MARKOV CELL STRUCTURES FOR EXPANDING MAPS IN DIMENSION TWO

BY

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ABSTRACT. Let $f : M^2 \to M^2$ be an expanding self-immersion of a closed 2-manifold, then for some positive integer $n$, $f^n$ leaves invariant a cell structure on $M^2$. A similar result is true when $M$ is a branched 2-manifold.

0. Introduction. Let $K$ be a topological space equipped with a continuous self-map $f$. It is known in many interesting cases that $K$ can be partitioned into cells; i.e., given a cell complex structure. For example, this is possible when $K$ is a smooth manifold. It is rarer for $K$ to support a cell structure with respect to which $f$ is a cellular map (i.e., leaves each skeleton invariant); for instance, this is impossible when $K$ is the circle and $f$ is rotation through an irrational angle.

Consider the situation when $K$ is a closed smooth 2-manifold and $f$ is an expanding immersion of $K$; i.e., $|df(X)| > |X|$ for all nonzero vectors $X$ tangent to $K$ and some Riemann metric on $K$. In this case, we show (Theorem 2.1) there is a positive integer $n$ such that $f^n$ (the composite of $f$ with itself $n$-times) is cellular relative to some cell structure on $K$. (We do not know whether $n$ can always be 1. See [3], [7] and [10] for other recent interesting constructions of invariant sets.)

We prove a similar (slightly weaker) result (Theorem 3.1) for expanding immersions of compact branched 2-manifolds satisfying Axioms 1, 2 and 3+ of Williams. (See [14] for the basic definitions.) These objects arise in Williams' study [14] of expanding attractors. We hope Theorem 3.1 will be useful in helping to understand 2-dimensional expanding attractors which are apparently more complicated (cf. [5]) than the 1-dimensional case where Williams has a very good theory [13].

Theorems 2.1 and 3.1 are clearly extensions (in a very special setting) of the theory of Markov partitions [1], [12], [2] and [9]. This paper is also an introduction to the results announced in [4].

1. A cellular embedding result. In this section, we formulate and prove a crucial result Lemma 1.1, a strong form of the cellular approximation
theorem for dimension 2. It is used in §2 to construct Markov cell structures for expanding maps on 2-manifolds.

Let $M^2$ denote a 2-dimensional Riemannian manifold. A special cell structure $C$ in $M^2$ is a filtration by closed subsets,

$$\emptyset = C^{-1} \subset C^0 \subset C^1 \subset C^2 = |C| \subset M^2,$$

such that $C^i - C^{i-1} (i = 0, 1, 2)$ has finitely many connected components, called the *i*-cells of $C$ with the following properties.

(i) The closure of each *i*-cell $\sigma$ is homeomorphic to $D^i = \{x \in \mathbb{R}^i \mid |x| < 1\}$ via a homeomorphism mapping $\sigma$ onto $\{x \in \mathbb{R}^i \mid |x| < 1\}$. (The closure of a cell is called a closed cell.)

(ii) The intersection of two closed cells is either empty or homeomorphic to $D^i$ (for some $i$).

(iii) Each vertex (0-cell) is contained in at least 2 and no more than 3 edges (closed 1-cells).

(iv) Edges are smooth curves.

If $C$ satisfies only properties (i), (ii) and (iii) of (1.1), it is a *topological special cell structure*.

For a closed subset $A$ of $M^2$, we define two types of thickenings $T(A, \varepsilon)$ and $\delta(A, C)$ where $\varepsilon > 0$ is a real number and $C$ is a special cell structure with $A \subset |C|$;

$$T(A, \varepsilon) = \{x \in M^2 \mid d(x, A) < \varepsilon\},$$

$$\delta(A, C) = \bigcup \{\sigma \mid \sigma \text{ a closed cell in } C, \sigma \cap A \neq \emptyset\} \quad (1.2)$$

where $d$ denotes the metric on $M^2$.

Choose base points $\{P_\sigma\}$ (where $P_\sigma \in \sigma$) for the 2-cells $\{\sigma\}$ of $C$ and let $d_0 > 0$ be the smaller of

$$d(|C^1|, \{P_\sigma\}) \quad \text{and} \quad \inf\{d(\sigma, \tau) \mid \sigma, \tau \text{ closed cells of } C, \sigma \cap \tau = \emptyset\}. \quad (1.3)$$

Associate sets $\{D_\sigma\}$, called auxiliary discs, to $\{\sigma\}$ satisfying

(i) $D_\sigma$ is homeomorphic to $D^2$,

(ii) $\sigma \subset \text{interior } D_\sigma$ and

(iii) $D_\sigma \subset T(\sigma, 10^{-1}d_0)$. \quad (1.4)

