TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 255, November 1979

MARKOV CELL STRUCTURES FOR EXPANDING MAPS IN DIMENSION TWO

BY

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ABSTRACT. Let f: $M^2 \to M^2$ be an expanding self-immersion of a closed 2-manifold, then for some positive integer n, f^n leaves invariant a cell structure on M^2. A similar result is true when M is a branched 2-manifold.

0. Introduction. Let K be a topological space equipped with a continuous self-map f. It is known in many interesting cases that K can be partitioned into cells; i.e., given a cell complex structure. For example, this is possible when K is a smooth manifold. It is rarer for K to support a cell structure with respect to which f is a cellular map (i.e., leaves each skeleton invariant); for instance, this is impossible when K is the circle and f is rotation through an irrational angle.

Consider the situation when K is a closed smooth 2-manifold and f is an expanding immersion of K; i.e., $|df(X)| > |X|$ for all nonzero vectors X tangent to K and some Riemann metric on K. In this case, we show (Theorem 2.1) there is a positive integer n such that f^n (the composite of f with itself n-times) is cellular relative to some cell structure on K. (We do not know whether n can always be 1. See [3], [7] and [10] for other recent interesting constructions of invariant sets.)

We prove a similar (slightly weaker) result (Theorem 3.1) for expanding immersions of compact branched 2-manifolds satisfying Axioms 1, 2 and 3^+ of Williams. (See [14] for the basic definitions.) These objects arise in Williams' study [14] of expanding attractors. We hope Theorem 3.1 will be useful in helping to understand 2-dimensional expanding attractors which are apparently more complicated (cf. [5]) than the 1-dimensional case where Williams has a very good theory [13].

Theorems 2.1 and 3.1 are clearly extensions (in a very special setting) of the theory of Markov partitions [1], [12], [2] and [9]. This paper is also an introduction to the results announced in [4].

1. A cellular embedding result. In this section, we formulate and prove a crucial result Lemma 1.1, a strong form of the cellular approximation

Received by the editors April 19, 1978 and, in revised form, October 17, 1978.
AMS (MOS) subject classifications (1970). Primary 58F15.
Key words and phrases. Expanding immersion, branched manifold, cell structure.
[1]Both authors were partially supported by grants from the National Science Foundation.

0002-9947/79/0000-0514/$04.75

theorem for dimension 2. It is used in §2 to construct Markov cell structures for expanding maps on 2-manifolds.

Let M^2 denote a 2-dimensional Riemannian manifold. A *special cell structure* C in M^2 is a filtration by closed subsets,

$$\varnothing = C^{-1} \subset C^0 \subset C^1 \subset C^2 = |C| \subset M^2,$$

such that $C^i - C^{i-1}$ $(i = 0, 1, 2)$ has finitely many connected components, called the *i-cells* of C with the following properties.

> (i) The closure of each i-cell σ is homeomorphic to $\mathbf{D}^i = \{x \in \mathbf{R}^i \mid |x| \leq 1\}$ via a homeomorphism mapping σ onto $\{x \in \mathbf{R}^i \mid |x| < 1\}$. (The closure of a cell is called a closed cell.)
> (ii) The intersection of two closed cells is either empty or homeomorphic to $\mathbf{D}^i$ (for some i).
> (iii) Each vertex (0-cell) is contained in at least 2 and no more than 3 edges (closed 1-cells).
> (iv) Edges are smooth curves.
> If C satisfies only properties (i), (ii) and (iii) of (1.1), it is a *topological special cell structure*. (1.1)

For a closed subset A of M^2, we define two types of thickenings $T(A, \varepsilon)$ and $\mathfrak{T}(A, C)$ where $\varepsilon > 0$ is a real number and C is a special cell structure with $A \subset |C|$;

$$T(A, \varepsilon) = \{x \in M^2 | d(x, A) \leq \varepsilon\},$$
$$\mathfrak{T}(A, C) = \bigcup \{\sigma | \sigma \text{ a closed cell in } C, \sigma \cap A \neq \varnothing\} \qquad (1.2)$$

where d denotes the metric on M^2.

Choose base points $\{P_\sigma\}$ (where $P_\sigma \in \sigma$) for the 2-cells $\{\sigma\}$ of C and let $d_0 > 0$ be the smaller of

$$d(|C^1|, \{P_\sigma\}) \quad \text{and} \quad \inf\{d(\sigma, \tau) | \sigma, \tau \text{ closed cells of } C, \sigma \cap \tau = \varnothing\}. \quad (1.3)$$

Associate sets $\{D_\sigma\}$, called auxiliary discs, to $\{\sigma\}$ satisfying

> (i) D_σ is homeomorphic to $\mathbf{D}^2$,
> (ii) $\sigma \subset$ interior D_σ and
> (iii) $D_\sigma \subset T(\sigma, 10^{-1}d_0)$. (1.4)

Assume M^2 is compact and let $d_1 > 0$ be a number such that, for each $x \in M^2$, the exponential map is a diffeomorphism from the disc of radius d_1 centered at the origin of T_xM^2 (the tangent space to M^2 at x) to $T(x, d_1)$. As is customary, let mesh C be the maximum distance between points belonging to a common closed cell in C; recall a map f: $|C| \to |K|$ (between cell structures) is cellular if $f(C^i) \subset K^i$ (for each i).

