**Z(2)**-KNOT COBORDISM IN CODIMENSION TWO, AND INVOLUTIONS ON HOMOTOPY SPHERES

BY

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Abstract. Let \( \mathbb{Z}(2) \) denote the ring of rational 2-adic integers. In this paper, we consider the group \( \Phi_k \) of \( \mathbb{Z}(2) \)-cobordism classes of \( \mathbb{Z}(2) \)-knot \( (\Sigma^{k+2}, K^k) \), where \( \Sigma \) is a 1-connected \( \mathbb{Z}(2) \)-sphere \( \mathbb{Z}(2) \)-cobordant to \( S^{k+2} \), and \( K \) is a 1-connected \( \mathbb{Z}(2) \)-sphere embedded in \( \Sigma \) with trivial normal bundle. For \( n > 3 \), we will prove that \( \psi_2 \equiv 0 \) and \( \psi_{2n-1} = C_2(\mathbb{Z}(2)), \epsilon = (-1)^n \). Also, we will show that the group \( \Theta_{4m+2}^{k+2} \) of \( L \)-equivalence classes of differentiable involutions on \((4m + 1)\)-homotopy spheres with codimension two fixed point sets defined by Bredon contains infinitely many copies of \( \mathbb{Z} \).

Let \( T \) be an orientation preserving differentiable involution on a homotopy sphere \( \Sigma^{k+2} \) with codimension two fixed point set \( F(T) = K \). Then it follows from Smith theory that \( K \) is a \( k \)-dimensional \( \mathbb{Z}(2) \)-sphere (a manifold having the same \( \mathbb{Z}(2) \) homology as \( S^k \)), where \( \mathbb{Z}(2) \) denotes the ring of rational 2-adic integers. We call the pair \( (\Sigma^{k+2}, K^k) \) a knot.

Two such involutions \( T_0 \) and \( T_1 \) are called \( L \)-equivalent if there exists an involution \( T' \) on \( \Sigma^{k+2} \times [0, 1] \) with \( F(T') = M^{k+1} \), a \( \mathbb{Z}(2) \)-cobordism between \( F(T_0) \) and \( -F(T_1) \). The set of \( L \)-equivalence classes of involutions on homotopy spheres forms an abelian group \( \Theta_k^{k+2} \) [4, pp. 339–340]. Ignoring the involutions, we call the two knots \( (\Sigma, F(T_0)) \) and \( (\Sigma, F(T_1)) \) \( L \)-equivalent. The set of \( L \)-equivalence classes also forms an abelian group \( \Theta_k^{k+2} \). Since the integral homology of \( K \) can be very complicated, we were unable to apply the methods in [6], [12] or [16] to compute \( \Theta_k^{k+2} \) (compare Lemma 4.2 below).

Let \( \Sigma^{k+2} \) denote a \( \mathbb{Z}(2) \)-sphere which is \( \mathbb{Z}(2) \)-cobordant to the standard sphere \( S^{k+2} \), and \( K^k \) a \( \mathbb{Z}(2) \)-sphere [1]. Throughout this paper, we will assume that both \( \Sigma \) and \( K \) are 1-connected for technical reasons (see (2.7) below). An embedding of \( K \) with trivial normal bundle in \( \Sigma \) is called a \( \mathbb{Z}(2) \)-knot, denoted by \( (\Sigma^{k+2}, K^k) \). Two \( \mathbb{Z}(2) \)-knots \( (\Sigma_1, K_1) \) and \( (\Sigma_2, K_2) \) are called \( \mathbb{Z}(2) \)-knot cobordant (or just \( \mathbb{Z}(2) \)-cobordant) if there exists \( (M^{k+3}, N^{k+1}) \) with \( \partial(M, N) = (\Sigma_1, K_1) \cup - (\Sigma_2, K_2) \) such that both \( M \) and \( N \) are 1-connected, the normal bundle of \( N \) in \( M \) is trivial, \( M \) is a \( \mathbb{Z}(2) \)-cobordism between \( \Sigma_1 \) and

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Let \( \Psi_k \) denote the abelian group of \( Z(2) \)-cobordant classes of \( Z(2) \)-knots \((\Sigma^{k+2}, K^k)\). Following [16], we let \( C_\varepsilon(Z(2)) \) (where \( \varepsilon = \pm 1 \)) denote the group of the cobordism classes of matrices over \( Z(2) \) (also called the cobordism classes of \( \varepsilon \)-symmetric structures over \( Z(2) \) in [12], [20]). The main result of the paper is the following theorem (see (2.5) and (3.4)).

