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MODULI OF PUNCTURED TORI AND THE ACCESSORY PARAMETER OF LAMÉ'S EQUATION

BY

L. KEEN,[1] H. E. RAUCH AND A. T. VASQUEZ[1]

ABSTRACT. To solve the problems of uniformization and moduli for Riemann surfaces, covering spaces and covering mappings must be constructed, and the parameters on which they depend must be determined. When the Riemann surface is a punctured torus this can be done quite explicitly in several ways. The covering mappings are related by an ordinary differential equation, the Lamé equation. There is a constant in this equation which is called the "accessory parameter". In this paper we study the behavior of this accessory parameter in two ways. First, we use Hill's method to obtain implicit relationships among the moduli of the different uniformizations and the accessory parameter. We prove that the accessory parameter is not suitable as a modulus–even locally. Then we use a computer and numerical techniques to determine more explicitly the character of the singularities of the accessory parameter.

0. Introduction. In the classical theory of uniformization of tori there are two distinct models for the space of moduli: the complex analytic τ (the period ratio) in the upper half plane and the real analytic trace parameters of Fricke for the Fuchsian uniformization of a punctured torus. Our original goal was to find an explicit relation between these models of Teichmüller space. They are related via an ordinary differential equation which is completely determined except for an unknown constant, called the "accessory parameter". Explicit computation of the accessory parameter then would give the desired relationship and also seemed to give another model for the moduli space. Indeed, in a modern version [3], the accessory parameter which arises in the quasi-Fuchsian uniformization of the punctured torus is a complex analytic parameter for the moduli space. Accessory parameters are interesting in their own right. There is an extensive discussion of accessory parameters in the classical literature; however, little of a concrete nature was known. We hope that what we have done here sheds some light on the general problem of accessory parameters.

In this paper, we prove that the accessory parameter is not even a local parameter for Teichmüller space and we give qualitative information about it;

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e.g., it is a nonholomorphic (real analytic) modular form of weight two for the full modular group. We treat the differential equation by Hill's method and determine the Fricke parameters as functions of τ and the accessory parameter. This approach leads to implicit formulas from which it is difficult to draw any conclusions. We then proceed to use a computer and numerical techniques to obtain information on the behavior of the accessory parameter. It was this approach that led us to the main theorem of the paper. We obtain, however, a great deal more qualitative information which we cannot prove in any other way. For example, the main theorem asserts only the existence of "folding" in the accessory parameter mapping. But we know much more; we know essentially where the folding occurs and that the fold is relatively simple. These results raise questions for which we do not yet have answers: what is the geometric significance of the fold for the underlying tori?

§1 is a brief description of the two models of Teichmüller space and §2 contains a discussion of the action of the modular group on each of these models and how it can be used to set up the correspondence between them. In §3 we use the geometry of the torus to relate the branch points to their images in the covering spaces which lead to the different models.

The accessory parameter is introduced in §4 and the transfomation laws it obeys are derived. The formulas for the Fricke parameters in terms of τ and the accessory parameter are obtained. These are at the heart of the numerical investigations. The theorem, that the accessory parameter is not a local parameter for Teichmüller space, is proved in §5.

§6 and §7 contain discussions of the numerical work: the methods, formulas and their derivations are given in §6, and a discussion of the results, as well as tables and graphs are given in §7.

1. Let $\mathfrak{M}$ = the Teichmüller space for a singly punctured torus. Two descriptions of $\mathfrak{M}$ are known. See **[6]** for a classical and **[10]** for a modern account. To set notation we briefly review these.

The first model for $\mathfrak{M}$ is U = the upper half plane = $\{u + iv \in \mathbf{C} | u, v$ real and $v > 0\}$. The equivalence Φ_1: $U \to \mathfrak{M}$ is described as follows. For $\tau \in U$, let $L = L(\tau)$ be the lattice generated by 1 and τ; i.e., $L = \mathbf{Z}1 + \mathbf{Z}\tau$. Let $T = T(\tau)$ be the corresponding torus $= \mathbf{C}/L$. Let $T' = T'(\tau)$ be obtained by removing from T a single point–to be specific, by removing the point determined by $0 \in \mathbf{C}$. Let t' = the point in T' determined by $(1 + \tau)/2$ in $\mathbf{C}$. We use t' as base-point to form the fundamental group $\pi_1 = \pi_1(T', t')$. Let $\alpha_1(s) = (1 + \tau)/2 + s$ for $0 \leqslant s \leqslant 1$; and $\beta_1(s) = (1 + \tau)/2 + s\tau$. α_1 and β_1 determine loops at t' in T'. Let α, β be the elements of $\pi_1(T', t')$ so represented. It is a familiar fact that π_1 is a *free* group freely generated by α and β. Finally, $\Phi_1(\tau) \in \mathfrak{M}$ is represented by the punctured torus T' together with the marking specified by α and β.

The second model for $\mathfrak{M}$ is $\mathfrak{FL} = \{(x, y, z) \in \mathbf{R}^3 | x, y, z$ are positive and $x^2 + y^2 + z^2 = xyz\}$. The equivalence Φ_2: $\mathfrak{FL} \to \mathfrak{M}$ is described as follows.

Let Δ = the interior of the unit disc. The triple $(x, y, z) \in \mathfrak{FL}$ will determine two matrices A_1 and B_1 which determine holomorphic automorphisms, A and B, of Δ.

$$A_1 = \begin{pmatrix} t & -T \\ -T & t \end{pmatrix}, \qquad B_1 = \begin{pmatrix} s & -Se^{i\alpha} \\ -Se^{-i\alpha} & s \end{pmatrix}, \tag{1.1}$$

where $2t = x$, $2s = y$, $T = \sqrt{(t^2 - 1)}$, $S = \sqrt{(s^2 - 1)}$ and $z = 2(st + ST \cos \alpha)$, $0 < \alpha < \pi$. Note that $x = \mathrm{Trace}(A_1)$, $y = \mathrm{Trace}(B_1)$ and $z = \mathrm{Trace}(A_1 B_1)$. (See **[10]**.) It is known that the group G, generated by A and B, is freely generated by A and B and that G acts freely and properly discontinuously on Δ such that Δ/G is a singly punctured torus.

Let $s \in \Delta/G$ be the point determined by $0 \in \Delta$. The natural projection $\Delta \to \Delta/G$ exhibits Δ as the universal covering space of Δ/G; G is the group of covering transformations. The point 0 determines, in the familiar fashion, an isomorphism of groups: $G \xrightarrow{\cong} \pi_1(\Delta/G, s)$. Let $a, b \in \pi_1$ be the elements corresponding via this isomorphism to $A, B \in G$. Finally $\Phi_2(x, y, z) \in \mathfrak{M}$ is represented by the punctured torus Δ/G and the marking specified by a and b.

Since Φ_1: $U \to \mathfrak{M}$ and Φ_2: $\mathfrak{FL} \to \mathfrak{M}$ are equivalences, so is $\varphi = \Phi_2^{-1} \circ \Phi_1$: $U \to \mathfrak{FL}$ and it is this mapping we wish to discuss.

The Teichmüller modular group operates on $\mathfrak{M}$ and hence (via Φ_i) on U and $\mathfrak{FL}$. Its action corresponds to the usual action of the classical modular group PSL(2, **Z**) on U. The action on $\mathfrak{FL}$ is also known (**[10]**). For example, the involution, $\pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, corresponds to $(x, y, z) \to (y, x, xy - z)$. The element of order 3, $\pm\left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, corresponds to $(x, y, z) \to (y, z, x)$. Since these elements generate PSL(2, **Z**), this gives the full correspondence.

We remark that this implies, writing $\varphi(\tau) = (x(\tau), y(\tau), z(\tau))$ that the functions x, y, z: $U \to \mathbf{R}^+$ are very much related to one another. Any one determines the other two in a simple fashion. For example, $y(\tau) = x(-1/\tau)$ and $z(\tau) = x(-1/(1 + \tau))$. Thus the function x determines the function φ.

It is convenient to enlarge the modular group slightly by adjoining the element of order 2 (which normalizes PSL(2, **Z**)) and is a reflection across the imaginary axis. The resulting super-modular group contains the modular group as a subgroup of index 2. Via φ, this new element acts on $\mathfrak{FL}$ by sending (x, y, z) to $(x, y, xy - z)$.

2. Qualitative information about φ. This super-modular equivariance gives a *great* deal of information about the map φ: $U \to \mathfrak{FL}$; e.g., the unique fixed point in U of $\tau \to -((1 + \tau)/\tau)$ is $\rho = (-1 + \sqrt{3}\, i)/2 = e(2\pi i/3)$. Hence $\varphi(\rho) = (x_1, y_1, z_1)$ is fixed by $(x, y, z) \to (y, z, x)$; and $\varphi(\rho) = (3, 3, 3)$. Simi-

larly $\varphi(i) = (2\sqrt{2}, 2\sqrt{2}, 4)$. Indeed one can calculate φ on the entire orbit of ρ and i under PSL(2, **Z**). In the same vein, the imaginary axis is fixed by $\tau \to -\bar{\tau}$ and so its φ image in $\mathfrak{F}\mathfrak{L}$ is fixed by $(x, y, z) \to (x, y, xy - z)$. This latter is the sublocus $xy = 2z$. Similarly φ takes the vertical line $\{\tau \in U | \operatorname{Re} \tau = -\frac{1}{2}\}$ onto the sublocus $y = z$. φ takes the unit circle onto the sublocus $x = y$ such that $|\tau| > 1$ corresponds to $x < y$, and the circle $|\tau + 1| = 1$ onto the sublocus $x = z$ such that $|\tau + 1| < 1$ corresponds to $x > z$.

REMARK. One can use φ to transport the Poincaré metric to the locus $\mathfrak{F}\mathfrak{L}$. It would be amusing to "see" this metric on $\mathfrak{F}\mathfrak{L}$. We have, of course, some clues concerning it; i.e., the action just described of the super-modular group is an action via *isometries*.

Let $\tau \in U$ and $\varphi(\tau) = (x, y, z) \in \mathfrak{F}\mathfrak{L}$. Then if $(T'; \alpha, \beta)(\tau)$ and $(\Delta/G; a, b)(x, y, z)$ are as above, they represent the same element of Teichmüller space $\mathfrak{M}$. This implies, of course, considerably more than just the conformal equivalence of T' and Δ/G. Indeed with the normalization above we can show:

PROPOSITION 1. *There is a holomorphic equivalence*: $\Delta/G \xrightarrow{\sim} T'$ *taking the point* $s \in \Delta/G$ *to* $t' \in T'$; *furthermore, the induced map of fundamental groups takes* a *to* α *and* b *to* β.