Lemma 1.1. *Let C be a special cell structure with $|C| = M^2$,* mesh $C <$ $(10)^{-1}d_1$ *and equipped with auxiliary discs $\{D_\sigma\}$; then there exists a number $\varepsilon > 0$ such that, given any other special cell structure K with $|K| = M^2$ and* mesh $K < \varepsilon$, *we can construct a cellular homeomorphism g: $|C| \to |K|$ with $g(\sigma) \subset D_\sigma$ for each closed cell σ in C and so that $g(\omega)$ contains a vertex of K for each open* 1*-cell ω in C.*

The proof of this result occupies most of §1. Pick a number $d_2 > 0$ such that $T(\sigma, 2d_2) \subset D_\sigma$ for each closed cell σ in C and satisfying the following extra constraint. For each edge ω in C and vertex v contained in ω, ω and the boundary of $T(v, r)$ intersect transversally in a single point provided $0 < r \leqslant 2d_2$. Consequently, we can smoothly parameterize each edge ω as a function ω: $[0, 3] \to M^2$ with the following properties

(i) $d\omega(t)/dt \neq 0$ for $t \in [0, 3]$,
(ii) $\omega([0, 1]) \subset T(\omega(0), 3d_2/2)$,
(iii) $\omega([2, 3]) \subset T(\omega(3), 3d_2/2)$ and
(iv) $\omega((1, 2)) \subset M^2 - \bigcup \{T(v, 3d_2/2) | v$ a vertex in $C\}$. (1.5)

(Fix such a choice of parameterizations for the remainder of §1.)

The construction of g uses the following elementary fact. (Its verification is left as an exercise.)

Lemma 1.2. *If A is a closed connected subset of M^2 and K is a special cell structure with $|K| = M^2$, then $\mathfrak{T}(A, K)$ is connected; in fact, any two vertices v_0 and v_1 in $\mathfrak{T}(A, K)$ can be joined by a simple polygonal arc in $\mathfrak{T}(A, K)$.*

(A *polygonal arc* is a concatenation of edges in a complex.)

The ε posited in Lemma 1.1 is any number smaller than $d_2/3$ satisfying

(i) $T(\omega[0, 1], \varepsilon) \subset T(\omega(0), 2d_2)$,
(ii) $T(\omega[2, 3], \varepsilon) \subset T(\omega(3), 2d_2)$,
(iii) $T(\omega[1, 2], \varepsilon) \subset M^2 - \bigcup \{T(v, d_2) | v$ a vertex in $C\}$,
(iv) $T(\omega_1, \varepsilon) \cap T(\omega_2, \varepsilon) \subset T(v, d_2)$ if $\omega_1 \cap \omega_2 = v$, (1.6)

where ω, ω_1, ω_2 are (parameterized) edges and v is a vertex of C.

First construct $g|C^1$; for each edge ω, we must determine $g(\omega)$. As an approximation to $g(\omega)$, we construct simple polygonal arcs ω': $[0, 3] \to \mathfrak{T}(\omega, K)$ with the following properties

(i) $\omega'[0, 1] \subset T(\omega(0), 2d_2)$,
(ii) $\omega'[2, 3] \subset T(\omega(3), 2d_2)$,
(iii) $\omega'[1, 2] \subset M^2 - \bigcup \{T(v, d_2) | v$ a vertex of $C\}$,
(iv) $\omega'[0, 3] \subset T(\omega, 2d_2)$ and
(v) if $\omega_1(0) = \omega_2(0)$ $(\omega_1(0) = \omega_2(3))$, then $\omega_1'(0) = \omega_2'(0)$ $(\omega_1'(0)$ $= \omega_2'(3))$, (1.7)

where ω_1, ω_2 are edges in C.

To construct ω', pick 4 vertices v_i $(i = 0, 1, 2, 3)$ from K with $v_i \in \mathfrak{T}(\omega(i), K)$; if α is a second edge in C with $\alpha(0) = \omega(0)$, make the same choice of v_0 in constructing α'. (If $\alpha(0) = \omega(3)$, then v_0 for α' should be the v_3 chosen for ω'). Now, using Lemma 1.2 connect successive vertices v_i, v_{i+1} by simple polygonal arcs γ_i in $\mathfrak{T}(\omega[i, i+1], K)$; the concatenation of these form a polygonal arc in $\mathfrak{T}(\omega[0, 3], K)$ connecting v_0 to v_3. The result may not be a simple arc; but, it is easy to find subarcs γ_i' of γ_i which concatenate to form a simple arc ω' connecting v_0 to v_3. (Note v_1 and v_2 need not be points on ω'.) The collection $\{\omega'\}$ thus constructed can be parameterized to satisfy (1.7).

