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GENERIC DIFFERENTIABILITY OF LIPSCHITZIAN FUNCTIONS

BY

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ABSTRACT. It is shown that, in separable topological vector spaces which are Baire spaces, the usual properties that have been introduced to study the local "first order" behaviour of real-valued functions which satisfy a Lipschitz type condition are "generically" equivalent and thus lead to a unique class of "generically smooth" functions.

These functions are characterized in terms of tangent cones and directional derivatives and their "generic" differentiability properties are studied. The results extend some of the well-known differentiability properties of continuous convex functions.

Introduction. This paper is devoted to the investigation of generic smoothness properties for locally Lipschitzian real-valued functions, the domain of which is an open subset of a topological vector space.

There has been a recent revival of this problem in the restricted case of convex functions [2], [3], [9], [10], [14]; the first result along this vein seems to be due to E. Asplund and asserts that every continuous convex function defined on an open convex subset of a Banach space is Gâteaux (or even Fréchet) differentiable on a dense G_δ subset of its domain, provided the existence of a dual norm which enjoys some nice geometrical properties.

Unfortunately, well-known examples show that such a property no longer holds for an arbitrary locally Lipschitzian real-valued function. Therefore, many authors have sought a generalization of the classical Rademacher theorem involving a new concept of theoric "measure-zero" set [1], [6], [13].

However, a lot of work can be done to extend the generic differentiability properties to a large class of real-valued functions. The main contribution of this note is to give, at least in "small" spaces (here it means separable spaces), a characterization of these funcions, expressed both in terms of directional derivatives and tangent cones.

The first section deals with the notion of a generalized gradient introduced by F. H. Clarke, [7], [8], from whom I have benefitted greatly. Theorem (1.7) is the cornerstone to our development and allows us to dispense with convexity assumptions in the proofs of Theorems (2.1) and (2.2). In the next section, we discuss the relation between smoothness properties of locally

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Lipschitzian functions and their generalized gradient. Finally, the results of these two sections are used to characterize generically smooth functions in separable Baire spaces.

Although we are mainly concerned with the Banach space problem, we have tried to keep the statements clear from useless assumptions; for instance, most of the results are meaningful in any topological vector space, although some technicalities could be avoided by considering locally convex spaces.

1. The generalized gradient and the mean value theorem. Throughout the following, f denotes a real-valued function defined on an open subset Ω of a real topological vector space E (Ω may be the whole space). For given $x \in \Omega$ and $t \in]0, 1]$, $f_t(x)$ is the real-valued function defined on $t^{-1}(\Omega - x)$ by $f_t(x)(h) = t^{-1}[f(x + th) - f(x)]$.

Now, we can define, for every $h \in E$,

$$\sigma_f(x)(h) = \limsup_{\substack{y \to x \\ t \downarrow 0}} f_t(y)(h).$$

f is said to be *locally Lipschitzian* on Ω if, for every $a \in \Omega$, there exists a neighbourhood A of a in Ω, and some circled neighborhood U of 0 in E such that $\bigcup_{x \in A} \sigma_f(x)(U)$ is a bounded subset of the real line R. It will be an easy consequence of Theorem (1.7) that, if E is locally convex, this notion coincides with the classical notion of a locally Lipschitzian function; in particular, in any topological vector space, every continuous convex function defined on an open convex subset Ω is locally Lipschitzian on Ω (cf. Proposition (1.10)).

The following is immediate.