Assume $M^2$ is compact and let $d_1 > 0$ be a number such that, for each $x \in M^2$, the exponential map is a diffeomorphism from the disc of radius $d_1$ centered at the origin of $T_x M^2$ (the tangent space to $M^2$ at $x$) to $T(x, d_1)$. As is customary, let mesh $C$ be the maximum distance between points belonging to a common closed cell in $C$; recall a map $f: |C| \to |K|$ (between cell structures) is cellular if $f(C^i) \subset K^i$ (for each $i$).
Lemma 1.1. Let \( C \) be a special cell structure with \( |C| = M^2 \), mesh \( C < (10)^{-1}d_1 \) and equipped with auxiliary discs \( \{ D_\sigma \} \); then there exists a number \( \varepsilon > 0 \) such that, given any other special cell structure \( K \) with \( |K| = M^2 \) and mesh \( K < \varepsilon \), we can construct a cellular homeomorphism \( g: |C| \rightarrow |K| \) with \( g(\sigma) \subset D_\sigma \) for each closed cell \( \sigma \) in \( C \) and so that \( g(\omega) \) contains a vertex of \( K \) for each open 1-cell \( \omega \) in \( C \).

The proof of this result occupies most of §1. Pick a number \( d_2 > 0 \) such that \( T(\sigma, 2d_2) \subset D_\sigma \) for each closed cell \( \sigma \) in \( C \) and satisfying the following extra constraint. For each edge \( \omega \) in \( C \) and vertex \( v \) contained in \( \omega \) and the boundary of \( T(v, r) \) intersect transversally in a single point provided \( 0 < r < 2d_2 \). Consequently, we can smoothly parameterize each edge \( \omega \) as a function \( \omega: [0, 3] \rightarrow M^2 \) with the following properties

\[
(i) \frac{d\omega(t)}{dt} \neq 0 \quad \text{for} \quad t \in [0, 3], \\
(ii) \omega([0, 1]) \subset T(\omega(0), 3d_2/2), \\
(iii) \omega([2, 3]) \subset T(\omega(3), 3d_2/2) \quad \text{and} \\
(iv) \omega((1, 2)) \subset M^2 - \bigcup \{ T(v, 3d_2/2) | v \text{ a vertex in } C \}. 
\]

(1.5)

(1.5)

(Fix such a choice of parameterizations for the remainder of §1.)

The construction of \( g \) uses the following elementary fact. (Its verification is left as an exercise.)

Lemma 1.2. If \( A \) is a closed connected subset of \( M^2 \) and \( K \) is a special cell structure with \( |K| = M^2 \), then \( \mathcal{F}(A, K) \) is connected; in fact, any two vertices \( v_0 \) and \( v_1 \) in \( \mathcal{F}(A, K) \) can be joined by a simple polygonal arc in \( \mathcal{F}(A, K) \).

(A polygonal arc is a concatenation of edges in a complex.)

The \( \varepsilon \) posited in Lemma 1.1 is any number smaller than \( d_2/3 \) satisfying

\[
(i) T(\omega(0), 1), \varepsilon) \subset T(\omega(0), 2d_2), \\
(ii) T(\omega(2), 3), \varepsilon) \subset T(\omega(3), 2d_2), \\
(iii) T(\omega[1, 2], \varepsilon) \subset M^2 - \bigcup \{ T(v, d_2) | v \text{ a vertex in } C \}, \\
(iv) T(\omega_1, \varepsilon) \cap T(\omega_2, \varepsilon) \subset T(\omega_1, d_2) \text{ if } \omega_1 \cap \omega_2 = v, \\
(v) \text{ if } \omega'(0) = \omega(0) (\omega_1(0) = \omega_2(3)), \text{ then } \omega'(0) = \omega'(0) (\omega_1(0) = \omega_2(3)),
\]

where \( \omega, \omega_1, \omega_2 \) are (parameterized) edges and \( v \) is a vertex of \( C \).

First construct \( g[C] \); for each edge \( \omega \), we must determine \( g(\omega) \). As an approximation to \( g(\omega) \), we construct simple polygonal arcs \( \omega': [0, 3] \rightarrow \mathcal{F}(\omega, K) \) with the following properties

\[
(i) \omega'[0, 1] \subset T(\omega(0), 2d_2), \\
(ii) \omega'[2, 3] \subset T(\omega(3), 2d_2), \\
(iii) \omega'[1, 2] \subset M^2 - \bigcup \{ T(v, d_2) | v \text{ a vertex of } C \}, \\
(iv) \omega'[0, 3] \subset T(\omega, 2d_2) \text{ and} \\
(v) \text{ if } \omega_1(0) = \omega_2(0) (\omega_1(0) = \omega_2(3)), \text{ then } \omega'_1(0) = \omega'_2(0) (\omega'_1(0) = \omega'_2(3)),
\]

where \( \omega_1, \omega_2 \) are edges in \( C \).
To construct \( \omega' \), pick 4 vertices \( v_i \) \( (i = 0, 1, 2, 3) \) from \( K \) with \( v_i \in \mathcal{F}(\omega(i), K) \); if \( \alpha \) is a second edge in \( C \) with \( \alpha(0) = \omega(0) \), make the same choice of \( v_0 \) in constructing \( \alpha' \). If \( \alpha(0) = \omega(3) \), then \( v_0 \) for \( \alpha' \) should be the \( v_3 \) chosen for \( \omega' \). Now, using Lemma 1.2 connect successive vertices \( v_i, v_{i+1} \) by simple polygonal arcs \( \gamma_i \) in \( \mathcal{F}(\omega[0, 3], K) \) connecting \( v_0 \) to \( v_3 \). The result may not be a simple arc; but, it is easy to find subarcs \( \gamma'_i \) of \( \gamma_i \) which concatenate to form a simple arc \( \omega' \) connecting \( v_0 \) to \( v_3 \). (Note \( v_1 \) and \( v_2 \) need not be points on \( \omega' \).)