Note, if ω_1 and ω_2 are nonintersecting edges of C, then $\omega_1'[0, 3]$ does not intersect $\omega_2'[0, 3]$. Unfortunately, when ω_1 and ω_2 are distinct but share a common vertex, possibly ω_1' and ω_2' meet in more than a common endpoint. However, by an elementary combinatorial argument, this particular collection of $\{\omega'\}$ (constructed above) can be modified to form a new collection $\{\omega''\}$ of simple polygonal arcs having the following properties

(i) $\{\omega''\}$ satisfies (1.7),
(ii) $\omega''(t) = \omega'(t)$ for $t \in [1, 2]$ and
(iii) $\omega''[0, 3] \cap \alpha''[0, 3]$ contains at most one point, if ω and α are distinct edges in C. (1.8)

We leave this argument as an exercise. Hint. The $\{\omega'\}$ can be chosen so that $\bigcup \{\omega''[0, 3]\} \subset \bigcup \{\omega'[0, 3]\}$ but neither of the following pair of equations need be valid

$$\omega''(0) = \omega'(0), \qquad \omega''(3) = \omega'(3).$$

Figure 1 shows the hinted modification. In it, piecewise smooth curves are used instead of polygonal arcs for purposes of illustration. The dashed lines in the second picture indicate parts of $\bigcup \{\omega_i'\}$ deleted in forming $\{\omega_i''\}$; the large circle in both pictures is the boundary of $T(x, d_2)$ where $x = \omega_1(0)$ or $\omega_1(3)$ as the case requires.

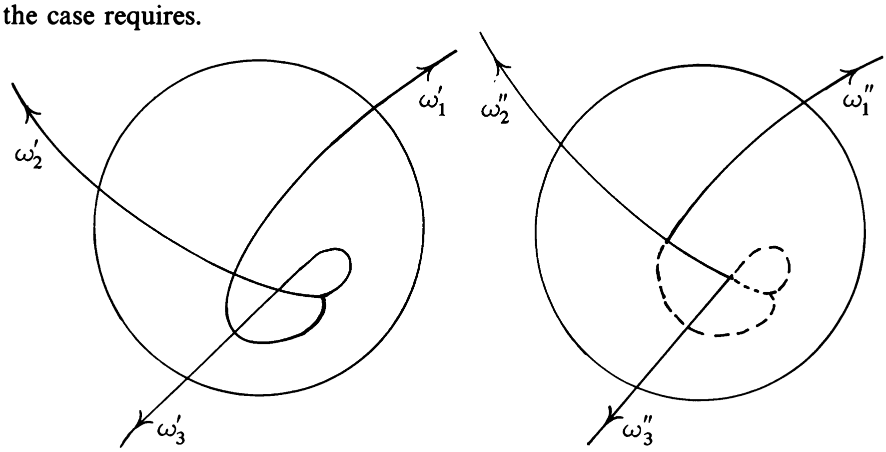

FIGURE 1

Now define $g|C^1$ by the formula $g(\omega(t)) = \omega''(t)$ for $t \in [0, 3]$ where ω is a (parameterized) edge; because of (1.8), this map is an embedding and $g(\omega) \subset D_\omega$. (Since mesh $K < d_2/3$, $g(\omega)$ contains a vertex of K for each open 1-cell ω in C.) In fact, if σ is a closed 2-cell and $\partial\sigma$ denotes its boundary, then $g(\partial\sigma) \subset D_\sigma$; hence, by Schoenflies' Theorem (cf. [8, p. 175]), $g|\partial\sigma$ can be extended to a homeomorphism of σ onto the closure of the interior component of $D_\sigma - g(\partial\sigma)$. In this way, extend $g|C^1$ to a cellular map g from C to K with $g(\sigma) \subset D_\sigma$ for each closed cell σ in C. This last fact (cf. (1.4)) implies

$$g|C^1\colon C^1 \to M^2 - \{P_\sigma\} \tag{1.9}$$

is homotopic to the inclusion map; hence, by an elementary winding number argument, $g\colon M^2 \to M^2$ is a homeomorphism. This completes the proof of Lemma 1.1.

In §3 a relative version of this result is used. We now formulate it leaving its proof as an exercise.

Drop the compactness constraint on M^2 and consider the following condition for special cell structures C in M^2.

For each $x \in |C|$, the exponential map is a diffeomorphism from the disc of radius 10 mesh C centered at the origin of T_xM^2 to $T(x, 10 \text{ mesh } C)$. (1.10)

LEMMA 1.3. *Let C be a special cell structure in M^2 satisfying* (1.10) *and equipped with auxiliary discs $\{D_\sigma\}$ (cf.* (1.4)); *let $U \subset V \subset C$ be subcomplexes such that any closed cell of C which meets $|U|$ is contained in $|V|$. Then there exists a second collection of auxiliary discs $\{D'_\sigma\}$ (cf.* (1.4)) *and a real number $\varepsilon > 0$ such that, for any other special cell structure K in M^2 with* mesh $K < \varepsilon$ *and* $T(|C|, \text{mesh } C) \subset |K|$ *and any cellular embedding $h\colon |V| \to |K|$ satisfying $h(\sigma) \subset D'_\sigma$ for all cells σ in V, we can construct a cellular embedding $g\colon |C| \to |K|$ such that*

(i) *$g(x) = h(x)$ for $x \in |U|$,*

(ii) *$g(\sigma) \subset D_\sigma$ for all cells σ in C and*

(iii) *$g(\omega)$ contains a vertex of K for each open* 1*-cell ω in C.*

2. Markov cell structures (2-manifold case). Let M^2 be a closed 2-manifold equipped with a map $f\colon M^2 \to M^2$; a *Markov cell structure* for (M^2, f) is a topological special cell structure $\mathcal{C}$ in M (cf. (1.1)) with $|\mathcal{C}| = M$ and such that $f^n\colon |\mathcal{C}| \to |\mathcal{C}|$ is cellular for some positive integer n. This section is devoted to proving the following result.