LEMMA (1.1) *For every $x \in \Omega$, the function $h \to \sigma_f(x)(h)$ is a sublinear functional on E.*

PROOF. Obviously, $\sigma_f(x)(th) = t\sigma_f(x)(h)$ for every positive real number t; furthermore, from the equality

$$f_t(y)(h + k) = f_t(y + th)(k) + f_t(y)(h)$$

we derive

$$\sigma_f(x)(h + k) < \limsup_{\substack{y \to x \\ t \downarrow 0}} f_t(y + th)(k) + \sigma_f(x)(h).$$

Thus, $\sigma_f(x)(h + k) < \sigma_f(x)(k) + \sigma_f(x)(h)$. Hence $\sigma_f(x)$ is sublinear. $\square$

PROPOSITION (1.2) *Every $a \in \Omega$ has a neighbourhood A in Ω such that $\{\sigma_f(x) | x \in A\}$ is a uniformly equicontinuous subset of sublinear functionals on E.*

PROOF. By assumption there exist a neighbourhood A of a in Ω and a circled neighbourhood U of 0 in E such that $\bigcup_{x\in A}\sigma_f(x)(U)$ is bounded in R. Let $M = \sup \bigcup_{x\in A} \sigma_f(x)(U)$; for every $\varepsilon > 0$ and $x \in A$, we have that

$$h - k \in \varepsilon M^{-1}U \Rightarrow \sigma_f(x)(h) - \sigma_f(x)(k) \leqslant \sigma_f(x)(h - k).$$

Hence

$$\sigma_f(x)(h) - \sigma_f(x)(k) \leqslant \varepsilon M^{-1} \sup \bigcup_{x\in A} \sigma_f(x)(U) = \varepsilon.$$

Similarly, $\sigma_f(x)(k) - \sigma_f(x)(h) \leqslant \varepsilon$; thus, for every $\varepsilon > 0$, there exists a neighbourhood $V = \varepsilon M^{-1}U$ of 0 in E such that, for $x \in A$,

$$h - k \in V \Rightarrow |\sigma_f(x)(h) - \sigma_f(x)(k)| \leqslant \varepsilon$$

which gives the expected result. □

The previous results lead to a very "natural" definition of the generalized gradient of a locally Lipschitzian function.

DEFINITION (1.3) Let f be a locally Lipschitzian function on $\Omega \subset E$, the *generalized gradient* of f at $x \in \Omega$ is the set $\partial f(x)$ of continuous linear functionals u on E which satisfy: $\forall h \in E$,

$$u(h) \leqslant \sigma_f(x)(h).$$

This defines a set-valued mapping $x \to \partial f(x)$ from Ω into the topological dual E^* of E. For fixed x in Ω, $\sigma_f(x)$ is namely the "support function" of $\partial f(x)$.

To show the consistency of this definition, we need to prove the nonvacuity of $\partial f(x)$ at every $x \in \Omega$. The following proposition gives, in fact, a somewhat deeper result which will be used in the proof of Theorem (1.7).

PROPOSITION (1.4) *For every $x \in \Omega$, $\bar{h} \in E$, and every real number s satisfying: $-\sigma_f(x)(-\bar{h}) \leqslant s \leqslant \sigma_f(x)(\bar{h})$, there exists $u \in \partial f(x)$ such that $u(\bar{h}) = s$. In particular, for every $x \in \Omega$, $\partial f(x) \neq \varnothing$.*

PROOF. By assumption, we define a continuous linear functional u on the closed subspace $M = \{x \in E | x = t\bar{h},\ t \in R\}$ of E by setting $u(\bar{h}) = s$; clearly u satisfies $u(h) \leqslant \sigma_f(x)(h)$ for every $h \in M$. Thus, it follows from Lemma (1.1) and a well-known generalization of the Hahn-Banach theorem [**15**, Chapter 2, Example 6, p. 69] that u has a continuous extension to E, still denoted by u, which satisfies $u(h) \leqslant \sigma_f(x)(h)$ for every $h \in E$; whence we derive $u \in \partial f(x)$. □

For finite dimensional spaces, F. H. Clarke gave the following definition of the generalized gradient of a locally Lipschitzian function [**7**, (1.1)]: Let f be a locally Lipschitzian function defined on an open subset Ω of R^n; by the well-known Rademacher theorem, f is derivable on the complementary subset of a null set in Ω (with respect to the Lebesgue measure on R^n restricted to Ω). If N is any such null set in Ω, we can define, for every $x \in \Omega$,

$$\partial f(x) = \overline{\mathrm{co}}\left\{ \lim_{y \to x} f'(y) | y \in \Omega \setminus N \right\}.$$

Namely $\partial f(x)$ is the convex closure of the "limit points" of the derivative $f'(y)$ of f at y when $f'(y)$ is defined and y converges to x in Ω. It can be shown [7, (1.4)] that this definition does not depend on the choice of the null set N and always coincides with our definition of the generalized gradient of f on Ω.