PROOF. Let $\tilde{T}'$ = the universal covering space of T'. By the uniformization theorem, $\tilde{T}'$ is holomorphically equivalent to one of the following three surfaces: $P_1(\mathbf{C})$ = the Riemann sphere, $\mathbf{C}$ = the complex plane, or Δ = the interior of the unit disc. Covering space theory implies that $\pi_1(T', t')$ acts freely and properly discontinuously as a group of holomorphic equivalences of $\tilde{T}'$. Since π_1 is free (not free abelian), this rules out the first two possibilities and we conclude that there is a holomorphic map π: $\Delta \to T'$ which is a universal covering.

Let G' = the group of covering transformations of $\pi = \{h: \Delta \to \Delta | h$ is a homeomorphism and $\pi \circ h = \pi\}$. Such h's are automatically holomorphic automorphisms of Δ and covering space theory yields a specific isomorphism $G' \xrightarrow{\cong} \pi_1(T', t')$ once a z' in $\pi^{-1}(t')$ is specified. Let $A', B' \in G'$ correspond to α, β via this isomorphism. Since $(T'; \alpha, \beta)$ and some $(\Delta/G; a, b)$ are equivalent it follows that A' and B' are conjugate to the hyperbolic transformations a and b. Thus A' and B' are hyperbolic transformations.

LEMMA 1. *The axes of* A' *and* B' *intersect at* z'.

PROOF. Let N_1: $T' \to T'$ be the involution $N_1([z]) = [-z]$ for $z \in \mathbf{C}' = C - L$. It is easy to see that $N_1(t') = t'$ and hence that N_{1*}: $\pi_1(T', t') \to \pi_1(T', t')$ is defined. It is straightforward to verify that

$$N_{1*}(\alpha) = \alpha^{-1} \quad \text{and} \quad N_{1*}(\beta) = \beta^{-1}. \tag{2.1}$$

By covering space theory there is a continuous N: $\Delta \to \Delta$ such that $\pi \circ N = N_1 \circ \pi$ and $N(z') = z'$. Since $N_1^2 = \mathrm{id}$ it readily follows that $N^2 = \mathrm{id}$. Hence N is a homeomorphism. Since π is a local holomorphic equivalence, N is holomorphic. Thus N is an element of order two.

We claim

$$NA'N^{-1} = A'^{-1} \quad \text{and} \quad NB'N^{-1} = B'^{-1}. \tag{2.2}$$

Consider the maps NA' and $A'^{-1}N$: $\Delta \to \Delta$. Composing each with π yields $N_1 \circ \pi$: $\Delta \to T'$. Therefore, these maps are equal if they agree at a single point. Thus it suffices to show that $N(A(z')) = A'^{-1}(N(z')) = A'^{-1}(z')$. The isomorphism $G' \xrightarrow{\simeq} \pi_1(T', t')$ which we are using says that $A'^{-1}(z')$ is the end point of $\tilde{\gamma}$ where $\tilde{\gamma}$ is any path starting at z' and $\gamma = \pi \circ \tilde{\gamma}$ represents α^{-1}. Equation (2.1) says that $N_1 \circ \alpha$ represents α^{-1}. Now if $\tilde{\alpha}$ is a lift of α starting at z', then $N \circ \tilde{\alpha}$ is an appropriate $\tilde{\gamma}$. But the isomorphism again tells us that $\tilde{\alpha}$ ends at $A(z')$. Thus $N \circ \tilde{\alpha}$ ends at $N(A(z'))$. The second part of the claim is analogous.

The proof of the lemma is completed by invoking the following geometric lemma.

GEOMETRIC LEMMA. *Let E, X be holomorphic equivalences of Δ. Suppose E is of order 2 and X is hyperbolic. Then the unique fixed point in Δ of E lies on the axis of X if and only if $EXE^{-1} = X^{-1}$.*

We return now to the proof of Proposition 1.

By conjugating with an appropriate holomorphic equivalence we may assume: $z' = 0$, axis of $A' =$ real axis, attracting fixed point of $A' = -1$. Let $G =$ this conjugate of G' and let A and B be the elements corresponding to A' and B'. Clearly A and B will be represented by matrices A_1 and B_1 as in (1.1). Let $x = \mathrm{Trace}(A_1)$, $y = \mathrm{Trace}(B_1)$ and $z = \mathrm{Trace}(A_1B_1)$. Since the axes of A_1 and B_1 intersect, it follows from the trace calculus (see **[11]**) that $x^2 + y^2 + z^2 = xyz$ if and only if $ABA^{-1}B^{-1}$ is parabolic. The parabolicity follows from the discussion below (see the remark in §4, and also **[6]**). Hence $(x, y, z) \in \mathfrak{F}\mathfrak{L}$ and $\varphi(\tau) = (x, y, z)$. $\square$

REMARK. Having normalized so that $z' = 0$, we see that $N(z) = -z$. We note that $p \circ N = N_2 \circ p$ for $N_2(z) = -z + (1 + \tau)/2$ since N_2 covers N_1: $T' \to T'$ and fixed $(1 + \tau)/2$.

3. Further geometry. We note that $\mathbf{C}' = \mathbf{C} - L$ is a covering space of $T' = \mathbf{C}'/L$. L is the group of covering transformations, and of course it is well known that we have a canonical identification $L \cong \pi_1(T', t')^{\mathrm{ab}} = \pi_1$–made–abelian. Put another way, the covering $\mathbf{C}' \to T'$ corresponds to the commutator subgroup of π_1. Since π: $\Delta \to \Delta/G \simeq T'$ is the universal

covering we can factor π through $\mathbf{C}'$, i.e., we have a commutative diagram:

$$\begin{tikzcd} \Delta \arrow[d, "\pi"] \arrow[dr, "p"] & \\ \Delta/G = T' & \mathbf{C}'/L \arrow[l] \end{tikzcd}$$

We may assume (and we do) that $p(0) = (1 + \tau)/2$. Of course, this diagram exhibits Δ as the universal covering space of $\mathbf{C}'$.

PROPOSITION 2. *$p(A(z)) = p(z) + 1$ and $p(B(z)) = p(z) + \tau$.*

PROOF. It is convenient to let c: $\mathbf{C}' \to T'$ denote the canonical map so far unnamed. Let h: $\mathbf{C}' \to \mathbf{C}'$ be $h(z) = z + 1$. Then the first assertion is that $p \circ A = h \circ p$. Since c: $\mathbf{C}' \to T'$ is a covering we could prove this if we show: $c \circ (p \circ A) = c \circ (h \circ p)$ and $p \circ A(0) = h \circ p(0) \in \mathbf{C}'$. The first is trivial since LHS $= \pi \circ A = \pi$ and RHS $= (c \circ h) \circ p = c \circ p = \pi$. By the definition of A, $A(0) =$ end point of $\tilde{\alpha}$ where $\tilde{\alpha}$ is a path in Δ starting at 0 and covering a representative of $\alpha \in \pi_1$. Hence $p(A(0)) =$ end point of $p \circ \tilde{\alpha}$. But $p \circ \tilde{\alpha}$ is a path in $\mathbf{C}$ which lifts a representative of α. We defined α by means of just such a path; specifically α_1 and $\alpha_1(1) = (1 + \tau)/2 + 1 = h((1 + \tau)/2) = h(p(0))$. $\square$

We now introduce a fundamental domain, $\mathfrak{D}$, for the action of G on Δ. We use the Dirichlet fundamental domain at the intersection of the axes of A and B.

The boundary of $\mathfrak{D}$ consists of four (non-Euclidean) lines. These are the loci of points equidistant from 0 and $X(0)$ as $X = A, B, A^{-1}$ and B^{-1}.

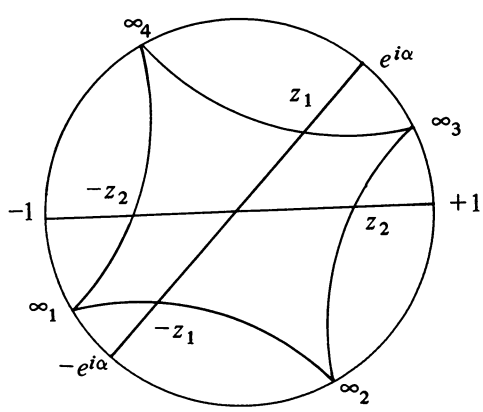

FIGURE 1

As the diagram indicates, z_2 is the intersection of the axis of A and the side of $\mathfrak{D}$ associated with A^{-1} (hereafter called the right edge). The angle between

them is $\pi/2$. z_1 is the intersection of the axis of B and the edge associated with B^{-1} (hereafter called the top edge); etc.

Since the involution N fixes 0, $N(z) = -z$ and the picture is, as it appears, symmetric around the origin.

The cusp ∞_3 is between 1 and $e^{i\alpha}$ and is $(st + ST\cos\alpha)/(St + sTe^{-i\alpha})$. It is clear from the picture that $A(\infty_3) = \infty_4 = -\infty_2 = -B(\infty_3)$. (See [10].)

The condition $x^2 + y^2 + z^2 = xyz$ is expressible as $ST\sin\alpha = 1$ or $\sinh(d_2)\sinh(d_1)\sin\alpha = 1$ where d_i = (non-Euclidean) distance from 0 to z_i. Note $\tanh(d_i/2) = z_i$, $x = 2\cosh d_1$, $z_2^2 = (x-2)/(x+2)$, $z_1^2 = (y-2)/(y+2)$.

PROPOSITION 3. $p(z_1) = 1/2$; $p(z_2) = \tau/2$; $p(z_3) = (1+\tau)/2$ $(z_3 = 0)$.

PROOF. The last assertion was part of our normalization. Consider z_2. Clearly $NA(z_2) = N(-z_2) = z_2$. So $p(z_2) = p(NA(z_2)) = N_2(p(A(z_2))) = N_2(p(z_2)+1) = -(p(z_2)+1) + 1 + \tau = -p(z_2) + \tau$; etc. $\square$

REMARK. Of course, $1/2$, $\tau/2$ and $(1+\tau)/2$ are the points usually denoted ω_1, ω_2 and ω_3 in the Weierstrass theory. Their images in T' are the fixed points of the involution N_1 (N_1 is the "sheet interchange"; generically this is the only nontrivial automorphism supported by T'). Their images under the $\wp$-function are denoted by e_1, e_2, e_3. Thus $\wp \circ p\colon \Delta \to \mathbf{C}$ is a branched covering, the branching is of order 2 at e_1, e_2, e_3 (since the preimages are z_1, z_2, z_3). The cusps of G map to ∞ where the branching is parabolic and so of infinite order.

Alternatively z_1, z_2, z_3 can be characterized as the fixed points of the involutions (see (2.2)) $E_1 = NB$, $E_2 = NA$, $E_3 = N$. Yet again they can be characterized (see geometric lemma) as the intersections of the axes of the pairs $(B^{-1}, B^{-1}A)$, $(A^{-1}B, A)$, (A, B) $(= (E_1E_3, E_1E_2), (E_2E_1, E_3E_2), (E_3E_2, E_3E_1))$.

The cusp ∞_3 is fixed by $E_2E_3E_1$ $((E_2E_3E_1)^2 = A^{-1}B^{-1}AB)$. Similarly for the others.