The collection \( \{\omega'\} \) thus constructed can be parameterized to satisfy (1.7).

Note, if \( \omega_1 \) and \( \omega_2 \) are nonintersecting edges of \( C \), then \( \omega_1[0, 3] \) does not intersect \( \omega_2[0, 3] \). Unfortunately, when \( \omega_1 \) and \( \omega_2 \) are distinct but share a common vertex, possibly \( \omega'_1 \) and \( \omega'_2 \) meet in more than a common endpoint.

However, by an elementary combinatorial argument, this particular collection of \( \{\omega'\} \) (constructed above) can be modified to form a new collection \( \{\omega''\} \) of simple polygonal arcs having the following properties

(i) \( \{\omega''\} \) satisfies (1.7),
(ii) \( \omega''(t) = \omega'(t) \) for \( t \in [1, 2] \) and
(iii) \( \omega''[0, 3] \cap \alpha''[0, 3] \) contains at most one point, if \( \omega \) and \( \alpha \) are distinct edges in \( C \).

Figure 1 shows the hinted modification. In it, piecewise smooth curves are used instead of polygonal arcs for purposes of illustration. The dashed lines in the second picture indicate parts of \( U \{\omega'_i\} \) deleted in forming \( \{\omega''\} \); the large circle in both pictures is the boundary of \( T(x, d_3) \) where \( x = \omega_1(0) \) or \( \omega_2(3) \) as the case requires.

![Figure 1](https://www.ams.org/journal-terms-of-use)
Now define $g|C^1$ by the formula $g(\omega(t)) = \omega''(t)$ for $t \in [0, 3]$ where $\omega$ is a (parameterized) edge; because of (1.8), this map is an embedding and $g(\omega) \subset D_\omega$. (Since mesh $K < d_2/3$, $g(\omega)$ contains a vertex of $K$ for each open 1-cell $\omega$ in $C$.) In fact, if $\sigma$ is a closed 2-cell and $\partial \sigma$ denotes its boundary, then $g(\partial \sigma) \subset D_\sigma$; hence, by Schoenflies' Theorem (cf. [8, p. 175]), $g|\partial \sigma$ can be extended to a homeomorphism of $\sigma$ onto the closure of the interior component of $D_\sigma - g(\partial \sigma)$. In this way, extend $g|C^1$ to a cellular map $g$ from $C$ to $K$ with $g(\sigma) \subset D_\sigma$ for each closed cell $\sigma$ in $C$. This last fact (cf. (1.4)) implies

$$g|C^1: C^1 \to M^2 - \{P_0\}$$

is homotopic to the inclusion map; hence, by an elementary winding number argument, $g: M^2 \to M^2$ is a homeomorphism. This completes the proof of Lemma 1.1.

In §3 a relative version of this result is used. We now formulate it leaving its proof as an exercise.

Drop the compactness constraint on $M^2$ and consider the following condition for special cell structures $C$ in $M^2$.

For each $x \in |C|$, the exponential map is a diffeomorphism from the disc of radius 10 mesh $C$ centered at the origin of $T_x M^2$ to $T(x, 10 \text{ mesh } C)$. (1.10)

**Lemma 1.3.** Let $C$ be a special cell structure in $M^2$ satisfying (1.10) and equipped with auxiliary discs $\{D_\sigma\}$ (cf. (1.4)); let $U \subset V \subset C$ be subcomplexes such that any closed cell of $C$ which meets $|U|$ is contained in $|V|$. Then there exists a second collection of auxiliary discs $\{D'_\sigma\}$ (cf. (1.4)) and a real number $\epsilon > 0$ such that, for any other special cell structure $K$ in $M^2$ with mesh $K < \epsilon$ and $T(|C|, \text{ mesh } C) \subset |K|$ and any cellular embedding $h: |V| \to |K|$ satisfying $h(\sigma) \subset D'_\sigma$ for all cells $\sigma$ in $V$, we can construct a cellular embedding $g$: $|C| \to |K|$ such that

(i) $g(x) = h(x)$ for $x \in |U|$,  
(ii) $g(\sigma) \subset D_\sigma$ for all cells $\sigma$ in $C$ and  
(iii) $g(\omega)$ contains a vertex of $K$ for each open 1-cell $\omega$ in $C$.