**Theorem.** For \( n > 3 \), there exists an isomorphism \( \rho_n^\varepsilon : \Psi_{2n-1} \rightarrow C_\varepsilon(Z(2)), \varepsilon = (-1)^n \). Also, \( \Psi_{2n} = 0 \).

Later, we will see that every element of \( \Psi_{2n-1} \) has a representative of the form \((S^{2n+1}, K^{2n-1})\) such that \( K \) is \((n-1)\)-connected and bounds an \((n-1)\)-connected \( 2n \)-manifold in \( S^{2n+1} \). Following [7] and [10], we will call it a simple knot. In the last section, we will study the correlation between involutions and the Seifert matrices for simple knots, and show that the group \( \Theta_{4m+1}^n \) contains infinitely many copies of \( Z \) (see Theorem 4.3 below).

Everything mentioned here will be in the differentiable category. The p.l. case can be treated in the same way by using [3].

1. Let \( \theta_k^{(2)} \) denote the group of \( Z(2) \)-cobordism classes of \( k \)-dimensional \( Z(2) \)-spheres. If \( f : K^k \rightarrow S^k \) is a \( Z(2) \)-homology equivalence [2, p. 3], then it induces a normal map with normal invariant contained in \([S^k, G(2)/O] \)[2]. We also write \( f \) for this normal map.

According to [1, (2.1)] or [2, p. 135], every element of \( \theta_k^{(2)} \) can be represented by a 1-connected \( Z(2) \)-sphere.

Let \( bP_{2n+1}^{(2)} \) denote the subgroup of \( \theta_k^{(2)} \) consisting of all classes represented by \( k \)-dimensional \( Z(2) \)-spheres which bound compact manifolds admitting odd frames [1]. We have the following lemma from [1, pp. 41-42].

**Lemma 1.1.** Let \( K_0 \) represent an element of \( bP_{2n+1}^{(2)} \), \( k > 5 \). Then \( K_0 \) is \( Z(2) \)-cobordant to a homotopy sphere \( K_1 \) if \( k \neq 4n-1 \), and to a \((2n-2)\)-connected \( Z(2) \)-sphere \( K_1 \) which bounds a parallelizable manifold if \( k = 4n-1 \).

**Lemma 1.2.** Let \( K_0 \) and \( K_1 \) be as in Lemma 1.1. If \( K_0 \) is 1-connected, then we may choose a \( ((k-1)/2)\)-connected \( Z(2) \)-cobordism \( Q \) between \( K_0 \) and \( K_1 \).

**Proof.** Let \( I \) denote the interval \([0, 1]\), and \( P \) the \( Z(2) \)-cobordism between \( K_0 \) and \( K_1 \) in Lemma 1.1. We apply the argument in the proof of [3, Theorem 1.3] to make \( P \) 1-connected (also see [1, (2.1)]). The normal map \( f : P \rightarrow S^k \times I \) with \( f^{-1}(S^k \times i) = K_i \) \((i = 0, 1)\) inducing a normal map \( F : P \times I \rightarrow S^k \times I \times I \). Write \( Y = S^k \times I \times I, X = S^k \times I \times 1, \) and \( X_+ = S^k \times I \times 0 \cup \partial(S^k \times I) \times I \). We note that the map \( F|M_+ = M_+ : Q \times 0 \cup \partial Q \times I \rightarrow X_+ \) is a \( Z(2) \)-homology equivalence in the sense of [2, p. 3]. Then we perform surgery rel \( M_+ \) to make the map \( F : (Q \times I; Q \times 1, M_+) \rightarrow (Y; X, X_+) \) normally cobordant to \( G : (N^{k+2}; M^{k+1}, M_+) \rightarrow (Y; X, X_+) \) such that \( G|M \).
is \( [(k + 1)/2\)-connected and \( G|N \) is \( [(k + 2)/2\)-connected (see Corollary 1 in [2, p. 59]).