$E_2 = N$ normalizes G. Let $\hat{G}$ = the group generated by G and N. Index$(G, \hat{G}) = 2$. It is well known that $\hat{G}$ is the free product of the three subgroups of order 2, generated by E_1, E_2, E_3. G determines $\hat{G}$ and conversely. A fundamental domain for $\hat{G}$ is one-half of $\mathfrak{D}$, e.g., it can be obtained by bisecting $\mathfrak{D}$ with the line from ∞_4 to $\infty_2 = -\infty_4$.

Aside from Proposition 2 we know nothing, in general, about the relationship (via p) between the fundamental domain $\mathfrak{D}$ and $\mathfrak{D}'$, the usual fundamental domain for $L = \mathbf{Z}1 + \mathbf{Z}\tau$ acting in $\mathbf{C}'$. When τ is on the imaginary axis however ($\mathfrak{D}'$ is a rectangle), we can exploit the fact that T' supports an anticonformal involution to say more.

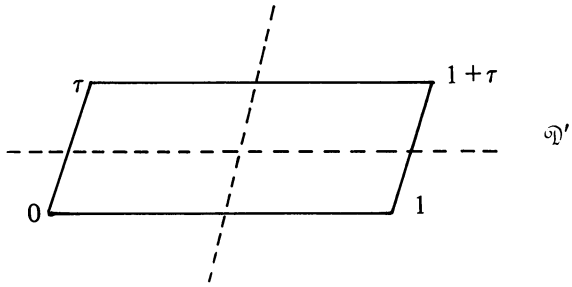

FIGURE 2

PROPOSITION 4. *In the rectangular case, the map* $p\colon \Delta \to \mathbf{C}'$ *maps the fundamental domain* $\mathfrak{D}$ *onto the fundamental domain* $\mathfrak{D}'$ *(the cusps correspond to corners). Furthermore the axis of* A *goes to the horizontal line through* $(1+\tau)/2$; *the axis of* B *goes to the vertical line through* $(1+\tau)/2$.

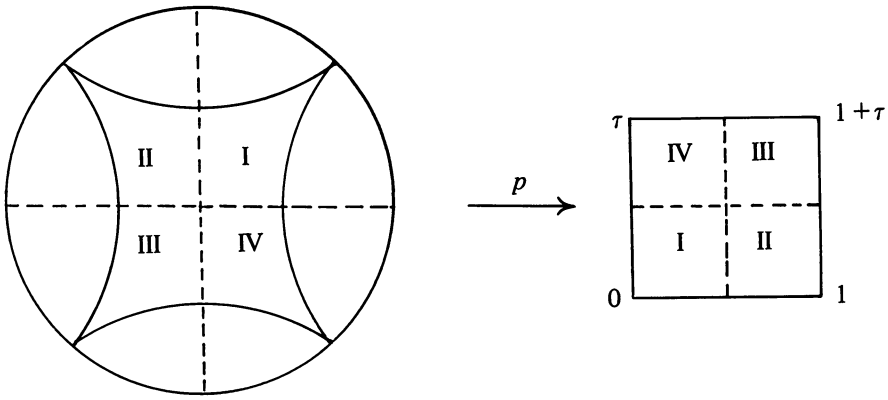

FIGURE 3

Before proving Proposition 4 we introduce some further notation.

Let l = the horizontal line in $\mathbf{C}'$ through $(1+\tau)/2$. Let R_2 be reflection across this line. $R_2(z) = \bar{z} + \tau$ (recall τ is pure imaginary). Let $R\colon \Delta \to \Delta$ be the involution covering R_2 and fixing 0 (produced as above). Since p is a local holomorphic equivalence, R is anticonformal.

LEMMA 2. $R(z) = \bar{z}$.

PROOF. As in Lemma 1, $R_2\colon \mathbf{C}' \to \mathbf{C}'$ induces $R_1\colon T' \to T'$. Clearly $R_1(t') = t'$ and $R_{1*}(\alpha) = \alpha$ ($R_{1*}\colon \pi_1(T', t') \to \pi_1(T', t')$). As in Lemma 1, this implies that $RAR^{-1} = A$.

Hence $R(A(0)) = A(R(0)) = A(0)$. Thus R and $z \to \bar{z}$ are both anticonformal automorphisms of Δ which fix both 0 and $A(0)$ and are therefore identical. $\square$

PROOF OF PROPOSITION 4. We now prove that $p(\text{axis of } A) = l =$ the horizontal line through $(1 + \tau)/2$. Note l is the set of fixed points of R_2 and axis of A is the set of fixed points of R. Trivially then, $p(\text{axis of } A) \subset l$. On the other hand, l is simply connected so there is a unique continuous section $s\colon l \to \Delta$ such that $s((1 + \tau)/2) = 0$. Since l is fixed by R_2, $R \circ s\colon l \to \Delta$ is another such section. Thus $R \circ s = s$, $R(s(l)) = s(l)$, and $s(l) \subset (\text{axis of } A)$; so $l = p(s(l)) \subset p(\text{axis of } A)$.

We now prove that $p(\text{right edge}) =$ the part of the imaginary axis between 0 and τ. The idea is first to show that right edge = set of fixed points of the map RAN.

To this end, we note that right edge = the geodesic perpendicular to the axis of A and passing through z_2. The involution $E_2 = NA$ fixes z_2 and thus reflects the axis of A through z_2. $RE_2 = RNA$ does the same but is anticonformal. This shows that RNA fixes the right edge, since it is an isometry, and fixes z_2 and the tangent to the right edge at z_2.

Since $p \circ (RNA) = (R_2 \circ N_2 \circ T_1) \circ p$ we see that $p(\text{right edge}) \subset$ fixed set of $R_2 \circ N_2 \circ T_1$. Computation shows this latter set to be $\mathbf{C}' \cap$ imaginary axis. Since the right edge is connected and $p(z_2) = \tau/2$, $p(\text{right edge}) \subset$ the part of the imaginary axis between 0 and τ. Since this latter set is connected and simply connected we can argue as above that the inclusion can be reversed.

The other assertions are handled similarly. For example, the top edge is fixed by RB, etc. $\square$

REMARK. The proposition shows that the quadrilaterals $\mathfrak{D}$ and $\mathfrak{D}'$ are conformally equivalent. Ahlfors [**1**, p. 52] calculated a conformal invariant of $\mathfrak{D}'$ which is τ/i. The numbers $(x, y, z) \in \mathfrak{F}\mathfrak{L}$ describe $\mathfrak{D}$ where $\varphi(\tau) = (x, y, z)$. If the invariant for $\mathfrak{D}$ could be calculated in terms of (x, y, z) we would have expressed τ in terms of x, y, z, i.e., calculated φ at least on the imaginary axis. Specifically what is wanted is the extremal length of the set Γ of curves in $\mathfrak{D}$ which connect the left and right sides.

4. The Lamé equation. Let L be any lattice, $\mathbf{C}' = \mathbf{C} - L$. Let $p\colon \Delta \to \mathbf{C}'$ be a universal covering map (any other is of the form $p \circ \gamma$ for γ a linear fractional automorphism of Δ). Let ν be any (locally defined) inverse for p. Consider the Schwartzian derivative $D(\nu)$ of ν. A fundamental feature of the operator D is that $D(\gamma \circ f) = D(f)$ for any linear fractional transformation γ. Hence $D(\nu)$ is independent of choices of p and ν; i.e., we have a well-defined holomorphic map $f_L\colon \mathbf{C} - L \to \mathbf{C}$.

LEMMA 3. *$f_L(z + l) = f_L(z)$ for all $l \in L$, $z \in \mathbf{C} - L$.*

PROOF. $T\colon \mathbf{C}' \to \mathbf{C}'$ defined by $T(z) = z + l$ is a holomorphic automorphism. So if $p\colon \Delta \to \mathbf{C}'$ is a universal covering map so is $T \circ p$. By the remark above there is an automorphism γ of Δ such that $T \circ p = p \circ \gamma$.

Hence for appropriate branches ν_i of the inverse of p, $\nu_1 \circ T = \gamma \circ \nu_2$. Now $D(\gamma \circ \nu_2)(z) = D(\nu_2)(z) = f_L(z)$. On the other hand, $D(\nu_1 \circ T)(z) = D(\nu_1) \circ T(z) = f_L(z + l)$ [5, p. 291]. □

It is well known [5] that ν can be expressed as the ratio w_1/w_2 of two linearly independent solutions of the differential equation,

$$d^2w/dz^2 + \tfrac{1}{2} f_L(z)w = 0, \qquad z \in \mathbf{C} - L. \tag{4.1}$$

Since we know a priori that p and hence ν are nicely behaved, classical theory [5, pp. 288 and 290] implies that $\frac{1}{2}f_L$ has a pole of order 2 at each $l \in L$; indeed that its principal part is $\frac{1}{4}(z - l)^{-2}$. In view of Lemma 3 we deduce:

$$\tfrac{1}{2} f_L(z) = \tfrac{1}{4}(\wp(z|L) + C(L))$$

for some constant $C(L)$. As usual $\wp(z|L)$ denotes the Weierstrass $\wp$-function associated with the lattice L. The following summarizes the discussion.

PROPOSITION 5. *Let L be a lattice, $\wp(z|L)$ the associated Weierstrass $\wp$-function. There is a constant $C(L)$ (the 'accessory parameter') such that local inverses of the universal covering map p: $\Delta \to \mathbf{C} - L$ are given as ratios of linearly independent solutions of the Lamé equation*

$$d^2w/dz^2 + \tfrac{1}{4}\,(\wp(z|L) + C(L))w = 0. \tag{4.2}$$

Furthermore we have the transformation laws:

$$\lambda^2 C(\lambda L) = C(L) \quad \textit{for } \lambda \in \mathbf{C} - 0, \tag{4.3a}$$

$$C(\bar{L}) = \overline{C(L)}. \tag{4.3b}$$

PROOF. It remains to prove (4.3a) and (4.3b). Let F: $\mathbf{C} - L \to \mathbf{C} - \lambda L$ be the holomorphic equivalence $F(z) = \lambda z$. Let p: $\Delta \to \mathbf{C} - L$ be a universal covering. Then $F \circ p$: $\Delta \to \mathbf{C} - \lambda L$ is also a universal covering. Thus if ν is a local inverse for $F \circ p$, $\nu \circ F$ is a local inverse for p. So

$$f_L(z) = D(\nu \circ F) = D(\nu) \circ F(z)(dF/dz)^2 + 0 = f_{\lambda L} \circ F(z)\lambda^2 = \lambda^2 f_{\lambda L}(\lambda z).$$

This yields: $\lambda^2(\wp(\lambda z|\lambda L) + C(\lambda L)) = \wp(z|L) + C(L)$; from which (4.3a) follows (see [20]).