2. Markov cell structures (2-manifold case). Let $M^2$ be a closed 2-manifold equipped with a map $f: M^2 \to M^2$; a Markov cell structure for $(M^2, f)$ is a topological special cell structure $C$ in $M$ (cf. (1.1)) with $|C| = M$ and such that $f^n: |C| \to |C|$ is cellular for some positive integer $n$. This section is devoted to proving the following result.

**Theorem 2.1.** If $f: M^2 \to M^2$ is an expanding endomorphism on a closed 2-manifold, then $(M^2, f)$ has a Markov cell structure.

We note Shub [11] has shown that, under the hypotheses of Theorem 2.1,
$M^2$ is either the torus on the Klein bottle and $f$ is topologically conjugate to a linear expanding map; but we will not use these facts in proving Theorem 2.1.

To prove this result, start by choosing a special cell structure $C$ on $M^2$ with $|C| = M^2$ and mesh $C < d_1/5$. (See the paragraph preceding the statement of Lemma 1.1 for the definition of $d_1$.) Such a complex $C$ can be constructed (for example) by using a dual cell structure to a smooth triangulation of $M^2$ having sufficiently small mesh.

Choose a collection of base points $\{P_\sigma\}$ and auxiliary discs $\{D_\sigma\}$ for $C$ (cf. (1.4)) and let $\varepsilon > 0$ be the number posited in Lemma 1.1. Let $n$ be a positive integer sufficiently large that

$$|df^n(X)| > \varepsilon^{-1}(\text{mesh } C)|X| \quad \text{and} \quad 2|X|$$

(2.1)

for each nonzero vector $X$ tangent to $M^2$ and let $F$ denote $f^n$. Since $F: M^2 \to M^2$ is a covering space, we can form a new special cell structure $K$ with $|K| = M^2$ by setting $K^i = F^{-1}(C^i)$; note mesh $K < \varepsilon$ because of (2.1).

Now, applying Lemma 1.1 to this setup, we obtain a cellular homeomorphism $g: |C| \to |K|$ with $g(\sigma) \subset D_\sigma$ for each closed cell $\sigma$ of $C$. Since $d(x,g(x)) < d_1$ for all $x \in M^2$ and $F$ is expanding, we can (by lifting $g$ via $F^s$) form a sequence of homeomorphisms $g_s: M^2 \to M^2$ (indexed by the nonnegative integers) having the following properties

(i) $Fg_s = g_{s-1}F$ for $s > 0$,
(ii) $g_0 = g$ and
(iii) $d(x,g_s(x)) < 2^{-s}d_1$ for all $x \in M^2$;

(2.2)

in fact, $x$ and $g_s(x)$ are in the same component of $F^{-s}T(F^s(x),d_1)$. Form the composites $G_s = g_sg_{s-1} \cdots g_0$; because of (2.2), the sequence $G_s$ converges uniformly to a continuous function $G: M^2 \to M^2$ satisfying the equation

$$FG = GFg.$$

(2.3)

Define $C_i$, closed subsets of $M^2$, by $C_i = G(C^i)$; because of (2.3) and the fact $Fg: |C| \to |C|$ is cellular, we have

$$F(C_i) \subset C_i \quad \text{for all } i.$$

(2.4)

To complete the demonstration of Theorem 2.1, it remains to show that the filtration $C$ (defined above) is a topological special cell structure. (Since $G$ is homotopic to the identity map, we note $|C| = G(M^2) = M^2$.) We will accomplish this by constructing a homeomorphism $G': M^2 \to M^2$ with $G'(C^i) = G(C^i)$ for all $i$; the construction of $G'$ occupies the remainder of this section.

Define a sequence of special cell structures $C(s)$ by the equations $C(s)^i = F^{-i}(C^i)$; note $C(0) = C$ and $C(1) = K$. Next, construct a sequence of cellular homeomorphisms $h_s: C(s)^i \to C(s)^i$ with $h_s(\sigma) = \sigma$ for each vertex or edge $\sigma$ of $C(s)$ and having the following property.
If $\omega$ is an edge of $C(s)$ parameterized by arc length $t$, then $g_s \omega(\Omega/2)$ is a vertex of $C(s + 1)$ ($\Omega$ denotes the total arc length of $\omega$) and the derivative of the arc length of $g_s \omega(t)$ with respect to $t$ exists except when $t = \Omega/2$ and is constant on $(0, \Omega/2)$ and $(\Omega/2, \Omega)$. 

(2.5)

It is easy to construct such homeomorphisms using the fact that $g_s: |C(s)| \rightarrow |C(s + 1)|$ is cellular and $g_s(\sigma)$ contains a vertex of $C(s + 1)$ for each open 1-cell $\sigma$ of $C(s)$.

Define a sequence of embeddings $H_i$ by the equations

$$H_i = g_i h_i g_{i-1} h_{i-1} \cdots g_0 h_0;$$

(2.6)

it follows in a straightforward fashion that this sequence converges uniformly to a continuous function $H: C^1 \rightarrow M^2$ such that $H(C^i) = G(C^i)$ for $i = 0$ and $1$.