If f is a continuous convex function defined on an open convex subset, we shall see later that $\partial f(x)$ is exactly the subdifferential of f at x (cf. Proposition (1.10)).

The following proposition shows that, in the general case, the mapping $x \to \partial f(x)$ still behaves as a subdifferential.

PROPOSITION (1.5) *$x \to \partial f(x)$ is an upper-semi-continuous mapping from Ω into the w^*-compact convex subsets of E^* (endowed with the weak*-topology). Moreover, every $a \in \Omega$ has a neighbourhood A in Ω such that $\bigcup_{x \in A} \partial f(x)$ is contained in a w^*-compact convex subset of E (local compactness property).*

PROOF. If U is any circled neighbourhood of 0 in E and $h \in U$, we have, for every $x \in \Omega$, $-\sigma_f(x)(-h) \leqslant u(h) \leqslant \sigma_f(x)(h)$ hence $|u(h)| \in \sigma_f(x)(U)$. Thus it easily follows from Lemma (1.2) that every $a \in \Omega$ has a neighbourhood A in Ω such that $\bigcup_{x \in A} \partial f(x)$ is contained in an equicontinuous, hence w^*-compact, subset of E^*. Thus we have proved the local compactness property. Then, to prove the semi-continuity, it suffices to show that the set $\{(x, u) \in \Omega \times E^* | u \in \partial f(x)\} = G$ is closed in $\Omega \times E^*$ (E^* being endowed with the weak*-topology). To do this, take any generalized sequence $(x_i, u_i)_{i \in I}$ in G which converges to (x, u) in $\Omega \times E^*$; for every $i \in I$ and $h \in E$, we have $u_i(h) \leqslant \sigma_f(x_i)(h)$, hence $u(h) \leqslant \limsup_{i \in I} \sigma_f(x_i)(h) \leqslant \sigma_f(x)(h)$; since the function $x \to \sigma_f(x)(h)$ for fixed h in E is obviously upper-semi-continuous in Ω, and this shows that $u \in \partial f(x)$, thus $(u, x) \in G$.

To complete the proof it remains to show that $\partial f(x)$ is a closed convex subset of E for every $x \in \Omega$, but this follows obviously from Lemma (1.1) and Proposition (1.2). $\square$

We now pass to the statement of the main result of this section which is expressed in Theorem (1.7) below.

However, this requires a few preliminaries. Let $[a, b]$ (resp., $]a, b[$) denote the closed (resp., open) interval $\{x \in \Omega | x = sb + (1 - s)a,\ s \in [0, 1]$ (resp., $s \in]0, 1[)\}$ in Ω; let g be any continuous real-valued function defined on $[a, b]$, we say that c is a "*g-critical point*" for $[a, b]$ if and only if there exist sequences a_n and b_n in $[a, b]$ and $r_n > 0$ such that:

(i) $g(a_n) - g(b_n) = r_n[g(a) - g(b)]$, $a_n - b_n = r_n(a - b)$,

(ii) $a_n \to c$, $b_n \to c$, and $r_n \to 0$.

PROPOSITION (1.6) (EXISTENCE OF A CRITICAL POINT) *For every closed interval* $[a, b] \subset \Omega$ *and every continuous real-valued function* g *defined on* $[a, b]$, *there exists a "g-critical point"* $c \in]a, b[$.