For (4.3b) let J: $\mathbf{C} - L \to \mathbf{C} - \bar{L}$ be the map $J(z) = \bar{z}$. If p_1: $\Delta \to \mathbf{C} - L$ is a holomorphic universal covering map, so is $p_2 = J \circ p \circ J$: $\Delta \to \mathbf{C} - \bar{L}$. If ν_1 is a local inverse of p_1, $J \circ \nu_1 \circ J$ is a local inverse for p_2. Since $(J \circ f \circ J)' = J \circ f' \circ J$ for any holomorphic f, $D(\nu_2) = J \circ D(\nu_1) \circ J$ or $\overline{f_{\bar{L}}(\bar{z})} = f_L(\bar{z})$. Thus

$$\overline{\wp(z|\bar{L}) + C(\bar{L})} = \wp(\bar{z}|L) + C(L).$$

From its definition, $\wp(\bar{z}|L) = \overline{\wp(z|\bar{L})}$ and (4.3b) follows. □

REMARK. Since the principal part (at $l \in L$) of $\frac{1}{4}(\wp(z|L) + C(L))$ is $\frac{1}{4}(z - l)^{-2}$, it follows that the "monodromy" of equation (4.2) is parabolic. (See p. 292 of [5]. The discussion just prior to that of Hills' formula (equation (4.7)) relates this "monodromy" to the commutator $ABA^{-1}B^{-1}$.)

It is convenient to work with functions on U, the upper half plane, so we define $C(\tau) = C(L_\tau)$ for $\tau \in U$ when $L_\tau = \mathbf{Z}1 + \mathbf{Z}\tau$, the lattice associated with τ. We reformulate the transformation laws and record some obvious facts.

PROPOSITION 6.

$$C\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2\, C(\tau) \quad \textit{for} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbf{Z}). \tag{4.4a}$$

$$C(-\bar{\tau}) = \overline{C(\tau)}\,. \tag{4.4b}$$

Furthermore,

(a) *C is real valued on the imaginary axis;*

(b) *C is real on the vertical lines* $\mathrm{Re}\ \tau = \pm\frac{1}{2}$;

(c) *$C(\tau_0) = 0$ for $\tau_0 = i$, $-\frac{1}{2} + (\sqrt{3}/2)i$ or any point equivalent to these under the action of the modular group* $\mathrm{PSL}(2, \mathbf{Z})$.

PROOF. Equation (4.4b) yields (a). For (b), note that if $\mathrm{Re}\,\tau = -\frac{1}{2}$, τ is fixed by the map $\tau \to (-\bar{\tau} - 1)$, so $C(\tau) = C(\bar{\tau} - 1) = C(-\bar{\tau}) = \overline{C(\tau)}$. For (c), let $\tau \in U$ be fixed by $g \in \mathrm{SL}(2, \mathbf{Z})$. Then $C(\tau) = C(g(\tau)) = (1/g'(\tau))C(\tau)$. Hence if $C(\tau) \neq 0$, $g(\tau) = \tau$ and $g'(\tau) = 1$; this implies that $g = \pm I$. $\square$

Of course, further results of the same sort can be readily produced. For example, the unit circle consists of the points fixed by the map $\tau \to -1/\tau$. This leads to $\overline{C(\tau)} = \tau^2 C(\tau)$ for such τ.

We come now to the connection between Proposition 5 and the calculation of $\varphi\colon U \to \mathfrak{F}\mathfrak{L}$.

Let $p\colon \Delta \to \mathbf{C} - L$ be the specific map discussed in §3. Proposition 2 asserts that $p(Az) = p(z) + 1$ and $p(Bz) = p(z) + \tau$ where $L = L_\tau = \mathbf{Z}1 + \mathbf{Z}\tau$. Let Ω be a small ε-neighborhood of l, the horizontal line in $\mathbf{C}' = \mathbf{C} - L$ through the point $(1 + \tau)/2$. Since (for all small enough ε) Ω is simply-connected there is a unique holomorphic $\nu\colon \Omega \to \Delta$ such that $p \circ \nu(z) = z$ for all $z \in \Omega$ and $\nu((1 + \tau)/2) = 0$. Hence $\nu(z + 1) = A(\nu(\tau))$ for all $z \in \Omega$. By Proposition 5, $\nu = w_1/w_2$ for $w_i\colon \Omega \to \mathbf{C}$ where the w_i are independent solutions of equation (4.2) (in Ω).

From the form of equation (4.2) it is clear that $z \to w_i(z + 1)$ are also solutions of (4.2) in Ω. Since the space of solutions is spanned by w_1 and w_2 there are constants α, β, γ and δ such that

$$w_1(z + 1) = \alpha w_1(z) + \beta w_2(z) \quad \text{for } z \in \Omega, \tag{4.5a}$$

$$w_2(z+1) = \gamma w_1(z) + \delta w_2(z) \quad \text{for } z \in \Omega. \tag{4.5b}$$

Let $\mathrm{wr}(z)$ be the Wronskian of w_1 and $w_2 = w_1'(z)w_2(z) - w_1(z)w_2'(z)$. It is standard that wr is a nonzero constant in Ω. But direct calculation from (4.5) shows that $\mathrm{wr}(z+1) = (\alpha\delta - \beta\gamma)\mathrm{wr}(z)$ and hence $\alpha\delta - \beta\gamma = 1$.

Comparing equation (4.5) with $\nu(z+1) = A(\nu(z))$, $\nu = w_1/w_2$ we see that A is represented by the matrix $\pm\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{C})$.

Thus (up to sign) the matrix representing A (of §§2 and 3) is determined by equations (4.5), i.e., by the behavior of a basis of solutions to Lamé's equation (4.2). The matrix depends on the basis $\{w_1, w_2\}$ which we have no way of knowing directly from equation (4.2). However, the *trace of the matrix is basis independent* and can be calculated directly from equation (4.2). This is relevant to the calculation of $\varphi\colon U \to \mathfrak{F}\mathfrak{L}$. If $\varphi(\tau) = (x, y, z)$, x is the absolute value of the trace of a matrix representing A (where $L = L_\tau = \mathbf{Z}1 + \mathbf{Z}\tau$).

We will make use of a formula, due to Hill, which expresses the trace in terms of the data appearing in (4.2).

For notation, we review some well-known material ([**14**], [**20**]). Let

$$d^2w/dt^2 + Q(t)w = 0 \tag{4.6}$$

be a periodic differential equation on $\mathbf{R}$ of period π; i.e., $Q(t+\pi) = Q(t)$ (for convenience, assume that Q is C^∞-complex valued). Let $\mathfrak{S} = S(Q)$ be the $\mathbf{C}$-vector space of solutions of (4.6). Define $F = F_Q\colon S \to S$ by $(Fw)(t) = w(t+\pi)$. This linear map is usually called the *Floquet Transformation* associated with equation (4.2). It is well known that $\det(F) = 1$ (Wronskian argument as above). The invariant we are interested in is $\mathrm{Trace}(F_Q) \in \mathbf{C}$.

The formula for $\mathrm{Trace}(F_Q)$ is due to Hill; it uses the Fourier coefficients of Q. Let $Q(t) = \theta_0 + \sum_{-\infty}^{\prime +\infty} g_n e^{2int}$ (the $'$ indicates that the $n = 0$ term has been omitted).

$$\mathrm{Trace}(F_Q) = 2 - 4\sin^2\left(\pi/2\sqrt{\theta_0}\right)\Delta(\theta), \tag{4.7}$$

where $\Delta(\theta)$ is the determinant of the $(\infty \times \infty)$ matrix $\| g_{n-m}/(\theta - 4n^2) + \delta_{n,m}\|_{-\infty}^{+\infty}$ ($g_0 = 0$ and δ is the Kronecker delta).

For numerical work (see §§6 and 7 below) it is convenient to pass from the Weierstrass form of the Lamé equation to the Jacobi form. We will use standard notation for all of this (see [**17**], [**20**]). The change of independent variable is given by $j(z) = \alpha = iK' + z\sqrt{e_1 - e_2}$. This takes the lattice L in the z-plane to the lattice L' in the α-plane where $L' = \mathbf{Z}2K + \mathbf{Z}2iK'$. The Lamé equation (4.2) becomes

$$d^2w/d\alpha^2 + \tfrac{1}{4}\left(\lambda\,\mathrm{sn}^2(\alpha, \lambda) + B\right)w = 0 \tag{4.8}$$

where $\lambda = (e_2 - e_3)/(e_1 - e_3)$ and we have written $B = B(\tau)$ for $(C(\tau) + e_3(\tau))/(e_1(\tau) - e_3(\tau))$.

Note that $j \circ p\colon \Delta \to \mathbf{C} - j(L) = \mathbf{C} - L'$ is a holomorphic universal covering map. We have

$$(j \circ p)(Az) = (jp)(z) + 2K, \qquad (j \circ p)(Bz) = (jp)(z) + 2iK'. \tag{4.9}$$

The argument sketched above leads to the following theorem.

THEOREM 1. *The map* $\varphi\colon U \to \mathfrak{F}\mathfrak{L}$ *can be described as follows*: $\varphi(\tau) = (x, y, z) \in \mathbf{R}^3$ *where* x, y *and* z *are positive and*

$$x = |\mathrm{trace}(F_{Q_1})|, \qquad y = |\mathrm{trace}(F_{Q_2})|, \qquad z = |\mathrm{trace}(F_{Q_3})|, \tag{4.10}$$

where the F_{Q_i} *are the Floquet Transformations associated with the periodic (period* π*) differential equations*

$$d^2w/dt^2 + Q_i(t)w = 0, \qquad t \in \mathbf{R}. \tag{4.11}$$

$$\begin{aligned} Q_1(t) &= \tfrac{1}{4}\left(\lambda\, \mathrm{sn}^2\left(\frac{2K}{\pi}\, t; \lambda\right) + B\right)\left(\frac{2K}{\pi}\right)^2 \\ &= \left(\frac{K}{\pi}\right)^2\left(1 - \frac{E}{K} + B\right) + \sum_{n=1}^{\infty} \frac{2n}{q_1^n - q_1^{-n}} \cos(2nt), \end{aligned} \tag{4.12a}$$

$$\begin{aligned} Q_2(t) &= \frac{1}{4}\left(\lambda\, \mathrm{sn}^2\left(K + iK' + \frac{2iK'}{\pi}\, t; \lambda\right) + B\right)\left(\frac{2iK'}{\pi}\right)^2 \\ &= \left(\frac{iK'}{\pi}\right)^2\left(B + \frac{E'}{K'}\right) + \sum_{n=1}^{\infty} \frac{2n}{q_2^n - q_2^{-n}} \cos(2nt), \end{aligned} \tag{4.12b}$$

$$\begin{aligned} Q_3(t) &= \frac{1}{4}\left(\lambda\, \mathrm{sn}^2\left(K + iK' + \frac{K + iK'}{\pi}\, 2t; \lambda\right) + B\right)\left(\frac{2(K + iK')}{\pi}\right)^2 \\ &= \left(\frac{K + iK'}{\pi}\right)^2\left(B + \frac{K - E + iK'}{K + iK'}\right) \\ &\quad + \sum_{n=1}^{\infty} \frac{2n}{q_3^n - q_3^{-n}} \cos(2nt), \end{aligned} \tag{4.12c}$$

where $q_\alpha = e^{\pi i \tau_\alpha}$; $\tau_1 = \tau$, $\tau_2 = -1/\tau$, $\tau_3 = -(1/(\tau + 1))$.