THEOREM 2.1. *If $f\colon M^2 \to M^2$ is an expanding endomorphism on a closed 2-manifold, then (M^2, f) has a Markov cell structure.*

We note Shub [11] has shown that, under the hypotheses of Theorem 2.1,

M^2 is either the torus on the Klein bottle and f is topologically conjugate to a linear expanding map; but we will not use these facts in proving Theorem 2.1.

To prove this result, start by choosing a special cell structure C on M^2 with $|C| = M^2$ and mesh $C < d_1/5$. (See the paragraph preceding the statement of Lemma 1.1 for the definition of d_1.) Such a complex C can be constructed (for example) by using a dual cell structure to a smooth triangulation of M^2 having sufficiently small mesh.

Choose a collection of base points $\{P_\sigma\}$ and auxiliary discs $\{D_\sigma\}$ for C (cf. (1.4)) and let $\varepsilon > 0$ be the number posited in Lemma 1.1. Let n be a positive integer sufficiently large that

$$|df^n(X)| > \varepsilon^{-1}(\text{mesh } C)|X| \text{ and } 2|X| \tag{2.1}$$

for each nonzero vector X tangent to M^2 and let F denote f^n. Since F: $M^2 \to M^2$ is a covering space, we can form a new special cell structure K with $|K| = M^2$ by setting $K^i = F^{-1}(C^i)$; note mesh $K < \varepsilon$ because of (2.1). Now, applying Lemma 1.1 to this set up, we obtain a cellular homeomorphism g: $|C| \to |K|$ with $g(\sigma) \subset D_\sigma$ for each closed cell σ of C. Since $d(x, g(x)) < d_1$ for all $x \in M^2$ and F is expanding, we can (by lifting g via F^s) form a sequence of homeomorphisms g_s: $M^2 \to M^2$ (indexed by the nonnegative integers) having the following properties

$$\begin{aligned}&\text{(i) } Fg_s = g_{s-1}F \text{ for } s > 0,\\ &\text{(ii) } g_0 = g \text{ and}\\ &\text{(iii) } d(x, g_s(x)) \leqslant 2^{-s}d_1 \text{ for all } x \in M^2;\end{aligned} \tag{2.2}$$

in fact, x and $g_s(x)$ are in the same component of $F^{-s}T(F^s(x), d_1)$. Form the composites $G_s = g_s g_{s-1} \cdots g_0$; because of (2.2), the sequence G_s converges uniformly to a continuous function G: $M^2 \to M^2$ satisfying the equation

$$FG = GFg. \tag{2.3}$$

Define $\mathcal{C}^i$, closed subsets of M^2, by $\mathcal{C}^i = G(C^i)$; because of (2.3) and the fact Fg: $|C| \to |C|$ is cellular, we have

$$F(\mathcal{C}^i) \subset \mathcal{C}^i \quad \text{for all } i. \tag{2.4}$$

To complete the demonstration of Theorem 2.1, it remains to show that the filtration $\mathcal{C}$ (defined above) is a topological special cell structure. (Since G is homotopic to the identity map, we note $|\mathcal{C}| = G(M^2) = M^2$.) We will accomplish this by constructing a homeomorphism G': $M^2 \to M^2$ with $G'(C^i) = G(C^i)$ for all i; the construction of G' occupies the remainder of this section.

Define a sequence of special cell structures $C(s)$ by the equations $C(s)^i = F^{-s}(C^i)$; note $C(0) = C$ and $C(1) = K$. Next, construct a sequence of cellular homeomorphisms h_s: $C(s)^1 \to C(s)^1$ with $h_s(\sigma) = \sigma$ for each vertex or edge σ of $C(s)$ and having the following property.

If ω is an edge of $C(s)$ parameterized by arc length t, then $g_s h_s \omega(\Omega/2)$ is a vertex of $C(s+1)$ (Ω denotes the total arc length of ω) and the derivative of the arc length of $g_s h_s \omega(t)$ with respect to t exists except when $t = \Omega/2$ and is constant on $(0, \Omega/2)$ and $(\Omega/2, \Omega)$. (2.5)

It is easy to construct such homeomorphisms using the fact that g_s: $|C(s)| \to |C(s+1)|$ is cellular and $g_s(\sigma)$ contains a vertex of $C(s+1)$ for each *open* 1-cell σ of $C(s)$.