By applying Theorem 1 in [2, p. 82], we may make \( G \) a homology equivalence of triad over \( Z_{(2)} \). We take \( Q \) to be \( M \). Since \( G|M: M \to S^k \times I \) is \( [(k + 1)/2\)-connected, \( Q \) is \( [(k - 1)/2\)-connected. Q.E.D.

**Corollary 1.3.** If a simply-connected \( Z_{(2)} \)-sphere \( K^k \) is \( Z_{(2)} \)-cobordant to \( S^k \), then \( K \) bounds a \((k + 1)\)-dimensional \( [(k - 1)/2\)-connected \( Z_{(2)} \)-disk.

2. Let \( (\Sigma^{k+2}, K^k) \) denote a \( Z_{(2)} \)-knot, that is, an embedding of a 1-connected \( Z_{(2)} \)-sphere \( K \) with trivial normal bundle in a 1-connected \( Z_{(2)} \)-sphere \( \Sigma \) which is \( Z_{(2)} \)-cobordant to \( S^{k+2} \).

Let \( X \) denote the closure of \( \Sigma^{k+2} - K^k \times D^2 \). Since \( H^1(K) = H^1(\Sigma) = H^2(\Sigma) = 0 \), the map \( H^1(X) \to H^1(\partial X) \) is onto from the Mayer-Vietoris sequence. Therefore, we may extend the projection map \( \partial X = K \times S^1 \to S^1 \) to a map \( g: X \to S^1 \), and thus we have the following lemma.

**Lemma 2.1.** There exists a degree 1 normal map \( f \) from \( (\Sigma^{k+2}, K^k) \) to the trivial knot \( (S^{k+2}, S^k) \) such that \( f^{-1}(S^k) = K \) and \( f \) is a bundle map on \( K \times D^2 \).

*Furthermore, by making the map \( g \) transverse to a point \( x \) in \( S^1 \), we have an oriented Seifert manifold \( F^{k+1} = g^{-1}(x) \) in \( \Sigma \) with \( \partial F = K \).

**Lemma 2.2.** The \( Z_{(2)} \)-sphere \( K^k \) in a \( Z_{(2)} \)-knot \( (\Sigma^{k+2}, K^k) \) represents an element in \( \langle \partial F^k \rangle \).

*Proof. It was shown in [1, (1.1)] that \( \Sigma^{k+2} \) admits an odd framing (that is, its stable normal bundle is \( Z_{(2)} \)-parallelizable). Therefore, \( F \) also admits an odd framing. Q.E.D.

**Lemma 2.3.** For \( k > 5 \), a \( Z_{(2)} \)-knot \( (\Sigma^{k+2}, K^k) \) is \( Z_{(2)} \)-cobordant to \( (\Sigma, K_1) \) such that \( \pi_1(\Sigma - K_1) = Z \).

*Proof. Same proof as in [14]—by making the Seifert manifold \( F \) 1-connected. Also, we note that \( K_1 \) is diffeomorphic to \( K \). Q.E.D.

As in [6], we see that \( f: \Sigma - K_1 \to S^{k+2} - S^k \) is not a homology equivalence over \( Z_{(2)}[Z] \) [2, p. 3], but rather a homology equivalence over \( Z_{(2)}[e] = Z_{(2)} \). Using the notation of [6], we consider the surgery obstruction group \( \Gamma_{k+3}(Z[Z] \to Z_{(2)}[e]) \), where the map \( Z[Z] \to Z_{(2)}[e] \) is the composite \( Z[Z] \to Z_{(2)}[Z] \to Z_{(2)}[e] \). We may set up the surgery problem and construct the surgery group from the surgery group \( L_m(Z, Z_{(2)}) \) considered by Anderson [2] \( (\Gamma(Z[Z] \to Z_{(2)}[Z]) \) in the notation of [6]) and the map \( Z_{(2)}(Z) \to Z_{(2)}[e] \), in the same way as the Cappell-Shaneson's surgery group \( \Gamma_m(Z[Z] \to Z[e]) \) is related to the Wall group \( L_m(Z) \) [22] and the map \( Z[Z] \to Z[e] \).