PROOF. The proof for (4.12a) has essentially been given above; what is missing is the classical determination of the Fourier coefficients of sn^2. (See **[20]**.) (4.12b) and (4.12c) are treated by analogy with (4.12a). For (4.12c), one uses w_1 and w_2 defined in a small ε-neighborhood of the line in the α-plane,

$$t \to \alpha(t) = K + iK' + (2(K + iK')/\pi)t.$$

The map $t \to t + \pi$ corresponds to $\alpha(t) \to \alpha(t) + 2K + 2iK'$ while $j(ABz) = jp(z) + 2K + 2iK'$. Thus $|\mathrm{trace}(F_{Q_3})| = |\mathrm{trace}(A_1B_1)| = z$. $\square$

Note that the accessory parameter $B = B(\tau)$ in the Jacobi form of the

equation is given by

$$B(\tau) = (C(\tau) + e_3(\tau))/(e_1(\tau) - e_3(\tau)). \tag{4.13}$$

The transformation laws for C give rise to the following transformation laws for B.

PROPOSITION 7 (TRANSFORMATION LAWS). (a) *B: $U \to \mathbf{C}$ factors through the usual elliptic modular function λ: $U \to \mathbf{C} - \{0,1\}$, i.e., there exists b: $\mathbf{C} - \{0, 1\} \to \mathbf{C}$ such that $B(\tau) = b(\lambda(\tau))$. Equivalently $B(\tau) = B(g(\tau))$ if $g \in \Gamma(2)$, the congruence subgroup of level two (λ induces an equivalence $U/\Gamma(2) \simeq \mathbf{C} - \{0, 1\}$)*

(b) $b(\lambda/(\lambda - 1)) = (b(\lambda) + \lambda)/(1 - \lambda)$,
(c) $b(1 - \lambda) = -(b(\lambda) + 1)$,
(d) $b(1/(1 - \lambda)) = (b(\lambda) + 1)/(\lambda - 1)$,
(e) $b((\lambda - 1)/\lambda) = -(b(\lambda) + \lambda)/\lambda$,
(f) $b(1/\lambda) = b(\lambda)/\lambda$
(g) $b(\bar{\lambda}) = \overline{b(\lambda)}$.

PROOF. Recall the transformation laws for the $e_i(\tau)$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbf{Z})$, there is associated a permutation $\sigma = \sigma(g)$ of the set $\{1, 2, 3\}$; and $e_i(g(\tau)) = (c\tau + d)^2 e_{\sigma(i)}(\tau)$. σ is a homomorphism and is determined therefore by the values, $\sigma(\tau) = \sigma\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right)$ and $\sigma(S) = \sigma\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right)$; see **[17]**. $\Gamma(2)$ is the kernel of the permutation map. This implies (a) since it is well known that λ: $U \to \mathbf{C} - \{0, 1\}$ factors through $U/\Gamma(2)$ (indeed it induces a holomorphic equivalence).

The next five laws correspond to the nontrivial permutations, the so-called λ-group. The calculations are straightforward although tedious, e.g.

$$\begin{aligned} B(S(\tau)) &= \frac{C(S\tau) + e_3(S\tau)}{e_1(S\tau) - e_3(S\tau)} = \frac{C(\tau) + e_1(\tau)}{e_3(\tau) - e_1(\tau)} \\ &= \frac{C + e_1}{e_3 - e_1} = \frac{C + e_3}{e_3 - e_1} + \frac{e_1 - e_3}{e_3 - e_1} = -B(\tau) - 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} \lambda(S\tau) &= \frac{e_2(S\tau) - e_3(S\tau)}{e_1(S\tau) - e_3(S\tau)} = \frac{e_2(\tau) - e_1(\tau)}{e_3(\tau) - e_1(\tau)} \\ &= \frac{e_2 - e_1}{e_3 - e_1} = \frac{e_2 - e_3}{e_3 - e_1} + \frac{e_3 - e_1}{e_3 - e_1} \\ &= -\lambda(\tau) + 1; \end{aligned}$$

thus $b(1 - \lambda) = -b(\lambda) - 1$.

(g) follows from $C(-\bar{\tau}) = \overline{C(\tau)}$ and $\lambda(-\bar{\tau}) = \overline{\lambda(\tau)}$. $\square$

5. In [12] it is proved that the mappings ρ: $U \to \mathfrak{F}\mathfrak{L}$ and C: $U \to \mathbf{C}$ are real analytic. It is the purpose of this section to show that neither the function $C(\tau)$ nor the function $b(\lambda)$ is locally schlicht.

THEOREM 2(a). *$C(\tau)$ is not a one-to-one function in a neighborhood of some point on the line segment* $\operatorname{Re}\tau = -\frac{1}{2}$, $\frac{1}{2} \leqslant \operatorname{Im}\tau \leqslant \sqrt{3}/2$.

PROOF. Proposition 6 shows that the real analytic function $C(\tau)$ is real valued on this line segment and is equal to zero at its endpoints. □

From Proposition 7 we immediately obtain the following corollary:

PROPOSITION 8. (i) $b(\frac{1}{2}) = -\frac{1}{2}$,

(ii) *for* $\rho = -1/2 + i\sqrt{3}/2$, $b(-\rho) = \rho/(1-\rho)$ *and* $b(-\bar{\rho}) = -1/(1-\rho)$,

(iii) $b(-1) = 0$,

(iv) $b(2) = -1$,

(v) *for* $\operatorname{Re}\lambda = \frac{1}{2}$, $\operatorname{Re} b(\lambda) = -\frac{1}{2}$,

(vi) *for real* λ, $b(\lambda)$ *is real.*

THEOREM 2(b). *There is a real number* λ_0 $(-1 \leqslant \lambda_0 < 0)$ *such that b is not one-to-one on a real interval containing* λ_0. *In particular, b is not locally one-to-one and hence is not a local parameter for Teichmüller space.*

PROOF. Proposition 7(g) and (f) shows that b is real on the real axis and that $b(-1) = 0$.

If the theorem is false, b is locally one-to-one and hence monotonic on the interval $[-1, 0)$.

Thus for some $l \in [-\infty, +\infty]$ we have $\lim_{\lambda\uparrow 0} b(\lambda) = l$.

Monotonicity rules out $l = 0$; Lemma 4 rules out $l > 0$ and Lemma 5 rules out $l < 0$.

LEMMA 4. $l \leqslant 0$.

PROOF. The classical modular function λ maps $\{\tau \in U | \tau = -1 + iv, v > 0\}$ so that $v\uparrow\infty$ corresponds to $\lambda\uparrow 0$. Recall that, by definition, $b(\lambda(\tau)) = B(\tau)$.

Equation (4.12a) in Theorem 1 shows that Q_1 is a real valued function for $\operatorname{Re}\tau = -1$ and that its 0th Fourier coefficient is $(K/\pi)^2(1 - E/K + B)$. It is classical that $\lim_{v\uparrow\infty} E/K = 1$ and hence if the lemma fails, $\theta_0(\tau)$, the 0th Fourier coefficient, is positive for all sufficiently large v. We will use a theorem of Borg ([4]; see also [14]) to arrive at a contradiction.

On the one hand Borg's theorem implies that any (real valued) solution of equation (4.11) has infinitely many zeros when the 0th Fourier coefficient is positive, on the other hand, our interest in equation (4.11) stems from the fact that, if $\nu = w_1/w_2$ is the ratio of any two independent solutions then ν is "essentially" a partial right-inverse of the universal covering map $j \circ p$:

$\Delta \to \mathbf{C} - L'$. This implies that ν can have at most one zero. Together these facts imply that w_1 and w_2 must have common zeros, which is absurd. □

REMARK. In the language of Hill's equation, the λ of the equation which is of interest to us lies in the zeroth region of instability. This is in contrast to the more usual situation in which the λ's of interest lie in the regions of stability.

LEMMA 5. $l \geqslant 0$.

PROOF. We show that $l < 0$ leads to a contradiction. Note first that modular equivariance of the map φ: $U \to \mathfrak{F}\mathfrak{L}$ implies that $x(\tau + 1) = x(\tau)$. Thus $x(-1 + iv) = x(iv)$.

The qualitative information in §2 readily implies that $\lim_{v\uparrow\infty} x(iv) = 2$. On the other hand, equations (4.10a) and (4.7) yield

$$\pm x(\tau) = 2 - 4\Delta(\tau)\sin^2\left((\pi/2)\sqrt{\theta_0(\tau)}\,\right).$$

Since

$$\lim_{v\uparrow\infty} \theta_0(-1 + iv) = ((\pi/2)/\pi)\left(1 - 1 + \lim_{\lambda\uparrow 0} b(\lambda)\right) = l/4,$$

we will get a contradiction from $\lim_{v\uparrow\infty} \Delta(\tau) = 1$. This follows from Lemma 6 below.

For $n \geqslant 0$ let $d_n(\tau)$ be the determinant of the central $(2n + 1) \times (2n + 1)$ submatrix (see equation (4.7)) so that, by definition, $\Delta(\tau) = \lim_{n\uparrow\infty} d_n(\tau)$.

LEMMA 6. *Assume* $\lim_{\lambda\uparrow 0} b(\lambda) = l < 0$. *Let* $\varepsilon > 0$. *Then, for all sufficiently large* v, *we have*: *for all* n, $|d_n(-1 + iv) - 1| < \varepsilon$.

PROOF. The matrix d_n has 1's on the diagonal and if $i \neq j$,

$$A_{i,j} = \frac{1}{4i_i^2 - \theta_0(\tau)} \frac{i - j}{q^{i-j} - \bar{q}^{(i-j)}}$$

where $q = e^{\pi i\tau}$. It is elementary that for some nonnegative α, for all large enough v we have $|A_{i,j}| \leqslant b_i\eta(\tau)$ where $b_i = 1/(4_i^2 + \alpha)$ and $\tau = -1 + iv$, $\eta(\tau) = 1/(e^{\pi v} - \bar{e}^{\pi v})$.

The lemma now follows from an elementary estimate of the determinant. □

6. Numerical investigations–the method. One of our principal motivations in conducting the present research was to obtain insight into the behavior of the functions $C(\tau)$ and $b(\lambda)$. Finding ourselves unable, after a certain amount of effort, to obtain this insight analytically, we turned to numerical methods.

We find that a foundation for a numerical analysis is given by the basic formulas (equations (4.10), (4.12), and (4.7)) for the Fricke parameters x, y and z. These equations express the (real-valued) functions $x(\tau)$, $y(\tau)$ and $z(\tau)$

in terms of the intermediate (complex-valued) function $B(\tau)$, the so-called accessory parameter.