Define a sequence of embeddings H_i by the equations

$$H_i = g_i h_i g_{i-1} h_{i-1} \cdots g_0 h_0; \tag{2.6}$$

it follows in a straightforward fashion that this sequence converges uniformly to a continuous function H: $C^1 \to M^2$ such that $H(C^i) = G(C^i)$ for $i = 0$ and 1.

LEMMA 2.2. *The function H: $C^1 \to \mathcal{C}^1$ is a homeomorphism.*

PROOF. It remains only to show that H is one-to-one. Let x and y be distinct points in C^1 and choose a simple piecewise smooth arc α: $[0, 1] \to C^1$ with $x = \alpha(a)$ and $y = \alpha(b)$ where $0 < a < b < 1$ (cf. Lemma 1.2). Because of (2.5) there exists an integer $s > 0$ such that the simple arc $H_s\alpha$ contains at least four vertices v_i $(i = 1, 2, 3, 4)$ of $C(s+1)$ with $v_i = H_s\alpha(t_i)$ where $t_1 < a < t_2 < t_3 < b < t_4$; in particular, $H_s(x)$ and $H_s(y)$ belong to nonintersecting edges ω_x and ω_y of $C(s+1)$. From this remark, it is easily seen (use the defining properties of h_i) that $H(x) \in \hat{G}_{s+1}(\omega_x)$ and $H(y) \in \hat{G}_{s+1}(\omega_y)$ where $\hat{G}_{s+1}$ is the composite GG_s^{-1}; hence, Lemma 2.2 is a consequence of the following result.

LEMMA 2.3. *If σ_1 and σ_2 are nonintersecting closed cells in $C(s)$, then $\hat{G}_s(\sigma_1) \cap \hat{G}_s(\sigma_2) = \varnothing$.*

PROOF. First observe, using Lemma 1.1, 1.4, 2.1, that $G(\sigma) \subset T(\sigma, 5^{-1}d_0)$ for each closed cell σ of C; also using (2.2), that $F^s\hat{G}_s = GF^s$; hence if $F^s\sigma_1 \cap F^s\sigma_2 = \varnothing$, then $\hat{G}_s(\sigma_1) \cap \hat{G}_s(\sigma_2) = \varnothing$. On the other hand, if $F^s\sigma_1$ and $F^s\sigma_2$ intersect, then $T(F^s\sigma_1 \cup F^s\sigma_2, 5^{-1}d_0) \subset T(x, d_1)$ for some point $x \in M^2$. Since $T(x, d_1)$ is homeomorphic to a closed 2-disc, it is evenly covered by F^s; in particular, $\sigma_i \subset S_i$ $(i = 1, 2)$ where S_1 and S_2 are distinct components of $F^{-s}T(x, d_1)$. Finally observe $\sigma_i \cup \hat{G}_s\sigma_i \subset S_i$ $(i = 1, 2)$; hence $\hat{G}_s\sigma_1$ and $\hat{G}_s\sigma_2$ cannot intersect.

Because of Lemma 2.2 (and the paragraph preceding it) we define $G'|C^1$ to be H. Let σ be a closed 2-cell of C with boundary denoted by $\partial\sigma$; then $H(\partial\sigma)$ is contained in the interior of $T(P_\sigma, d_1)$ since

$$H(\partial\sigma) = G(\partial\sigma) \subset G(\sigma) \subset T(\sigma, 5^{-1}d_0). \tag{2.7}$$

Since $\partial\sigma$ is homeomorphic to a circle, $T(P_\sigma, d_1)$ to a closed 2-disc and H is an embedding, Schoenflies' Theorem allows us to extend $G'|\partial\sigma$ to a homeomorphism of σ to the closure of the interior component of $T(P_\sigma, d_1) - G'(\partial\sigma)$; thus, we extend $G'|C^1$ to all of M^2.

To complete the demonstration of Theorem 2.1, it remains to verify that G' is a one-to-one function. (Note any one-to-one continuous self-map of a closed connected manifold must be onto.) To do this, first observe that $G'(\sigma) \subset G(\sigma)$ for each closed 2-cell σ of C; this is a consequence of the fact that $G'|\partial\sigma$ and $G|\partial\sigma\colon \partial\sigma \to G'(\partial\sigma)$ are homotopic (hence $G|\partial\sigma$ is essential). Therefore, it suffices to show for all pairs of distinct 2-cells σ_1 and σ_2 in C that the following containment is true

$$G(\sigma_1) \cap G(\sigma_2) \subset G'(\sigma_1 \cap \sigma_2). \tag{2.8}$$

Let $z \in G(\sigma_1) \cap G(\sigma_2)$; hence, there exist points $x \in \sigma_1$ and $y \in \sigma_2$ with $G(x) = z = G(y)$; consequently, $d(G_i(x), G_i(y)) \to 0$ as $i \to \infty$. Since G_i is a homeomorphism, there must exist points $x_i \in \partial\sigma_1$ and $y_i \in \partial\sigma_2$ such that $G_i(x_i) \to z$ and $G_i(y_i) \to z$ as $i \to \infty$ (cf. Figure 2); but $\partial\sigma_i$ $(i = 1, 2)$ is compact; hence $z \in G(\partial\sigma_1) \cap G(\partial\sigma_2)$.