We call a \( Z_{(2)} \)-knot \( (\Sigma^{k+2}, K^k) \) simple, if \( K \) is \( ([k/2] - 1) \)-connected, \( \pi_j(\Sigma - K) = \pi_j(S^1) \) for \( j < [k/2] \), and \( \Sigma \) is \([k/2]\)-connected.
PROPOSITION 2.4. For \( k > 5 \), a \( Z(\Sigma)^{-} \) knot is \( Z(\Sigma)^{-} \) cobordant to a simple one.

PROOF. Let \( (\Sigma^{k+2}, K^k) \) be a \( Z(\Sigma)^{-} \) knot satisfying the condition in (2.3), that is, \( \Sigma - K \) is 1-connected. We construct a normal map \( f: (\Sigma^{k+2}, K^k) \to (S^{k+2}, S^k) \) as in the first paragraph of the section. Define \( F: f \times \text{id}: (\Sigma^{k+2}, K^k) \times I \to (S^{k+2}, S^k) \times I \). Let \( g: \Omega \to S^k \times I \) be the \( Z(\Sigma)^{-} \) cobordism constructed in Lemma 1.2 between \( K \) and \( K_x \) such that \( g|K = f|K \). By the cobordism extension theorem, we may extend \( g \) to \( Q \times D^2 \) and glue it to \( F \) on \( \Sigma \times I \) along \( K \times D^2 \times 1 \) to get a normal map \( G: V \to S^{k+2} \times I \), with \( Q \subset V, G^{-1}(S^k \times I) = Q, G|Q = g, \) and \( G \) is a bundle map on a neighborhood of \( Q \). We note that \( G \) is a normal cobordism from \( f: (\Sigma, K) \to (S^{k+2}, S^k) \times 0 \) to \( h: (P, K_x) \to (S^{k+2}, S^k) \times 1 \).

Let \( E = D^{k+1} \times S^1 \) denote the closure of \( S^{k+2} - S^k \times D^2 \). We write \( Y \) for \( E \times X_+ \) for \( E \times 1 \), and \( X_+ \) for the closure of \( \partial Y - X_+ \). Considering the induced map \( H = G|(N^{k+2}, M^{k+2}, M^{k+2}_+) \to (Y; X, X_+) \), where \( N = G^{-1}(Y), M = G^{-1}(X), \) and \( M_+ = G^{-1}(X_+) \). We note that \( P = M \cup K_1 \times D^2 \) and \( V = Q \times D^2 \cup N \). The map \( G|M_+ \) induces an isomorphism on \( \pi_1 \), and is a \( Z(\Sigma)^{-}[e] \)-homology equivalence. By performing surgery rel \( M_+ \), we may assume that \( H|M \) is \( [(k + 2)/2]\)-connected, and \( H|N \) is \( [(k + 3)/2]\)-connected. Then we perform the relative surgery as in [2, p. 82] to make \( H: (N, M, M_+) \to (Y; X, X_+) \), a homology equivalence of triads over \( Z(\Sigma)[e] \) (here we use the surgery with coefficient associated with \( Z[Z] \to Z(\Sigma)[Z] \to Z(\Sigma)[e] \)). By gluing back \( G|Q \times D^2 \) to \( H \), we have a \( Z(\Sigma)^{-} \) cobordism between \( (\Sigma, K) \) and a \( Z(\Sigma)^{-} \) knot \( (\Sigma_1, K_1) \). If \( k \) is odd, then \( (\Sigma_1, K_1) \) is a simple \( Z(\Sigma)^{-} \) knot.

If \( k = 2n \), we may take \( K_1 \) to be a homotopy sphere \(((1.1) \text{ and } (1.2))\), and the above proof shows that \( H|M \to S^1 \times D^{2n+1} \) is \( (n + 1) \)-connected. Hence \( G: \Sigma_1 \cup S^{2n} \times D^2 \to S^{2n+2} \) is \( (n + 1) \)-connected. Therefore \( \Sigma_1 \) is \( n \)-connected, and \( H^{n+1}(\Sigma_1) = H_{n+1}(\Sigma_1) \) is free abelian. But \( G \) is a \( Z(\Sigma)^{-} \) homology equivalence. Thus \( G \) is a homotopy sphere. The argument in [11] shows that \( (\Sigma_1, K_1) \) is knot cobordant to a knot with complement having the homotopy type of a circle. Q.E.D.