**Lemma 2.2.** The function $H: C^1 \rightarrow \mathcal{C}^1$ is a homeomorphism.

**Proof.** It remains only to show that $H$ is one-to-one. Let $x$ and $y$ be distinct points in $C^1$ and choose a simple piecewise smooth arc $a: [0, 1] \rightarrow C^1$ with $x = a(a)$ and $y = a(b)$ where $0 < a < b < 1$ (cf. Lemma 1.2). Because of (2.5) there exists an integer $s > 0$ such that the simple arc $H_s a$ contains at least four vertices $v_i$ ($i = 1, 2, 3, 4$) of $C(s + 1)$ with $v_1 = H_s a(t_1)$ where $t_1 < a < t_2 < t_3 < b < t_4$; in particular, $H_s(x)$ and $H_s(y)$ belong to nonintersecting edges $\omega_x$ and $\omega_y$ of $C(s + 1)$. From this remark, it is easily seen (use the defining properties of $h_i$) that $H(x) \in \hat{G}_{s+1}(\omega_x)$ and $H(y) \in \hat{G}_{s+1}(\omega_y)$ where $\hat{G}_{s+1}$ is the composite $(G G_{s+1})^{-1}$; hence, Lemma 2.2 is a consequence of the following result.

**Lemma 2.3.** If $\sigma_1$ and $\sigma_2$ are nonintersecting closed cells in $C(s)$, then $\hat{G}_{s}(\sigma_1) \cap \hat{G}_{s}(\sigma_2) = \emptyset$.

**Proof.** First observe, using Lemma 1.1, 1.4, 2.1, that $G(\sigma) \subset T(\sigma, 5^{-1}d_0)$ for each closed cell $\sigma$ of $C$; also using (2.2), that $F^* \hat{G}_{s} = GF^*$; hence if $F^*\sigma_1 \cap F^*\sigma_2 = \emptyset$, then $\hat{G}_{s}(\sigma_1) \cap \hat{G}_{s}(\sigma_2) = \emptyset$. On the other hand, if $F^*\sigma_1$ and $F^*\sigma_2$ intersect, then $T(F^*\sigma_1 \cup F^*\sigma_2, 5^{-1}d_0) \subset T(x, d_i)$ for some point $x \in M^2$. Since $T(x, d_i)$ is homeomorphic to a closed 2-disc, it is evenly covered by $F^*$; in particular, $\sigma_i \subset S_i$ ($i = 1, 2$) where $S_1$ and $S_2$ are distinct components of $F^{-1}T(x, d_i)$. Finally observe $\sigma_1 \cup \hat{G}_{s}\sigma_i \subset S_i$ ($i = 1, 2$); hence $\hat{G}_{s}\sigma_1$ and $\hat{G}_{s}\sigma_2$ cannot intersect.

Because of Lemma 2.2 (and the paragraph preceding it) we define $G'|C^1$ to be $H$. Let $\sigma$ be a closed 2-cell of $C$ with boundary denoted by $\partial \sigma$; then $H(\partial \sigma)$ is contained in the interior of $T(P_\sigma, d_1)$ since

$$H(\partial \sigma) = G(\partial \sigma) \subset G(\sigma) \subset T(\sigma, 5^{-1}d_0).$$

(2.7)
Since $\partial \sigma$ is homeomorphic to a circle, $T(P_0, d_1)$ to a closed 2-disc and $H$ is an embedding, Schoenflies' Theorem allows us to extend $G'|\partial \sigma$ to a homeomorphism of $\sigma$ to the closure of the interior component of $T(P_0, d_1) - G'(\partial \sigma)$; thus, we extend $G'|C^1$ to all of $M^2$.

To complete the demonstration of Theorem 2.1, it remains to verify that $G'$ is a one-to-one function. (Note any one-to-one continuous self-map of a closed connected manifold must be onto.) To do this, first observe that $G'(\sigma) \subset G(\sigma)$ for each closed 2-cell $\sigma$ of $C$; this is a consequence of the fact that $G'|\partial \sigma$ and $G|\partial \sigma$: $\partial \sigma \to G'(\partial \sigma)$ are homotopic (hence $G|\partial \sigma$ is essential). Therefore, it suffices to show for all pairs of distinct 2-cells $\sigma_1$ and $\sigma_2$ in $C$ that the following containment is true

$$G(\sigma_1) \cap G(\sigma_2) \subset G'(\sigma_1 \cap \sigma_2). \quad (2.8)$$

Let $z \in G(\sigma_1) \cap G(\sigma_2)$; hence, there exist points $x \in \sigma_1$ and $y \in \sigma_2$ with $G(x) = z = G(y)$; consequently, $d(G_i(x), G_i(y)) \to 0$ as $i \to \infty$. Since $G_i$ is a homeomorphism, there must exist points $x_i \in \partial \sigma_1$ and $y_i \in \partial \sigma_2$ such that $G_i(x_i) \to z$ and $G_i(y_i) \to z$ as $i \to \infty$ (cf. Figure 2); but $\partial \sigma_i$ ($i = 1, 2$) is compact; hence $z \in G(\partial \sigma_1) \cap G(\partial \sigma_2)$.

