim_{\substack{t \rightarrow \infty \\ t \in R_+}} \left| \sum_{m=1}^{\infty} \frac{\alpha_m}{\mu_m} W_t(m) - \sum_{m=1}^{\infty} \frac{\alpha_m}{\sqrt{2\pi}} \cdot \frac{1}{B_m} \exp \left\{ \frac{-(t - m\mu)^2}{2B_m^2} \right\} \right| = 0.$$

Let $\bar{\mu}_m = (\mu/(\alpha + 1))(\alpha_m + 1)$; hence $\mu/(\bar{\mu} + 1) < \bar{\mu}_m < \mu(\bar{\mu} + 1)$. Let $\bar{A}_n = \sum_{k=1}^n \bar{\mu}_k$ and $\gamma_n = \sum_{k=1}^n \alpha_k$; therefore

$$\frac{\bar{A}_n - n\mu}{\sqrt{n}} = \frac{1}{\sqrt{n}} \left[\frac{\mu}{(\alpha + 1)} \gamma_n + \frac{n\mu}{\alpha + 1} - n\mu \right] = \frac{\mu}{\alpha + 1} \cdot \frac{(\gamma_n - \alpha n)}{\sqrt{n}} \rightarrow 0$$

as $n \rightarrow \infty$. Next since $\sigma^2 n > B_n^2 \geq (\underline{\sigma})^2 n$ for all n there is a subsequence n_k such that $\sigma_{n_k}^2 > (\underline{\sigma})^2$ for all k . We may construct an i.i.d. sequence $\{L_{n_k}\}_{k=1}^{\infty}$ of uniformly distributed random variables with mean 0 and variance v where $\sqrt{3}v < \frac{1}{2}\mu/(\bar{\mu} + 1)$ (i.e., uniform on $[-\sqrt{3}v, \sqrt{3}v]$). Set $L_n = 0$ off the subsequence. We may construct a sequence $\{Y_n\}_{n=1}^{\infty}$ of independent random variables, also independent of $\{L_{n_k}\}_{k=1}^{\infty}$, such that Y_n has mean $\bar{\mu}_n$, has support on $(+\frac{1}{2}\mu/(\bar{\mu} + 1), \infty)$ and such that the variance of $\bar{T}_n = Y_n + L_n$ is σ_n^2 . It is quite easy to check that $\{\bar{T}_n\}_{n=1}^{\infty}$ satisfies Condition C(a) since \bar{T}_{n_k} has a bounded density. By construction \bar{T}_n is positive.

Applying Theorem 2 to $\{\bar{T}_k\}_{k=1}^{\infty}$ we have

$$\lim_{t \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\bar{\mu}_m}{\sqrt{2\pi}} \cdot \frac{1}{B_m} \exp \left\{ \frac{-(t - m\mu)^2}{2B_m^2} \right\} = 1.$$

Hence

$$\lim_{t \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\alpha_m + 1}{\sqrt{2\pi}} \cdot \frac{1}{B_m} \exp \left\{ \frac{-(t - m\mu)^2}{2B_m^2} \right\} = \frac{\alpha + 1}{\mu}.$$

But setting $0 = \alpha = \alpha_1 = \alpha_2 = \dots$ gives

$$\lim_{t \rightarrow \infty} \sum_{m=1}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{B_m} \exp \left\{ \frac{-(t - m\mu)^2}{2B_m^2} \right\} = \frac{1}{\mu},$$

so that

$$\lim_{t \rightarrow \infty} \sum_{m=1}^{\infty} \frac{\alpha_m}{\sqrt{2\pi}} \cdot \frac{1}{B_m} \exp \left\{ \frac{-(t - m\mu)^2}{2B_m^2} \right\} = \frac{\alpha}{\mu}.$$

This gives the result. \square

COROLLARY 1. *In the lattice case if Conditions S(a)–S(c) are verified, then*

$$\lim_{\substack{t \rightarrow \infty \\ t \in R_+}} P \{ \text{renewal at } t \} = \frac{1}{\mu}.$$

PROOF. Define $N_t = \inf\{n - 1 | S_{n-1} \leq t < S_n\}$.

The process $V_t = t - S_{N_t}$, $t \in R_+$, is a nonstationary regenerative process which regenerates itself when $V_t = 0$. Also $V_t = 0$ if and only if $S_{N_t} = t$. The mean time $V_t = 0$ per cycle is 1, so applying Theorem 3

$$\lim_{\substack{t \rightarrow \infty \\ t \in R_+}} P \{ \text{renewal at } t \} = \frac{1}{\mu}. \quad \square$$

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