The last sentence in the above proof gives us the following.

COROLLARY 2.5. For \( n > 3 \), \( \Psi_{2n} = 0 \).

PROPOSITION 2.6. If a \( Z(\Sigma)^{-} \) knot \( (\Sigma^{k+2}, K^k) \) is \( Z(\Sigma)^{-} \) cobordant to the trivial knot \( (S^{k+2}, S^k) \), \( k > 5 \), then \( (\Sigma, K) \) bounds a pair of \( Z(\Sigma)^{-} \) disk \( (W^{k+3}, B^{k+1}) \) such that \( B \) is \( [(k - 1)/2]\)-connected and \( W \) is \( [(k + 1)/2]\)-connected.

PROOF. The proof is very similar to the ones given in (1.2) and (2.4). Hence we only give an outline here.

Let \( (N, Q) \) be a \( Z(\Sigma)^{-} \) cobordism between \( (S^{k+2}, S^k) \) and \( (\Sigma, K) \). Recall that
both $N$ and $Q$ are 1-connected. By applying the same argument in (2.3) rel $\partial N$, we may require that $\pi_j(N - Q) = \pi_j(S^1)$. Furthermore, both $\pi_j(S^{k+2} - S^k) \to \pi_j(N - Q)$ and $\pi_j(\Sigma - K) \to \pi_j(N - Q)$ are isomorphisms. We then apply the argument in (1.2) rel $\partial Q$ to construct a $Z(2)$-cobordism $R$ between $Q$ and $Q_1$ such that $\partial R = \partial Q \times I$ and $Q_1$ is $\lfloor (k - 1)/2 \rfloor$-connected. Applying the cobordism extension theorem as in (2.4) rel $\partial Q$, we have a normal cobordism $(P^{k+4}, R^{k+2})$ between $(N, Q)$ and $(N_1, Q_1)$ rel $\partial(N, Q) \times I$. As in (2.4), we have $\pi_j(N - Q) = \pi_j(S^1)$ for $j < \lfloor (k + 1)/2 \rfloor$ and $\pi_j(P - R) = \pi_j(S^1)$ for $j \leq \lfloor (k + 2)/2 \rfloor$. Then we make $P - R \times D^2$ and $N_1 - Q_1 \times D^2 Z(2)$-cobordisms by using [2, p. 82] in the interiors as in (2.4) (as in (2.4), $N_1 - Q_1 \times D^2$ is only homology equivalent to $S^1$ over $Z(2)[e]$, but not over $Z(2)[Z]$). Gluing $Q_1 \times D^2$ back to $N_1 - Q_1 \times D^2$, we obtain a $Z(2)$-cobordism $(N_1, Q_1)$ between $(S^{k+2}, S^k)$ and $(\Sigma, K)$ such that $Q_1$ is $\lfloor (k - 1)/2 \rfloor$-connected and $N$ is $\lfloor (k + 1)/2 \rfloor$-connected.

Finally, we construct $(W, B)$ by gluing the standard disks pair $(D^{k+3}, D^{k+1})$ to $(N, Q)$ along $(S^{k+2}, S^k)$. Q.E.D.

Remark 2.7. The restriction of both $\Sigma$ and $K$ being 1-connected is used in producing a normal map $f: (\Sigma^{k+2}, K^k) \to (S^{k+2}, S^k)$ in (2.1) and a Seifert manifold $F$ for $K$ (Lemma 2.2). Furthermore, unlike [6], the definition of a homology equivalence $f: M \to X$ in [2, p. 3] requires that $f_*: \pi_1(M) \to \pi_1(X)$ is an isomorphism. This condition is contained in the statement of Theorem 1 in [2, p. 82], which was used in the proof of our (1.2) and (2.4) above. If that theorem holds true without the restriction on $\pi_1$, then the proof of (1.2) would show that any $Z_p$-sphere $K^k$ is $Z_p$-cobordant to a $\lfloor (k/2) - 1 \rfloor$-connected one, where $P$ is a set of primes (compare [2, p. 135]).

3. Let $C_r(Z(2))$, $\epsilon = \pm 1$, denote the group of cobordism classes of matrices over $Z(2)$ as defined in [20, §1] (also see [12], [16]).