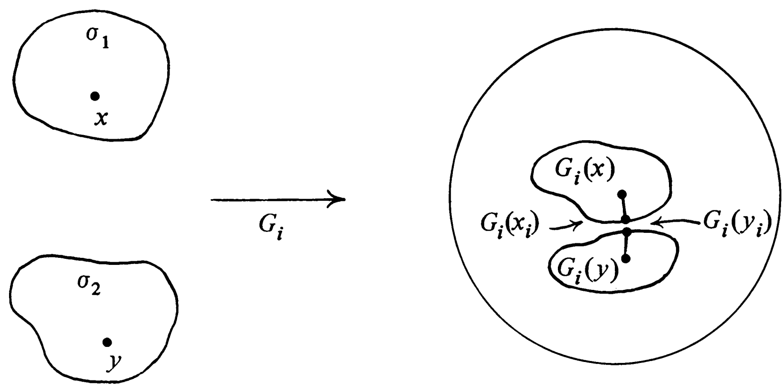

FIGURE 2

Since $G(\partial\sigma_i) = H(\partial\sigma_i)$ $(i = 1, 2)$ and H is a one-to-one function (Lemma 2.2), we have

$$z \in H(\partial\sigma_1 \cap \partial\sigma_2) = G'(\sigma_1 \cap \sigma_2) \tag{2.9}$$

which verifies (2.8) and completes the proof of Theorem 2.1.

REMARK 2.4. Let $f\colon T^2 \to T^2$, where T^2 is the torus, be an expanding endomorphism induced by a 2×2 matrix A with integral entries and whose eigenvalues are real but irrational numbers having distinct absolute values. It is implicit in Franks' paper [6] that a proper closed f-invariant subset of T^2 cannot contain a C^1-arc; i.e., a continuously differentiable arc. In particular, if

$$A = \begin{pmatrix} 0 & 11 \\ -1 & 7 \end{pmatrix},$$

then the 1-skeleton of the Markov cell structure constructed in Theorem 2.1 for (T^2, f) cannot contain a C^1-arc. (See also Bowen's paper **[3]**.)

3. Markov cell structures (branched 2-manifold case). This section is devoted to proving an analogue of Theorem 2.1 in the branched 2-manifold case. (See Williams' paper **[14]** for the basic definitions and facts concerning branched manifolds.) A *cell structure* C for a compact branched 2-manifold M^2 is a filtration of M^2 by closed subsets

$$\varnothing = C^{-1} \subset C^0 \subset C^1 \subset C^2 = |C| = M^2 \tag{3.1}$$

such that $C^i - C^{i-1}$ $(i = 0, 1, 2)$ has a finite number of components, called i-cells, satisfying property (i) in (1.1). If M^2 is equipped with a map f, a *Markov cell structure* for (M^2, f) is a cell structure C for M^2 such that f^n: $|C| \to |C|$ is cellular for some positive integer n.

THEOREM 3.1. *Let f: $M^2 \to M^2$ be an immersion satisfying Axioms* 1, 2 *and* 3^+ *of* **[14]** *where M^2 is a compact branched 2-manifold, then (M^2, f) is shift equivalent to a pair (N^2, g) having a Markov cell structure.*

The proof of Theorem 3.1 is similar to, but technically more complicated than, that of Theorem 2.1; hence, we sketch it, going into detail only where it differs from the previous argument.

By **[14**, Lemma 5.6], (M^2, f) is shift equivalent to a pair (N^2, g) where N^2 is normally branched; hence, we will assume that M^2 is normally branched. Let βM denote the branch set of M^2; as a consequence of **[14**, §8], M^2 contains a 2-dimensional compact submanifold (with boundary) A and a 2-dimensional compact branched submanifold (with boundary) B' satisfying

$$\begin{aligned} &\text{(i) interior } A \cup \text{interior } B' = M^2 \text{ and} \\ &\text{(ii) } \beta M \subset \text{interior } B' \cap (M^2 - A); \end{aligned} \tag{3.2}$$

furthermore, there is a compact 2-manifold B (with boundary), a surjective immersion p: $B' \to B$ which maps βM homeomorphically onto $p(\beta M)$, and an immersion φ: $B \to M^2$ with $f(x) = \varphi p(x)$ for all $x \in B'$. By deleting short collar neighborhoods from A, B and B', we obtain compact submanifolds A_i, B_i and B_i' $(i = -1, 0, 1)$ with

$$\begin{aligned} &A_{-1} \subset \text{interior } A_0 \subset \text{interior } A_1 \subset \text{interior } A \quad \text{and} \\ &B_{-1} \subset \text{interior } B_0 \subset \text{interior } B_1 \subset \text{interior } B \end{aligned} \tag{3.2.1}$$

such that $B_i' = p^{-1}B_i$ and (3.2) is satisfied when A and B' are replaced by A_{-1} and B'_{-1}. We also assume that the Riemannian metrics on M^2, A, B' and B fit together consistently; i.e., A and B' have the metrics induced from M^2

and dp: $TB' \to TB$ preserves the Riemann metric.