PROOF. Since the set of "g-critical points" of $[a, b]$ is exactly the set of "$\bar{g}$-critical points" of $[a, b]$, where $\bar{g}$ is the continuous function defined on $[a, b]$ by setting: for $x \in [a, b]$,

$$\bar{g}(x) = g(x) - g(a) - \theta[g(b) - g(a)]$$

for $x = \theta b + (1 - \theta)a$ we can always assume that $g(a) = g(b) = 0$; then we have only to find a point c in $]a, b[$ and sequences a_n and b_n in $[a, b]$ such that $a_n \to c$, $b_n \to c$, $a_n \neq b_n$, and $g(a_n) = g(b_n)$. Let us consider the subset M of $[a, b]$ of the points at which the continuous function $|g|$ achieves its maximum on $[a, b]$. If M contains some nonempty subinterval of $]a, b[$, every interior point c of $M \cap]a, b[$ is a "g-critical point" for $[a, b]$. Namely, $|g|$, and therefore g, is constant on some neighbourhood of c in $]a, b[$ and it suffices to take sequences a_n and b_n in this neighbourhood which converge to c. Otherwise take any $c \in M \cap]a, b[$, and let $(V_n)_{n \in N}$ be a countable basis of neighbourhoods of c in $]a, b[$; since M does not contain a nonempty open subinterval of $]a, b[$, there exist necessarily a_n and b_n in V_n such that $a_n < c < b_n$ and $|g(a_n)| = |g(b_n)| < |g(c)|$. Finally, since $|g(c)| > 0$, $|g|$ coincides with g or $-g$ on some neighbourhood of c in $]a, b[$, and, therefore, $g(a_n) = g(b_n)$ for n sufficiently large. This concludes the proof. □

We are now ready to prove

THEOREM (1.7) (MEAN VALUE THEOREM) *For every locally Lipschitzian real-valued function* f *defined on an open subset* Ω *of* E *which contains* $[a, b]$, *there exist* $c \in]a, b[$ *and* $u \in \partial f(c)$ *such that* $f(a) - f(b) = u(a - b)$.

PROOF. Let $c \in]a, b[$ be an "f-critical point" for $[a, b]$. There exist sequences a_n and b_n in $[a, b]$ and t_n in $R^+ \setminus \{0\}$ such that

$$t_n^{-1}[f(a_n) - f(b_n)] = f(a) - f(b),$$

$$a_n - b_n = t_n(a - b), \qquad a_n \to c, \qquad b_n \to c, \qquad t_n \to 0.$$

Then

$$\begin{aligned} f(a) - f(b) &= \lim_n t_n^{-1}[f(a_n) - f(b_n)] \\ &= \lim_n t_n^{-1}[f(b_n + t_n(a - b)) - f(b_n)] \\ &\leq \limsup_{\substack{y \to c \\ t \downarrow 0}} t^{-1}[f(y + t(a - b)) - f(y)] = \sigma_f(c)(a - b). \end{aligned}$$

A similar computation leads to $f(b) - f(a) \leq \sigma_f(c)(b - a)$. The theorem is now a straightforward consequence of Proposition (1.4). □

We shall now prove the minimality of the previous construction with

respect to a definite group of properties (cf. Theorem (1.8) below).

Let M be a mapping from Ω into the subsets of E^*; M is said *to have the mean value property with respect to* f if and only if, for every closed interval $[a, b] \subset \Omega$, there exist $c \in]a, b[$ and $u \in M(c)$ such that $f(a) - f(b) = u(a - b)$.