It follows from the discussion in §4 that if we define $x(\tau, B)$, $y(\tau, B)$ and $z(\tau, B)$ by the preceding formulas where we take B to be an *arbitrary* complex number, the resulting (complex-valued) functions x, y and z still satisfy the fundamental equation

$$x^2 + y^2 + z^2 = xyz. \tag{6.1}$$

We emphasize that for a given τ when $B = B(\tau)$ the quantities x, y and z are real.

Hence our point of departure is to attempt to solve the implicit equations

$$\operatorname{Im} x = 0, \qquad \operatorname{Im} y = 0, \qquad \operatorname{Im} z = 0, \tag{6.2}$$

for $B(\tau)$. Perhaps we should remark that the conditions $\operatorname{Im}(x(\tau, B)) = 0$, etc., imply that the *monodromy group* of

$$d^2w/d\alpha^2 + \tfrac{1}{4}\left(\operatorname{sn}^2(\alpha; \lambda) + B\right)w = 0$$

(viewed as an equation on the punctured torus $\mathbf{C} - L'/L'$, $\lambda = \lambda(\tau)$) determines a *Fuchsian* group in PSL(2, **C**).

Noting that $B(\tau)$ is a pair of real numbers, it seems at first glance that (6.2) is overdetermined. It occurs to one then that (6.1) can be used to reduce the three equations of (6.2) to two. Simple calculations however show that no such simple scheme works globally; no two of the equations in (6.2) imply (even in the presence of (6.1)) the third. Fortunately experience showed that, in the vicinity of a given τ, a judicious choice of two of the three equations of (6.2) in fact permitted numerical evaluation of $B(\tau)$. Indeed most of the numerical work we report consists of finding $b(\lambda) = B(\lambda(\tau))$ for λ transversing special analytic arcs, along which we knew a priori that $b(\lambda)$ is specified by a *single* real parameter. Thus for the most part one of the equations of (6.2) sufficed.

It should be emphasized that our remarks about the implicit equations are purely heuristic and that the actual computation proceeded, as it were, fortuitously without any a priori justification. For example, as Proposition 8 says, on $\operatorname{Re}\lambda = -\frac{1}{2}$, $\operatorname{Re} b(\lambda) = -\frac{1}{2}$. Writing $b(\lambda) = -\frac{1}{2} + ri$ we found experimentally that z was real and $x = \bar{y}$. Hence, it turned out to be sufficient to solve $\operatorname{Im} x = 0$. A posteriori, it was easy to verify these relationships.

As the preceding paragraph implicitly suggests we sometimes view τ and sometimes view λ as the basic parameter. We wish to point out how the known parameters (E, K, E', K', q) in the implicit equation (6.2) are determined from the given λ.

The first problem is to determine q, not only because it appears explicitly in the determinants Δ_1, Δ_2 and Δ_3 but also because the only accessible way of

computing K, K', E, E' explicitly is in terms of theta functions whose argument is q. We used the formula $q = \varepsilon + 2\varepsilon^5 + 15\varepsilon^9 + 150\varepsilon^{13} + O(\varepsilon^{17})$, $2\varepsilon = (1 - (1-\lambda)^{1/4})/(1 + (1-\lambda)^{1/4})$ (see [**20**, p. 486] or [**17**, p. 265]).

It should be observed that for the calculation of $b(\lambda)$ where λ is on the specific analytic arcs already mentioned, we proceed to define all quantities directly in terms of τ because the λ-arcs are the images of certain simple analytic arcs in the τ-plane. In other words, for these calculations τ, not λ is given. For K, K' we use $K = \frac{\pi}{2}\,\theta^2\left[{0 \atop 0}\right]$, $K' = -\,i\tau K$. For E we use

$$E = -KZ'(K + iK') = \frac{1}{4K}\,Z'\begin{bmatrix}0\\1\end{bmatrix}\left(\frac{1}{2} + \frac{\tau}{2}\right) = -\,\frac{1}{4K}\,Z'\begin{bmatrix}1\\0\end{bmatrix}(0)$$

$$= -\,\frac{1}{4K}\,\frac{\theta''\left[{1 \atop 0}\right]}{\theta\left[{1 \atop 0}\right]}\,(0, \tau)$$

$$= -\,\frac{\pi}{2}\left(\sum_{n=0}^{\infty}(2n+1)^2 q^{n^2+n}\right)\Big/\left(\sum_{0}^{\infty} q^{n^2+n}\right).$$

The first two equations are from [**17**, equation (8), p. 136 and line -7 on p. 135]. The next equality follows from the reduction formula and the substitution formula, [**17**, Chapter II] and the definition of Z as a logarithmic derivative of theta [**17**, line 16, p. 126]. The succeeding equality results from the fact that $\theta'\left[{1 \atop 0}\right](0) = 0$, since $\theta\left[{1 \atop 0}\right](s)$ is even; the final equality is obtained from simple manipulation of the theta series.

For E' we use

$$E' = \left(1 - \pi i\tau\theta'\left[{0 \atop 1}\right]/2\theta\left[{0 \atop 1}\right]\right)/\theta^2\left[{0 \atop 0}\right].$$

To obtain this formula we note [**17**, lines 4, 5, p. 135] that if we write $E = E(\lambda)$, then $E' = E(1-\lambda)$. The transition from λ to $1-\lambda$ is a consequence of the transformation $S\colon \tau \to -1/\tau$ [**17**, Table IV, p. 168], which also implies $K \to K' = -i\tau K$. If we then apply S to the next to last expression for E above, use the transformation law [**17**, line 12 of p. 182], logarithmically differentiate twice, and substitute, we obtain the desired formula.

Having made a decision to solve certain of the equations (6.2) numerically, we then resorted to the computer. Besides FORTRAN, the essential ingredients were two software packages, one used for computing the truncation of any of the infinite determinants, Δ, and another for solving the implicit equations (6.2).

The numerical solution then proceeds by the method of continuation. For example, to compute $b(\lambda)$ along $\operatorname{Re}\lambda = +\frac{1}{2}$ (i.e., $|\tau| = 1$) we let $\lambda = \frac{1}{2}$ (i.e., $\tau = i = e^{i\pi/2}$) and used as initial value $B(i) = b(\frac{1}{2}) = -\frac{1}{2}$ (Proposition 8(i)). We then incremented θ in $\tau = e^{i\theta}$ and the software routine found the corresponding $B(e^{i\theta})$ by an iterative procedure, using as initial guess the (computed) value of B at the previous θ.

7. Numerical investigation—results. Our initial conjecture was that $C(\tau)$ and $b(\lambda)$ were homeomorphisms and hence moduli for once punctured tori (see, however, Theorem 2, §4). In particular, then it seemed reasonable to compute the images of the boundaries of the τ and λ fundamental domains.

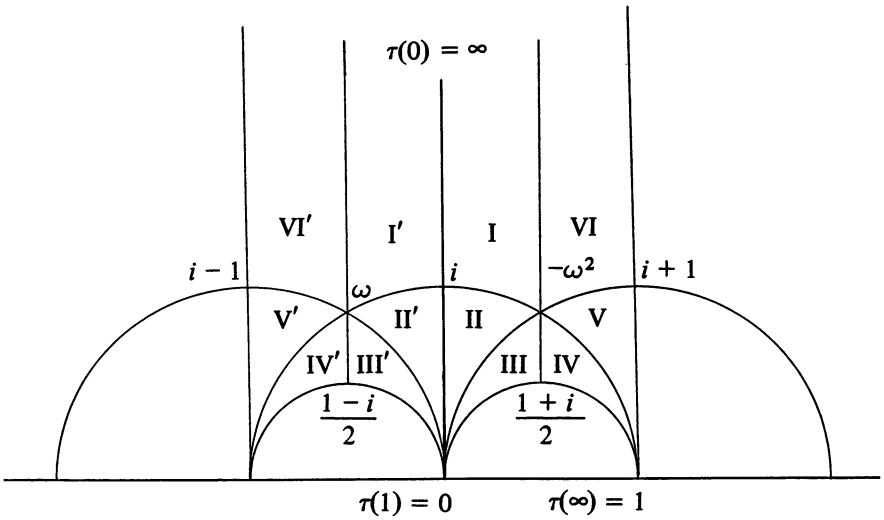

FIGURE 4. τ upper half-plane.

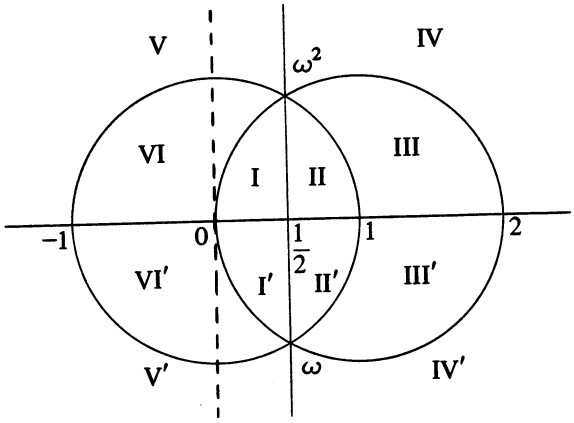

FIGURE 5. λ-plane.

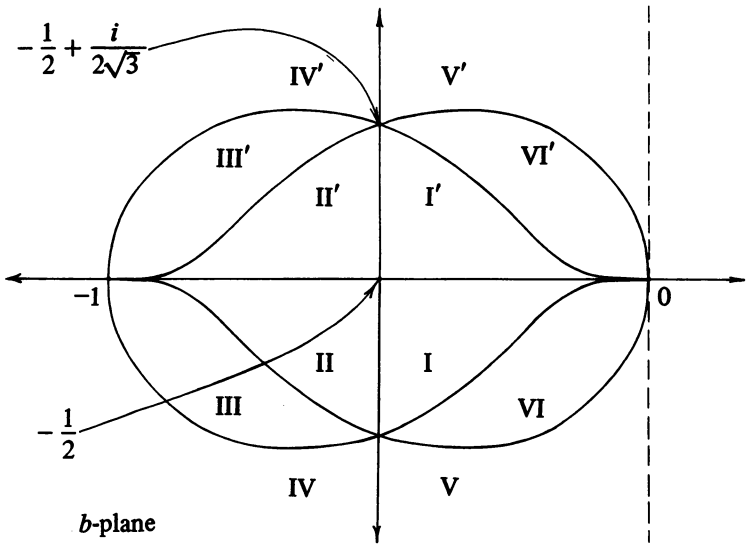

FIGURE 6.
Image under $b(\lambda)$ of boundaries of half-fundamental domains in λ-plane. Correspondence is shown by matching pairs of adjacent Roman numerals.