![Figure 2](https://example.com/figure2.png)

Since $G(\partial \sigma_i) = H(\partial \sigma_i)$ ($i = 1, 2$) and $H$ is a one-to-one function (Lemma 2.2), we have

$$z \in H(\partial \sigma_1 \cap \partial \sigma_2) = G'(\sigma_1 \cap \sigma_2) \quad (2.9)$$

which verifies (2.8) and completes the proof of Theorem 2.1.

**Remark 2.4.** Let $f: T^2 \to T^2$, where $T^2$ is the torus, be an expanding endomorphism induced by a $2 \times 2$ matrix $A$ with integral entries and whose eigenvalues are real but irrational numbers having distinct absolute values. It is implicit in Franks’ paper [6] that a proper closed $f$-invariant subset of $T^2$ cannot contain a $C^1$-arc; i.e., a continuously differentiable arc. In particular,
if

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & 7 \end{pmatrix}, \]

then the 1-skeleton of the Markov cell structure constructed in Theorem 2.1 for \((T^2, f)\) cannot contain a \(C^1\)-arc. (See also Bowen’s paper [3].)

3. Markov cell structures (branched 2-manifold case). This section is devoted to proving an analogue of Theorem 2.1 in the branched 2-manifold case. (See Williams’ paper [14] for the basic definitions and facts concerning branched manifolds.) A cell structure \(C\) for a compact branched 2-manifold \(M^2\) is a filtration of \(M^2\) by closed subsets

\[ \emptyset = C^{-1} \subset C^0 \subset C^1 \subset C^2 = |C| = M^2 \quad (3.1) \]

such that \(C^i - C^{i-1} (i = 0, 1, 2)\) has a finite number of components, called \(i\)-cells, satisfying property (i) in (1.1). If \(M^2\) is equipped with a map \(f\), a Markov cell structure for \((M^2, f)\) is a cell structure \(C\) for \(M^2\) such that \(f^n: |C| \to |C|\) is cellular for some positive integer \(n\).

**Theorem 3.1.** Let \(f: M^2 \to M^2\) be an immersion satisfying Axioms 1, 2 and 3+ of [14] where \(M^2\) is a compact branched 2-manifold, then \((M^2, f)\) is shift equivalent to a pair \((N^2, g)\) having a Markov cell structure.

The proof of Theorem 3.1 is similar to, but technically more complicated than, that of Theorem 2.1; hence, we sketch it, going into detail only where it differs from the previous argument.

By [14, Lemma 5.6], \((M^2, f)\) is shift equivalent to a pair \((N^2, g)\) where \(N^2\) is normally branched; hence, we will assume that \(M^2\) is normally branched. Let \(\beta M\) denote the branch set of \(M^2\); as a consequence of [14, §8], \(M^2\) contains a 2-dimensional compact submanifold (with boundary) \(A\) and a 2-dimensional compact branched submanifold (with boundary) \(B'\) satisfying

\[
\begin{align*}
(i) \text{ interior } A \cup \text{ interior } B' &= M^2 \\
(ii) \beta M \subset \text{ interior } B' \cap (M^2 - A) \\
\end{align*}
\quad (3.2)
\]

Furthermore, there is a compact 2-manifold \(B\) (with boundary), a surjective immersion \(p: B' \to B\) which maps \(\beta M\) homeomorphically onto \(p(\beta M)\), and an immersion \(\phi: B \to M^2\) with \(f(x) = \phi p(x)\) for all \(x \in B'\). By deleting short collar neighborhoods from \(A\), \(B\) and \(B'\), we obtain compact submanifolds \(A_i, B_i\) and \(B'_i\) \((i = -1, 0, 1)\) with

\[
\begin{align*}
A_{-1} \subset \text{ interior } A_0 \subset \text{ interior } A_1 \subset \text{ interior } A \\
B_{-1} \subset \text{ interior } B_0 \subset \text{ interior } B_1 \subset \text{ interior } B \\
\end{align*}
\quad (3.2.1)
\]

such that \(B'_i = p^{-1}B_i\) and (3.2) is satisfied when \(A\) and \(B'\) are replaced by \(A_{-1}\) and \(B'_{-1}\). We also assume that the Riemannian metrics on \(M^2, A, B'\) and \(B\) fit together consistently; i.e., \(A\) and \(B'\) have the metrics induced from \(M^2\)
and \( dp : TB' \to TB \) preserves the Riemann metric.

A (topological) branched cell structure \( \mathcal{B} \) for \( M^2 \) is a filtration by closed subsets

\[
\emptyset = \mathcal{B}^{-1} \subset \mathcal{B}^0 \subset \mathcal{B}^1 \subset \mathcal{B}^2 = |\mathcal{B}| = M^2
\]

such that there exist (topological) special cell structures \( C \) in the interior of \( A \) and \( K \) in the interior of \( B \) satisfying the following conditions

(i) \( \mathcal{B}^i = C^i \cup p^{-1}K^i \),
(ii) \( A_{-1} \subset |C|, B_{-1} \subset |K| \) and
(iii) \( |C| \cap p^{-1}|K| \) is a subcomplex of \( C \); each cell of which maps homeomorphically via \( p \) onto a cell of \( K \).