A (*topological*) *branched cell structure* $\mathfrak{B}$ for M^2 is a filtration by closed subsets

$$\varnothing = \mathfrak{B}^{-1} \subset \mathfrak{B}^0 \subset \mathfrak{B}^1 \subset \mathfrak{B}^2 = |\mathfrak{B}| = M^2 \tag{3.3}$$

such that there exist (topological) special cell structures C in the interior of A and K in the interior of B satisfying the following conditions

(i) $\mathfrak{B}^i = C^i \cup p^{-1}K^i$,
(ii) $A_{-1} \subset |C|$, $B_{-1} \subset |K|$ and
(iii) $|C| \cap p^{-1}|K|$ is a subcomplex of C; each cell of which maps homeomorphically via p onto a cell of K. (3.4)

The components of $\mathfrak{B}^i - \mathfrak{B}^{i-1}$ are (in general) not cells when they intersect βM^2. A map f: $|\mathfrak{B}| \to |\mathfrak{B}|$ is *cellular* if $f(\mathfrak{B}^i) \subset \mathfrak{B}^i$ (for all i). We will derive Theorem 3.1 directly from the following result.

LEMMA 3.2. *There is a topological branched cell structure* $\mathfrak{B}$ *for* M^2 *of arbitrarily small mesh such that* f^n: $|\mathfrak{B}| \to |\mathfrak{B}|$ *is a cellular map for some positive integer* n.

Before proving this lemma, we use it together with the collapsing technique introduced by Williams (cf. [**14**, Lemmas 2.2 and 5.4]) to complete the demonstration of Theorem 3.1. Let K be the topological special cell structure for B posited in (3.4). Since $\mathfrak{B}$ can be constructed with arbitrarily small mesh, we can assume that K has a subcomplex K_0 such that

$$p(\beta M^2) \subset \text{interior } |K_0| \subset \text{interior } B_{-1}. \tag{3.5}$$

Form N^2 by collapsing $p^{-1}|K_0|$ under the immersion p: $B' \to B$ and let g: $N^2 \to N^2$ be the immersion induced from f, then (N^2, g) is shift equivalent to (M^2, f) and $\mathfrak{B}$ induces a cell structure on N^2 (via the canonical quotient map) with respect to which g^n is cellular; i.e., (N^2, g) has a Markov cell structure.

REMARK 3.3. The Markov cell structure constructed above may fail to satisfy property (ii) in (1.1); for instance, if K_0 is not a "full" subcomplex in K, this is possible. But, by constructing more carefully, we believe this property can also be satisfied.

It remains to prove Lemma 3.2; we start by constructing a branched cell structure $\mathfrak{B}(0)$ for M^2 of arbitrarily small mesh. First, triangulate M^2 by a smooth triangulation $\mathfrak{T}_1$ (with arbitrarily small mesh) so that B' is a subcomplex with its triangulation induced via p^{-1} from a smooth triangulation $\mathfrak{T}_2$ of B (cf. [**14**, Lemma 5.7]). Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be subcomplexes consisting of all closed simplices from $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively, which meet A_0 and B_1, respectively; if mesh $\mathfrak{T}_1$ is sufficiently small, then

$$|\mathfrak{F}_1| \subset \text{interior } A_1 \quad \text{and} \quad |\mathfrak{F}_2| \subset \text{interior } B. \tag{3.6}$$

Let C and K be the dual cell complexes to $\mathfrak{F}_1$ and $\mathfrak{F}_2$, respectively; then C, K satisfy conditions (ii) and (iii) of (3.4) and hence determine via formula (i) of (3.4) a branched cell structure $\mathfrak{B}(0)$ for M^2 with arbitrarily small mesh. (Note also $B_1 \subset |K|$ and $|C| \subset A_1$.)

We assume that

$$|df(X)| > 2|X| \tag{3.7}$$

for each vector X tangent to M^2. (To do this, we may have to replace f by a power of itself.) Define a second branched cell structure $\mathfrak{B}(1)$ for M^2 by letting $\mathfrak{B}(1)^i = f^{-1}(\mathfrak{B}(0)^i)$. (For this to be a branched cell structure, mesh $\mathfrak{B}(0)$ must be sufficiently small.) There are special cell structures $C(1)$ in the interior of A and $K(1)$ in the interior of B satisfying (3.4) (with $\mathfrak{B}$, C and K replaced by $\mathfrak{B}(1)$, $C(1)$ and $K(1)$) and having the following properties

(i) $C(1)^i \subset \mathfrak{B}(1)^i$ and $K(1)^i \subset \varphi^{-1}(\mathfrak{B}(0)^i)$;
(ii) $T(|C|, \text{mesh } \mathfrak{B}(0)) \subset |C(1)|$ and $T(|K|, \text{mesh } \mathfrak{B}(0)) \subset |K(1)|$. (3.8)

(In order to satisfy property (ii), mesh $\mathfrak{B}(0)$ must again be sufficiently small.)