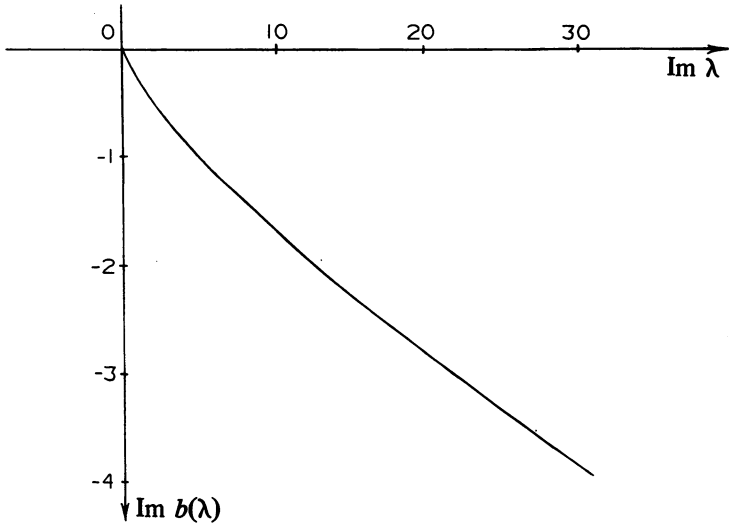

FIGURE 7
Graph of $\operatorname{Im} b(\lambda)$ versus $\operatorname{Im} \lambda$ for $\operatorname{Re} \lambda = 1/2$, based on Table 1.

The τ-upper half plane with part of a tesselation by fundamental domains of the modular group is shown in Figure 4. The union of regions numbered I through VI (and I′ through VI′) forms a fundamental domain for $\Gamma(2)$, the level 2 subgroup of the modular group. Figure 5 shows the regions in the λ-plane which correspond under the elliptic modular function λ. Each region is a half-fundamental domain for the so-called λ group [**17**, Chapter IV, §17].

Our first results consist of a map of the boundaries of the fundamental domain of the λ-group in Figure 5 by the map $b(\lambda)$. The results are shown in Figure 6, where again the correspondence is clearly indicated by the labeling.

Figure 6 was constructed as follows. From Proposition 8(i) in §5 we know that $B(i) = b(\frac{1}{2}) = -\frac{1}{2}$. We also know that, for $|\tau| = 1$ ($\operatorname{Re}\lambda = \frac{1}{2}$), $\operatorname{Re} B(\tau) = \operatorname{Re} b(\lambda(\tau)) = -\frac{1}{2}$. Starting with $b(\frac{1}{2}) = -\frac{1}{2}$ we solved the equation $\operatorname{Im} x = 0$ to tabulate b.

Figure 7 has a graph of $\operatorname{Im} b(\lambda)$ for $\lambda = \frac{1}{2} + si$, $0 < s < 30$, based on the values in Table 1.

TABLE 1

$\lambda = \frac{1}{2} + i \operatorname{Im}\lambda$		$b(\lambda) = -\frac{1}{2} + i \operatorname{Im} b(\lambda)$	
$\operatorname{Im}\lambda$	$\operatorname{Im} b(\lambda)$	$\operatorname{Im}\lambda$	$\operatorname{Im} b(\lambda)$
0.0	0.0	7.504	-1.358
1.069	-.3125	9.475	-1.609
1.931	-.5184	12.22	-1.940
2.962	-.7007	16.15	-2.388
4.130	-.8858	21.97	-3.010
4.965	-1.009	30.88	-3.903
6.055	-1.163		

From these values we derived the teardrop shaped lobe below the real b-axis with cusp at 0 by means of the following reasoning.

The transformation $\tau \to 1/(1-\tau)$ takes $e^{i\theta}$ to $\frac{1}{2} + (i/2)\cot\theta/2$, fixes the point $\frac{1}{2} + (\sqrt{3}/2)i$ and rotates (non-Euclideanly) the arc of $|\tau| = 1$ separating I and II onto the arc separating III and IV. The corresponding transformation in the λ group is $\lambda \to 1/(1-\lambda)$. Recall that $b(1/(1-\lambda)) = (b(\lambda) + 1)/(\lambda - 1)$. A table for the graph of the lobe is Table 2. The other teardrop lobe below the real axis is obtained from the transformation law $b(1-\lambda) = -(b(\lambda) + 1)$. From the law $b(\bar{\lambda}) = \overline{b(\lambda)}$ one obtains the remaining lobes.

To give more information about Figure 6 we calculated $b(\lambda)$ for $\operatorname{Re}\tau = 0$ $(0 < \lambda < 1)$. In fact we calculated $B(\tau)$ for $1 \leqslant \operatorname{Im}\tau \leqslant 4$ in 60 steps. Table 3

contains 10 of these. We used the transformation $\tau \to -1/\tau$ which corresponds to $\lambda - 1 - \lambda$ and the fact that $b(1 - \lambda) = (b(\lambda) + 1)$ to calculate $B(\tau)$ for $\frac{1}{4} < \operatorname{Im} \tau < 1$. The combined values are graphed in Figure 8.

An examination of Figure 6 suggested the first surprise of the investigation. Namely, the figure shows the image of the arc $1 < \lambda < 2$ has both endpoints at -1. Exploiting the transformation $\lambda \to 1/\lambda$ and the law $b(1/\lambda) = b(\lambda)/\lambda$

TABLE 2

"$.8839 - 2$" means $.8839 \times 10^{-2}$

Re λ	Im λ	Re $b(\lambda)$	Im $b(\lambda)$
2.000		-1.000	
1.993	.1203	- .9979	-.0355
1.971	.2387	- .9915	-.0705
1.936	.3533	- .9809	-.1046
1.887	.4623	- .9664	-.1372
1.826	.5642	- .9481	-.1679
1.753	.6575	- .9263	-.1965
1.672	.7049	- .9014	-.2224
1.582	.8135	- .8737	-.2455
1.485	.8745	- .8436	-.2655
1.383	.9236	- .8116	-.2822
1.279	.9604	- .7780	-.2956
1.172	.9850	- .7432	-.3056
1.0616	.9978	- .7078	-.3123
.9613	.9993	- .6720	-.3156
.8593	.9901	- .6362	-.3157
.7613	.9711	- .6009	-.3129
.6683	.9434	- .5662	-.3073
.5809	.9080	- .5325	-.2991
.5000	.8660	- .5000	-.2887
.4258	.8187	- .4688	-.2763
.3587	.7673	- .4391	-.2622
.2986	.7128	- .4109	-.2469
.2456	.6564	- .3844	-.2303
.1993	.5990	- .3595	-.2131
.1595	.5418	- .3362	-.1954
.1257	.4853	- .3145	-.1775

Table 2 continued

Re λ	Im λ	Re $b(\lambda)$	Im $b(\lambda)$
.09742	.4305	-.2942	-.1597
.07419	.3780	-.2753	-.1422
.05540	.3282	-.2577	-.1253
.04048	.2817	-.2412	-.1091
.02889	.2386	-.2258	-.09373
.02008	.1994	-.2113	-.07942
.01355	.1640	-.1976	-.06626
.8839-2	.1327	-.1846	-.04369
.5554-2	.1052	-.1721	-.04369
.3343-2	.08169	-.1601	-.03436
.1914-2	.06185	-.1486	-.02635
.1036-2	.04550	-.1375	-.01963
.5240-3	.03237	-.1266	-.01439
.2452-3	.02214	-.1160	-.9788-2
.1045-3	.01446	-.1057	-.6464-2
.3983-4	.8925-2	-.09571	-.4035-2
.1325-4	.5149-2	-.08595	-.2353-2
.3732-5	.2732-2	-.07468	-.1262-2
.8530-6	.1306-2	-.06732	-.6091-3
.1495-6	.5467-3	-.05851	-.2574-3
.1854-7	.1925-3	-.05006	-.9145-4
.1451-8	.5386-4	-.04204	-.2580-4
.6025-10	.1098-4	-.03447	-.5300-5
.1013-11	.1423-5	-.02744	-.6921-6
.4382-14	.9361-7	-.02100	-.4578-7
.2161-17	.2079-8	-.01524	-.3754-8

we constructed Table 4 (the values $b(\lambda)$, $1 \leqslant \lambda \leqslant 2$) and the corresponding graph (Figure 9) and found that b is not one-to-one on this interval. (In this calculation we solved $\operatorname{Im} z = 0$.) Thus our initial conjecture is false and b is not even a local parameter for Teichmüller space. (See Theorem 2(b) in §5.)

But going further, Figure 6 leads us to expect a folding of part of region III in the λ-plane under the map $b(\lambda)$. We pursued this and exhibited explicit folding of a part of region III. Indeed, Table 5 exhibits a coarse grid of

TABLE 3

$\tau = 0 + i \operatorname{Im} \tau$	λ real	$B(\tau) = b(\lambda)$ real
$\operatorname{Im} \tau$	λ	$B(\tau) = b(\lambda)$
1.000	.5000	-.5000
1.250	.2705	-.3443
1.550	.1156	-.2261
1.850	.04674	-.1591
2.150	.01848	-.1199
2.450	.7241-2	-.09516
2.750	.2828-2	-.07828
3.050	.1103-2	-.06598
3.350	.4299-3	-.05658
3.650	.1675-3	-.04916
3.950	.6529-4	-.04315

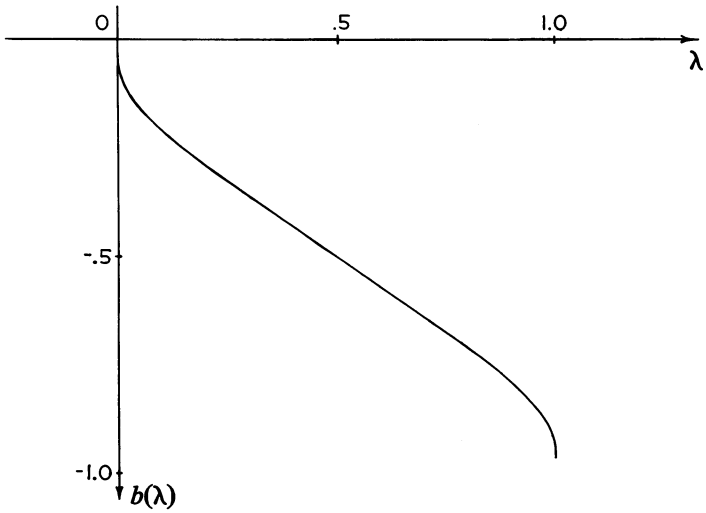

FIGURE 8

Graph of $b(\lambda)$ versus λ for $0 < \lambda < 1$ based on Table 3 ($\operatorname{Re} \tau = 0$, $B(\tau) = b(\lambda)$).