We first define a map $\rho: \Psi_{2n-1} \to C_r(Z(2))$, $\epsilon = (-1)^n$, modelled on [12] and [16]. Given an element $\gamma$ of $\Psi_{2n-1}$, we may choose a simple $Z(2)$-knot $(\Sigma^{2n+1}, K^{2n-1})$ representing $\gamma$ by (2.4), that is, $K$ is $(n - 2)$-connected, $\Sigma$ is $(n - 1)$-connected, and $\pi_j(\Sigma - K) = \pi_j(S^1)$ for $j \leq n - 1$. Let $F^{2n}$ be a Seifert manifold. We use the argument in [14] to make $F (n - 1)$-connected. Let $H = H_n(F^{2n})$, and denote by $A(x, y)$ the Seifert linking form on $H$ defined by computing the linking number of the cycle $x$ with the cycle $y$ "pushed" a small distance in the positive normal direction. Since $K$ is just a $Z(2)$-sphere, the intersection form on $F$: $\langle x, y \rangle = A(x, y) + eA(y, x)$ is not unimodular, but is invertible over $Z(2)$. We define $\rho(\gamma) =$ the cobordism class of $A$ in $C_r(Z(2))$. We call $A$ a Seifert matrix for $(\Sigma^{2n+1}, K^{2n-1})$.

A $Z(2)$-knot $(\Sigma^{k+2}, K^k)$ is called null-cobordant if it is $Z(2)$-cobordant to the trivial knot. Being null-cobordant is equivalent to the fact that $(\Sigma^{k+2}, K^k)$ bounds a pair of 1-connected $Z(2)$-disk $(W^{k+3}, B^{k+2})$. 

In order to show that \( \rho \) is well defined, it suffices to prove the following lemma.

**Lemma 3.1.** If \((\Sigma^{2n+1}, K^{2n-1})\) is a simple null-cobordant \(Z(2)\)-knot, and \(A\) is a Seifert matrix for \(K\), then \(A\) is null-cobordant in \(C_*(Z(2))\).

**Proof.** It follows from (2.6) that \((\Sigma^{2n+1}, K^{2n-1})\) bounds a pair of \(Z(2)\)-disks \((W^{2n+2}, B^{2n})\) such that \(B\) is \((n-1)\)-connected, and \(W\) is \(n\)-connected. Then the proof of Lemma 2 in [16] (or [12, p. 89], [20, p. 77]) can be used here. Q.E.D.

**Lemma 3.2.** For \(n \geq 3\), the map \(\rho\) is onto.

**Proof.** Every element of \(C_*(Z(2))\) can be represented by an integral matrix \(A\) with \(\det(A + \epsilon A') = \text{an odd integer}\), where \(A'\) denotes the transpose of \(A\). Then we construct a manifold \(F^{2n}\) with intersection form \(\langle \cdot, \cdot \rangle = A + \epsilon A'\) by plumbing. We may take \(F^{2n}\) to be \((n-1)\)-connected and \(\partial F = K (n-1)\)-connected [5, Chapter V]. As in [11, pp. 255–257], we may embed \(F\) into \(S^{2n+1}\) and perform surgery on the complement \(S^{2n+1} - F\) to realize the Seifert matrix \(A\). We thus get a simple \(Z(2)\)-knot \((S^{2n+1}, K)\) with Seifert matrix \(A\). Q.E.D.

**Lemma 3.3.** For \(n \geq 3\), the map \(\rho\) is injective.

**Proof.** Let \((\Sigma^{2n+1}, K^{2n-1})\) be a simple \(Z(2)\)-knot, and \(A\) its Seifert matrix associated with an \((n-1)\)-connected Seifert manifold \(F^{2n}\) for \(K\).

As in [16, Lemma 5], it suffices to show that a simple \(Z(2)\)-knot \((\Sigma^{2n+1}, K^{2n-1})\) with a null-cobordant Seifert matrix \(A\) is null-cobordant. According to (1.3), \(\Sigma^{2n+1}\) bounds an \(n\)-connected \(Z(2)\)-disk \(W^{2n+2}\). Thus \((W^{2n+2}, \Sigma^{2n+1})\) is \(n\)-connected.