Let $\{P_\sigma\}$, $\{D_\sigma\}$ be base points and auxiliary discs for C; let $V = p^{-1}|K| \cap |C|$ and $U = p^{-1}\mathfrak{T}(B_0, K) \cap |C|$. If mesh $\mathfrak{B}(0)$ is sufficiently small, then $U \subset V \subset C$ satisfy the first hypothesis of Lemma 1.3; let $\{D'_\sigma\}$ be the auxiliary discs posited there. Replacing these by smaller discs (if necessary), we may assume there are auxiliary discs $\{\hat{D}_\tau\}$ for K such that $p(D'_\sigma) = \hat{D}_\tau$ for each closed cell σ of V and τ of K with $p(\sigma) = \tau$. Also let $\{\hat{P}_\tau\}$ be a collection of base points for K such that $p(P_\sigma) = \hat{P}_\tau$ for 2-cells σ of V and τ of K with $p(\sigma) = \tau$. Replacing f by a high power of itself (if necessary), we can assume that mesh $\mathfrak{B}(1)$ is small enough so that we can apply Lemma 1.3 (twice) to obtain cellular embeddings $\hat{g}\colon |K| \to |K(1)|$ and $g\colon |C| \to |C(1)|$ with the following properties

(i) $pg(x) = \hat{g}p(x)$ for $x \in |C| \cap p^{-1}|K(0)|$ where $K(0) = \mathfrak{T}(B_0, K)$;
(ii) $g(\sigma) \subset D_\sigma$ and $\hat{g}(\tau) \subset \hat{D}_\tau$ where σ and τ are closed cells in C and K, respectively. (3.9)

Of course, mesh $\mathfrak{B}(0)$ must be small enough that Lemma 1.3 applies; in particular, $T(x, 5c)$ should be homeomorphic to a closed 2-disc for both $x \in A_0$ and $x \in B_0$ where $c = 2$ mesh $\mathfrak{B}(0)$; also, assume $T(A_1, c) \subset A$ and $T(B_1, c) \subset B$. Now, define inductively a sequence of embeddings $g_s\colon A_1 \to A$ and $\hat{g}_s\colon B_1 \to B$ as follows. If $x \in A_1$ and $f(x) \in A_0$ (respectively, $f(x) \in B'_0$), let $g_s(x)$ be the unique point in the same component of $f^{-1}T(f(x), c)$, $(f^{-1}p^{-1}T(pf(x), c))$ as x such that

$$fg_s(x) = g_{s-1}f(x), \qquad (pfg_s(x) = \hat{g}_{s-1}pf(x)); \tag{3.10.1}$$

if $x \in B_1$ and $y = \varphi(x) \in A_0$ (respectively, $y \in B_0'$), let $\hat{g}_s(x)$ be the unique point in the same component of $\varphi^{-1}T(y, c)$, $(\varphi^{-1}p^{-1}T(p(y), c))$ as x such that

$$\varphi\hat{g}_s(x) = g_{s-1}(y), \qquad (p\varphi\hat{g}_s(x) = \hat{g}_{s-1}p(y)). \tag{3.10.2}$$

(When $s = 1$ use g and $\hat{g}$ in formulas 3.10.1, 3.10.2 in place of g_0 and $\hat{g}_0$, respectively.) Note the following analogue of 3.9(i) is true

$$pg_s(x) = \hat{g}_s p(x) \quad \text{for } x \in A_1 \cap B_1'. \tag{3.10.3}$$

Next, form the composite embeddings G_s: $|C(0)| \to A$ where $C(0) = \mathcal{T}(A_0, C)$ and $\hat{G}_s$: $|K(0)| \to B$ defined by the equations

$$G_s(x) = g_s g_{s-1} \cdots g_1 g(x) \quad \text{for } x \in |C(0)| \qquad \text{and}$$
$$\hat{G}_s(y) = \hat{g}_s \hat{g}_{s-1} \cdots \hat{g}_1 \hat{g}(y) \quad \text{for } y \in |K(0)|. \tag{3.11}$$

These sequences converge uniformly to continuous functions G: $|C(0)| \to A$ and $\hat{G}$: $|K(0)| \to B$ satisfying

$$pG(x) = \hat{G}p(x) \quad \text{for } x \in |C(0)| \cap p^{-1}|K(0)|. \tag{3.12}$$

Define filtrations $\mathcal{C}$ and $\mathcal{K}$ by the equations

$$\mathcal{C}^i = G\big(C(0)^i\big) \quad \text{and} \quad \mathcal{K}^i = \hat{G}\big(K(0)^i\big); \tag{3.13}$$

although G and $\hat{G}$ need not be embeddings, arguing as in §2, it can be shown that $\mathcal{C}$ and $\mathcal{K}$ are topological special cell structures in the interiors of A and B, respectively. Also, conditions (ii) and (iii) of (3.4) are satisfied when C and K are replaced by $\mathcal{C}$ and $\mathcal{K}$; hence, letting $\mathcal{B}^i = \mathcal{C}^i \cup p^{-1}\mathcal{K}^i$, we obtain a topological branched cell structure $\mathcal{B}$ for M^2 and, using (3.10.1), (3.10.2), it can be shown that f: $|\mathcal{B}| \to |\mathcal{B}|$ is cellular. This completes the proof of Lemma 3.2.

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