The components of \( \mathcal{B}^i - \mathcal{B}^{i-1} \) are (in general) not cells when they intersect \( \beta M^2 \). A map \( f : |\mathcal{B}| \to |\mathcal{B}| \) is cellular if \( f(\mathcal{B}^i) \subset \mathcal{B}^i \) (for all \( i \)). We will derive Theorem 3.1 directly from the following result.

**Lemma 3.2.** There is a topological branched cell structure \( \mathcal{B} \) for \( M^2 \) of arbitrarily small mesh such that \( f^n : |\mathcal{B}| \to |\mathcal{B}| \) is a cellular map for some positive integer \( n \).

Before proving this lemma, we use it together with the collapsing technique introduced by Williams (cf. [14, Lemmas 2.2 and 5.4]) to complete the demonstration of Theorem 3.1. Let \( K \) be the topological special cell structure for \( B \) posited in (3.4). Since \( \mathcal{B} \) can be constructed with arbitrarily small mesh, we can assume that \( K \) has a subcomplex \( K_0 \) such that

\[
p(\beta M^2) \subset \text{interior } |K_0| \subset \text{interior } B_{-1}.
\]

Form \( N^2 \) by collapsing \( p^{-1}|K_0| \) under the immersion \( p : B' \to B \) and let \( g : N^2 \to N^2 \) be the immersion induced from \( f \), then \( (N^2, g) \) is shift equivalent to \( (M^2, f) \) and \( \mathcal{B} \) induces a cell structure on \( N^2 \) (via the canonical quotient map) with respect to which \( g^n \) is cellular; i.e., \( (N^2, g) \) has a Markov cell structure.

**Remark 3.3.** The Markov cell structure constructed above may fail to satisfy property (ii) in (1.1); for instance, if \( K_0 \) is not a “full” subcomplex in \( K \), this is possible. But, by constructing more carefully, we believe this property can also be satisfied.

It remains to prove Lemma 3.2; we start by constructing a branched cell structure \( \mathcal{B}(0) \) for \( M^2 \) of arbitrarily small mesh. First, triangulate \( M^2 \) by a smooth triangulation \( \mathcal{T}_1 \) (with arbitrarily small mesh) so that \( B' \) is a subcomplex with its triangulation induced via \( p^{-1} \) from a smooth triangulation \( \mathcal{T}_2 \) of \( B \) (cf. [14, Lemma 5.7]). Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be subcomplexes consisting of all closed simplices from \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \), respectively, which meet \( A_0 \) and \( B_1 \), respectively; if mesh \( \mathcal{T}_1 \) is sufficiently small, then

\[
|\mathcal{F}_1| \subset \text{interior } A_1 \quad \text{and} \quad |\mathcal{F}_2| \subset \text{interior } B.
\]
Let $C$ and $K$ be the dual cell complexes to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively; then $C$, $K$ satisfy conditions (ii) and (iii) of (3.4) and hence determine via formula (i) of (3.4) a branched cell structure $\mathfrak{B}(0)$ for $M^2$ with arbitrarily small mesh. (Note also $B_1 \subset |K|$ and $|C| \subset A_1$.)

We assume that

$$|df(X)| > 2|X|$$  \hspace{1cm} (3.7)

for each vector $X$ tangent to $M^2$. (To do this, we may have to replace $f$ by a power of itself.) Define a second branched cell structure $\mathfrak{B}(1)$ for $M^2$ by letting $\mathfrak{B}(1)' = f^{-1}(\mathfrak{B}(0))$. (For this to be a branched cell structure, mesh $\mathfrak{B}(0)$ must be sufficiently small.) There are special cell structures $C(1)$ in the interior of $A$ and $K(1)$ in the interior of $B$ satisfying (3.4) (with $\mathfrak{B}$, $C$ and $K$ replaced by $\mathfrak{B}(1)$, $C(1)$ and $K(1)$) and having the following properties

\begin{itemize}
  \item[(i)] $C(1)' \subset \mathfrak{B}(1)'$ and $K(1)' \subset f^{-1}(\mathfrak{B}(0))$;
  \item[(ii)] $T(|C|, \text{ mesh } \mathfrak{B}(0)) \subset |C(1)|$ and $T(|K|, \text{ mesh } \mathfrak{B}(0)) \subset |K(1)|$.
\end{itemize}

(In order to satisfy property (ii), mesh $\mathfrak{B}(0)$ must again be sufficiently small.)