TABLE 4

λ	$b(\lambda)$	λ	$b(\lambda)$
1.0001	-.9579	1.371	-.8988
1.001	-.9350	1.446	-.9098
1.003	-.9243	1.540	-.9243
1.007	-.9114	1.658	-.9432
1.019	-.8967	1.808	-.9679
1.049	-.8821	2.000	-1.000
1.131	-.8751		

TABLE 5

Re λ	Im λ	.035	.040	.045	.050
-.01		-.1254	-.1293	-.1330	-.1364
		-.01519i	-.01728i	-.01936i	-.02142i
-.03		-.1241	-.1269	-.1296	-.1324
		-.01498i	-.01707i	-.01914i	-.02120i
-.05		-.1253	-.1272	-.1291	-.1310
		-.01474i	-.01681i	-.01887i	-.02092i
-.07		-.1266	-.1278	-.1291	-.1305
		-.01451i	-.01656i	-.01860i	-.02064i
-.09		-.1272	-.1281	-.1291	-.1301
		-.01430i	-.01633i	-.01830i	-.02035i
-.11		-.1272	-.1279	-.1287	-.1295
		-.01410i	-.01610i	-.01810i	-.02009i
-.13		-.1268	-.1273	-.1279	-.1285
		-.01392i	-.01590i	-.01787i	-.01984i

images of the indicated rectangular λ-grid. Figure 10 graphs the images of the indicated horizontal segments in the λ-plane. We point out that there are two folds coming together at a cusp.

It is important to observe that in the computations for Table 5 and Figure 10 we were not calculating along a single analytic arc in the λ-plane along which b could a priori be specified by a single real parameter. Rather the

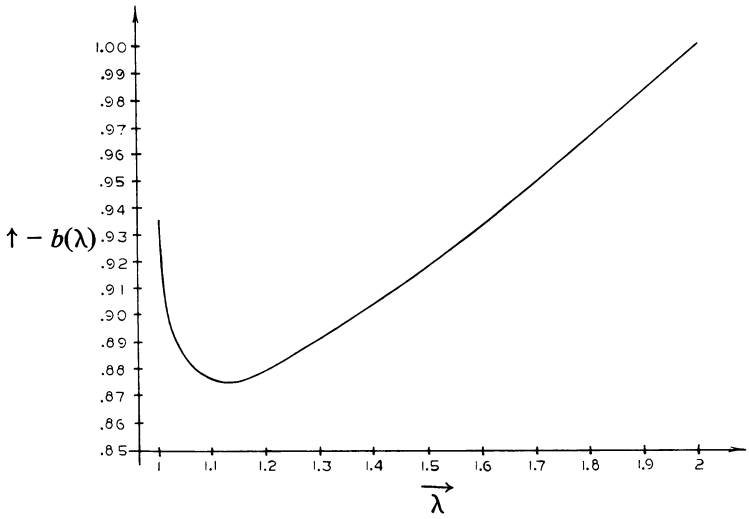

FIGURE 9

Graph of $-b(\lambda)$ versus λ for $1 < \lambda < 2$ showing the non-1-1 character of $b(\lambda)$. Graph based on Table 4.

calculation was truely two dimensional and hence it was necessary to use two of the equations in (6.2). The program automatically eliminated one of these three equations depending on the partial results available at the time. It is interesting that it did not consistently eliminate the same one.

We also calculated the image of the boundary of region I in the τ-plane under the mapping $C(\tau)$. Without exhibiting it explicitly we shall describe it. The two vertical segments both mapped on the positive real axis with apparently bounded monotonic graphs–different for the two segments. The image of the curved arc is a narow oval lying in the fourth quadrant, beginning and ending at the origin, with the branches meeting at a nonzero acute angle, one branch tangent to the imaginary axis. The far extreme point of the loop is approximately (.073, −.227).

The image under $C(\tau)$ of the vertical segment separating III and IV is still real, but it is zero at both ends (Proposition 6, (4.4a)), hence $C(\tau)$ cannot be locally one-to-one on that segment. In fact the graph seems to have the unique minimum value $-.34404$ at $\tau = \frac{1}{2} + .64240i$. The transformation laws indicate the mapping $C(\tau)$ is onto. Also the folding in the C-plane and the b-plane seems to occur at noncorresponding values of τ and λ.

Since $C(\tau)$ transforms as a modular form under the modular group PSL(2, **Z**) and is real (not complex) analytic, we looked through the literature to see if we could "guess" $C(\tau)$ by comparing it to a known form. The only appropriate form we found is due to Hecke [7]. We tabulated its values and

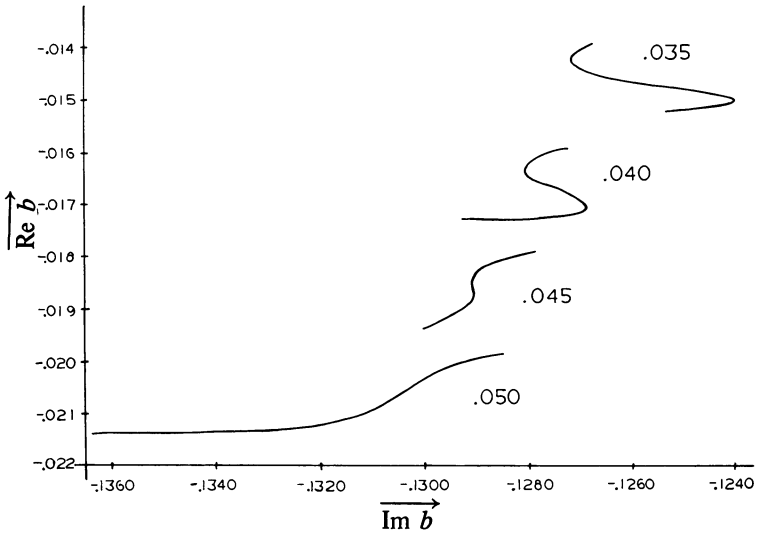

FIGURE 10

Image in the b-plane of the rectangular grid $-.01 < \operatorname{Re}\lambda < -.13$. $.035 < \operatorname{Im}\lambda < .050$ under $b(\lambda)$ (see Table 5). The images of only four horizontal coordinate lines are shown, each labeled with its value of $\operatorname{Im}\lambda$. The fold in the map is clearly evident. The two "creases" in the fold are the two curves (not shown) joining the corresponding "reversal" points. They clearly end in a cusp near the curve labeled .045. The λ-grid is in region VI near 0 in Figure 5, while the image shown here is in region VI near 0 in Figure 6.

compared them to our computed values for $C(\tau)$. There was no correlation between them at all.

As will be noted in the last section our calculations of the accessory parameter ($C(\tau)$ or $b(\lambda)$) routinely computed the values of the Fricke parameters x, y and z as functions of τ. In Figure 11 we give a graph of all three functions for $\tau = it$, $.2592 < t < 3.857$. We graphed $x(y, z)$ versus $s = \log t$ rather than t because of the symmetries revealed by the $\log t$ plot. These follow from the symmetry considerations in §1. On $\operatorname{Re}\tau = 0$, $2z = xy$ and $\tau \to -1/\tau$ $(s \to -s)$ interchanges x and y. Writing $X(s) = x(ie^s)$, $Y(s) = y(ie^s)$ etc. these imply $X(-s) = Y(s)$ and $Z(s) = \frac{1}{2}X(s)Y(s) = Z(-s)$.

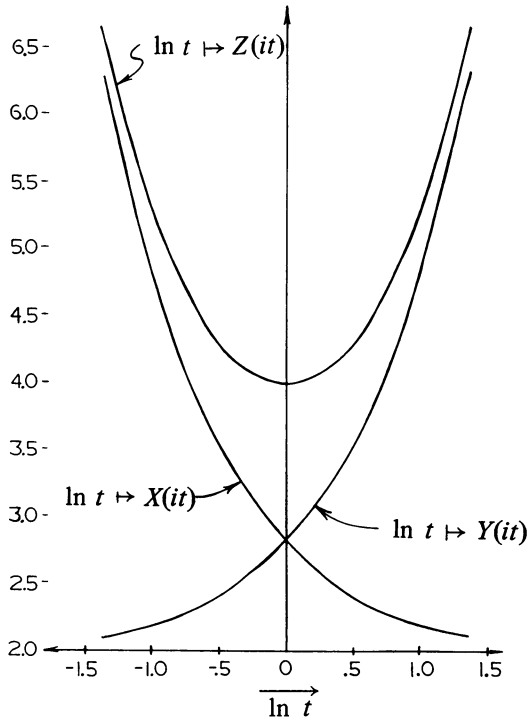

FIGURE 11

The three Fricke parameters, for purely imaginary $\tau = it$ as functions of $\ln t$.

8. Numerical investigations–discussion. The computations reported in §§6 and 7 were made on the 370/168 system at CUNY.

The FORTRAN programs are based on the use of two IMSL packages, LEQTIC to compute the truncated Hill determinants and ZSYSTM to solve the implicit equations obtained by setting imaginary parts of the appropriate Fricke parameters equal to zero. Because the IMSL packages seem to be operational only in double precision at CUNY we wrote the programs in double precision, but there seems to be no other reason for so doing. Copies of printouts of all programs used here are available from the authors.

The assigned error parameter in ZSYSTM was usually set at 10^{-6}. There were no general theoretical error estimates available to us. The only theoretical devices of some–limited–usefulness to us were: (i) the computation at every continuation step of $L = x^2 + y^2 + z^2 - xyz$, which is zero in theory and whose smallness of modulus was, therefore, reassuring (sometimes when x, y, z were all supposed to be real we divided L by $x^2 + y^2 + z^2$ to compute a relative error whose smallness of modulus was also reassuring); (ii)

after starting a continuation run with one of the few theoretically known values of, say, $b(\lambda)$, it was reassuring to reach another theoretically known value with great accuracy.

Other than this, we proceeded by testing consistency of our results as we lengthened the truncation in any given approximation. There are two sources of error due to approximation in the programs; that due to the finite truncation of the infinite Hill determinants, on the one hand, and that due to the truncation of the series ("q-series") for the theta constants on the other. Our computations were all performed with a 15×15 (i.e., index 7) truncation of the Hill determinants, and then some recomputation of a few selected results in sensitive regions with truncations up to 45×45 (index 22) was done. In each case the recomputed results coincide with the originals up to four significant figures, and we assume *ad hoc* that this is a reasonable guess for the accuracy of the remainder of our results. We used fixed truncations at index $n = 6$, thus at powers $n(n-1) = 30$ or $n^2 = 36$ of q for the appropriate theta constants. These truncations are quite accurate (as one readily verifies by hand computation) quite close to the real axis in the upper half-plane of τ. We did not investigate the effect of lengthening the truncation of these series. To do so would at most not visibly affect results shown in the graphs and figures we have included and, in particular, then, not the numbers recorded in our tables.

Finally, we point out that, if one leaves out debugging and some preliminary experimenting, the actual computations of which the results presented here are only an excerpt used only a few minutes of computing time on the 370/168. At the 45×45 truncation of the Hill determinants, by way of contrast, one or two continuation steps alone consume as much time.

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Department of Mathematics, Graduate School and University Center, The City University of New York, New York, New York 10036

Department of Mathematics, Lehman College, The City University of New York, Bronx, New York 10468