Since \(A\) is null-cobordant, there is a subspace \(G\) of \(H_n(F)\) of one-half the rank on which \(A\) is identically zero. Hence the intersection form \(A + \epsilon A'\) is also identically zero on \(G\). Therefore, there exist disjoint \(n\)-spheres \(\{S_i\}\) embedded in \(F\) representing a basis for \(G\). Since the corresponding linking numbers \(A(x, y)\) are zero, these embeddings extend to disjoint embeddings of disks in \(W^{2n+2}\) by Haefliger's embedding theorem [8]. Since we may construct \(F\) alternatively by making the normal map \(f: (\Sigma, K) \to (S^{2n+1}, S^{2n-1})\) transverse to the disk \(D^{2n}\) and then performing surgery to get \(F = f^{-1}(D^{2n})\), we see that the normal bundles of \(\{S_i\}\) are trivial, and the framings for the tubular neighborhoods of \(\{S_i\}\) can be extended because the self-linking is zero. Then we perform surgery inside \(W^{2n+2}\) to replace \(F\) by a \(Z(2)\)-disk (see [12], [16], also [20, (6.6)]). Q.E.D.

From the previous three lemmas, we have the following theorem:

**Theorem 3.4.** For \(n \geq 3\), \(\rho = \Psi_{2n-1} \to C_*(Z(2)), \epsilon = (-1)^n\), is an isomorphism.
The next corollary follows from (3.2) and (3.4).

**Corollary 3.5.** For \( n \geq 3 \), every element of \( \Psi_{2n-1} \) is represented by a simple knot [10, p. 145].

4. In this section, we will study involutions on homotopy spheres with codimension two fixed point sets [4, VI. 8].

According to (3.5), every element of \( \Psi_{2n-1}, n \geq 3 \), can be represented by a simple knot \((S^{2n+1}, K^{2n-1})\), where \( K^{2n-1} \) is \((n - 1)\)-connected, and bounds an \((n - 1)\)-connected manifold \( F^{2n} \subseteq S^{2n+1} \). Let \( A \) be a Seifert matrix (associated with \( F \)) for \( K \). The intersection form of \( F: \langle , \rangle = A + \varepsilon A' \) is not unimodular, but with odd determinant. It was noted in [7, p. 52] that Levine's classification theorem for simple spherical knots also holds in this more general context—two simple knots are isotopic in \( S^{2n+1} \) if and only if their Seifert matrices are related by a chain of congruence, elementary enlargements, and elementary reductions [17]. We will use Seifert matrices to describe which simple knots arise as fixed point sets of involutions on homotopy spheres.

We will let \( \Sigma \) and \( \Sigma' \) denote homotopy spheres in the rest of the paper. Also we let \( \Sigma_0 \) denote the generator of \( bP_{2n+1} \) [13]. Recall that \( \varepsilon = (-1)^n \).

**Theorem 4.1.** If \((S^{2n+1}, K^{2n-1}), n \geq 3, \) is a simple knot and \( \Sigma^{2n+1} \) a homotopy sphere, then \( \Sigma^{2n+1} = \Sigma^{2n+1} \neq S^{2n+1} \) admits an involution \( T \) with \( K \) as its fixed point set and with orbit space \( \Sigma' = \Sigma / T \) if and only if

(a) \((S^{2n+1}, K^{2n-1})\) has a Seifert matrix \( B \) of the form \( B = A(A - \varepsilon A')^{-1}A \) for some integral matrix \( A \) with \( \det(A + \varepsilon A') = \) an odd integer and \( \det(A - \varepsilon A') = \pm 1 \).

(b) For \( n \) odd, \( \Sigma = s \Sigma_0 + 2 \Sigma' \), where \( s = \text{signature}(A - \varepsilon A') \). For \( n \) even, \( \Sigma = a \Sigma_0 + 2 \Sigma' \), where \( a = \text{the Arf invariant of } A \).

**Proof.** This is just a slight modification of the proof for spherical simple knots given in [19].

The orbit map \( \Sigma \to \Sigma' \) is a 2-fold branched covering. Let \( A \) be a Seifert matrix for the simple knot \((\Sigma', K')\). Hence \( \det(A + \varepsilon A') = \) an odd integer. Corollary (5.7) of [10] shows that \( \Sigma \) bounds an \( n \)-connected parallelizable manifold with intersection form \( A - \varepsilon A' \). Therefore \( \Sigma \) is a homotopy sphere if and only if \( \det(A - \varepsilon A') = \pm 1 \).