Let $\{P_\sigma\}$, $\{D_\sigma\}$ be base points and auxiliary discs for $C$; let $V = p^{-1}|K| \cap |C|$ and $U = p^{-1}T(B_0, K) \cap |C|$. If mesh $\mathfrak{B}(0)$ is sufficiently small, then $U \subset V \subset C$ satisfy the first hypothesis of Lemma 1.3; let $\{D_\tau\}$ be the auxiliary discs posited there. Replacing these by smaller discs (if necessary), we may assume there are auxiliary discs $\{\hat{D}_\tau\}$ for $K$ such that $p(D_\sigma) = \hat{D}_\tau$ for each closed cell $\sigma$ of $V$ and $\tau$ of $K$ with $p(\sigma) = \tau$. Also let $\{\hat{P}_\sigma\}$ be a collection of base points for $K$ such that $p(P_\sigma) = \hat{P}_\tau$ for 2-cells $\sigma$ of $V$ and $\tau$ of $K$ with $p(\sigma) = \tau$. Replacing $f$ by a high power of itself (if necessary), we can assume that mesh $\mathfrak{B}(1)$ is small enough so that we can apply Lemma 1.3 (twice) to obtain cellular embeddings $\hat{g}: |K| \to |K(1)|$ and $g: |C| \to |C(1)|$ with the following properties

\begin{itemize}
  \item[(i)] $pg(x) = \hat{g}p(x)$ for $x \in |C| \cap p^{-1}|K(0)|$ where $K(0) = \mathfrak{B}(0)$;
  \item[(ii)] $g(\sigma) \subset D_\sigma$ and $\hat{g}(\tau) \subset \hat{D}_\tau$ where $\sigma$ and $\tau$ are closed cells in $C$ and $K$, respectively.
\end{itemize}

(3.9)

Of course, mesh $\mathfrak{B}(0)$ must be small enough that Lemma 1.3 applies; in particular, $T(x, 5c)$ should be homeomorphic to a closed 2-disc for both $x \in A_0$ and $x \in B_0$ where $c = 2$ mesh $\mathfrak{B}(0)$; also, assume $T(A_1, c) \subset A$ and $T(B_1, c) \subset B$. Now, define inductively a sequence of embeddings $g_i: A_1 \to A$ and $\hat{g}_i: B_1 \to B$ as follows. If $x \in A_1$ and $f(x) \in A_0$ (respectively, $f(x) \in B_0$), let $g_i(x)$ be the unique point in the same component of $f^{-1}T(f(x), c)$, $(f^{-1}p^{-1}T(pf(x), c))$ as $x$ such that
if $x \in B_1$ and $y = \varphi(x) \in A_0$ (respectively, $y \in B_0$), let $\hat{g}_s(x)$ be the unique point in the same component of $\varphi^{-1}T(y, c)$, $(\varphi^{-1}p^{-1}T(p(y), c))$ as $x$ such that

$$\varphi \hat{g}_s(x) = g_{s-1}(y), \quad (p \varphi \hat{g}_s(x) = \hat{g}_{s-1}p(y)).$$

(When $s = 1$ use $g$ and $\hat{g}$ in formulas 3.10.1, 3.10.2 in place of $g_0$ and $\hat{g}_0$, respectively.) Note the following analogue of 3.9(i) is true

$$pg_s(x) = \hat{g}_p(x) \quad \text{for} \quad x \in A_1 \cap B_1.'$$

Next, form the composite embeddings $G_s: |C(0)| \to A$ where $C(0) = \Theta(A_0, C)$ and $\hat{G}_s: |K(0)| \to B$ defined by the equations

$$G_s(x) = g_s g_{s-1} \cdots g_1 g(x) \quad \text{for} \quad x \in |C(0)| \quad \text{and} \quad \hat{G}_s(y) = \hat{g}_s \hat{g}_{s-1} \cdots \hat{g}_1 \hat{g}(y) \quad \text{for} \quad y \in |K(0)|.$$ 

(3.11)

These sequences converge uniformly to continuous functions $G: |C(0)| \to A$ and $\hat{G}: |K(0)| \to B$ satisfying

$$pG(x) = \hat{G}p(x) \quad \text{for} \quad x \in |C(0)| \cap p^{-1}|K(0)|.$$ 

(3.12)

Define filtrations $\mathcal{C}$ and $\mathcal{K}$ by the equations

$$\mathcal{C} = G(C(0)) \quad \text{and} \quad \mathcal{K} = \hat{G}(K(0));$$ 

(3.13)

although $G$ and $\hat{G}$ need not be embeddings, arguing as in §2, it can be shown that $\mathcal{C}$ and $\mathcal{K}$ are topological special cell structures in the interiors of $A$ and $B$, respectively. Also, conditions (ii) and (iii) of (3.4) are satisfied when $C$ and $K$ are replaced by $\mathcal{C}$ and $\mathcal{K}$; hence, letting $\mathcal{B} = \mathcal{C} \cup p^{-1}\mathcal{K}$, we obtain a topological branched cell structure $\mathcal{B}$ for $M^2$ and, using (3.10.1), (3.10.2), it can be shown that $f: |\mathcal{B}| \to |\mathcal{B}|$ is cellular. This completes the proof of Lemma 3.2.

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