The rest of the proof is almost the same as that of [19]. In the proof of Lemma 2 of [19], we multiply equation (3) by \((A + \varepsilon A')^{-1} \). Since \((A + \varepsilon A')^{-1} \) exists over the rationals \( Q \), the same proof carries through. We refer the readers to [19] for details. Q.E.D.

Let \( \Theta_k^{3+2} \) denote the group (under connected sum) of \( L \)-equivalence classes of involutions on homotopy spheres with codimension two fixed point sets (see [4, p. 340]).
Let $K^{2k-1}$ be a $\mathbb{Z}(2)$-sphere embedded in a homotopy sphere $\Sigma^{2n+1}$ with trivial normal bundle. Let $X$ denote the closure of $\Sigma - K \times D^2$. By using the Poincaré duality and Alexander duality, we see that $H^2(X, \partial X) = H_n(X) = H^1(K) = 0$. Therefore, there exists a Seifert manifold $F^{2n}$ for $(\Sigma, K)$ [14, Lemma 2]. As in [12], [16] or [20, p. 77], we may use $H_n(F^{2n})/\text{Torsion}$ to construct a Seifert matrix, and define a map $\rho': \Theta^{2n+1,2n-1}_{2n} \to C^s(\mathbb{Z}(2))$, where $e = (-1)^n$.

We may use the arguments in [12], [16] or [20, (6.6)] to show that $\rho'$ is well defined and surjective (we need a highly connected Seifert manifold to prove the injectivity in [12], [16] or [20]). Thus we have the following lemma:

**Lemma 4.2.** For $n \geq 3$, $\rho': \Theta^{2n+1,2n-1}_{2n} \to C^s(\mathbb{Z}(2))$, $e = (-1)^n$, is surjective.

**Theorem 4.3.** For $m \geq 2$, $\Theta^{4m+1}_{4m-1}$ contains infinitely many copies of $\mathbb{Z}$.

**Proof.** Because of (4.1) and (4.2), it suffices to show that there exist infinitely many linearly independent integral Seifert matrices $\{A_k\}$ in $C^1(\mathbb{Z}(2))$ satisfying $\det(A_k + A'_k) = \text{an odd integer}$ and $\det(A_k - A'_k) = \pm 1$. (Notice here $e = (-1)^{2m} = +1$.)

From [16, p. 243], we have the following sequence of linearly independent elements $\{A_k\}$ ($k = 1, 2, \ldots$) in $C^1(\mathbb{Z})$:

$$
A_k = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & k & 0 \\
0 & -k & 0 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

It is known that an integral matrix $A$ is null-cobordant over $\mathbb{Z}$ if and only if it is null-cobordant over the rationals $\mathbb{Q}$ [15, 16]. Therefore, $\{A_k\}$ are linearly independent over $\mathbb{Z}(2)$.

Because both $A_k + A'_k$ and $A_k - A'_k$ are unimodular, the fixed point sets of the involutions constructed are homotopy spheres [19]. Q.E.D.

Next we consider the question of whether every element of $bP^{(2)}_{4m+2}$ can be realized as the fixed point set of some involution on a $(4m+1)$-homotopy sphere $\Sigma$. (For $bP^{(2)}_{4m+2}$, see [18].) According to [1, p. 41] (also see (1.1)), all elements of $bP^{(2)}_{4m}$ can be realized as the $(2m-2)$-connected boundaries of $(2m-1)$-connected manifolds constructed by plumbing with respect to symmetric, even matrices with odd determinants. Over $\mathbb{Z}(2)$, these matrices can be expressed as sums of copies of $\left( \begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix} \right)$ and $\left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$ [9]. The argument in [21] will give us involutions on homotopy sphere $\Sigma^{4m+1}$ with elements of $bP^{(2)}_{4m}$ corresponding to the above two matrices as fixed point sets (also see [4, p. 341]). By taking connected sums, we can realize all of them.

**Proposition 4.4.** Every element $bP^{(2)}_{4m}$ has a representative which is the fixed point set of an involution on a homotopy sphere $\Sigma^{4m+1}$.
REFERENCES


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