Now, for a given couple (M, M') of two such mappings, let us say that M is smaller than M' if, for every $x \in \Omega$, $M(x) \subset M'(x)$; then, the generalized gradient of f appears as the smallest element of this ordering of mappings among all those which have the mean value property with respect to f:

THEOREM (1.8) *The generalized gradient is the "smallest" upper-semi-continuous mapping from Ω into the w^*-compact convex subsets of E^* (endowed with the weak*-topology) which has the mean value property with respect to f.*

PROOF. Suppose that M is a suitable mapping from Ω into the subsets of E^*. For every $a \in \Omega$ and $h \in E$, there exists $t \in]0, 1]$ such that $[a, a + th] \subset \Omega$. Then $A = \{x \in \Omega | [x, x + th] \subset \Omega\}$ is a neighbourhood of a in Ω. Now, for $s \in]0, 1]$, and $x \in A$, $f_s(x)(h) = u(h)$ where $u \in M(y)$ for some $y \in]x, x + sh[$ (apply the mean value property of M on the closed interval $[x, x + sh] \subset [x, x + th] \subset \Omega$). Thus,

$$\sigma_f(a)(h) \leqslant \limsup_{y \to a} \sup_{u \in M(y)} u(h).$$

But from the semicontinuity of M, this implies

$$\sigma_f(a)(h) \leqslant \sup_{u \in M(a)} u(h)$$

and, by the Hahn-Banach theorem, $\partial f(a) \subset M(a)$.

A trivial consequence of Theorem (1.8) is that, for every locally Lipschitzian function f continuously Gâteaux-differentiable on Ω (the continuity being taken with respect to the weak*-topology on E^*), the generalized gradient of f at every $x \in \Omega$ is reduced to a singleton, namely the Gâteaux-derivative $f'(x)$. Conversely, if $\partial f(x)$ is a singleton for every $x \in \Omega$, we shall see in the next section (cf. Proposition (2.1)) that f is continuously Gâteaux-differentiable on Ω. However, f may be Gâteaux-differentiable at some $x \in \Omega$ and $\partial f(x)$ not reduced to a singleton; furthermore, the following counterexample which is due to F. H. Clarke shows that even if E is taken to be the real line R, there exist locally-Lipschitzian real-valued functions, the generalized gradient of which is never reduced to a singleton, although, by Rademacher's theorem, such a function is necessarily differentiable on a dense subset of R.

PROPOSITION (1.9) *There exists a strictly increasing locally Lipschitzian real-valued function f defined on R such that $\partial f(x) = [0, 1]$ for every $x \in R$.*

We shall base our example on the existence of a measurable subset M of the real line R (equipped with the Lebesgue measure m) which intersects every nonempty open interval $I \subset R$ in a set of positive measure $0 < m(M \cap I)$ (see for instance [11]). Now let f be the indicator function of M and g the continuous function defined on R by

$$g(x) = \int_0^x f(t)\, dt.$$

Clearly g is well-defined, strictly increasing, and Lipschitzian since, for every $x < x'$, we have

$$g(x') - g(x) = \int_x^{x'} f(t)\, dt = x' - x - m([x, x'] \cap R \setminus M) < x' - x$$

(where m denotes the Lebesgue measure on R).

Since f is measurable, g admits almost everywhere a finite derivative which coincides almost everywhere with f [5, §6, Example 14, p. 122]. Thus the derivative of g is almost everywhere 0 or 1 and each value is achieved on a dense subset of R. Therefore, by a result of F. H. Clarke already mentioned in this section, the generalized gradient of g is always the closed interval $[0, 1]$. □

Finally, we end this section by giving the proof of a result concerning continuous convex functions that has been previously claimed.

PROPOSITION (1.10) *Let f be any continuous convex function defined on some convex open subset Ω of E; for every $x \in \Omega$, the generalized gradient $\partial f(x)$ of f at x is exactly the set of the subgradients of f at x.*

PROOF. We first show that f is locally Lipschitzian in the sense defined at the beginning of this section. To do this, take any $a \in \Omega$; since f is continuous at a, there exist a neighbourhood A of a in Ω, a circled neighbourhood U of 0 in E and a positive constant M in R such that $x \in A$, $h \in U$, $f(x + h) \in [-M, M]$.

Now, using the convexity of f, we compute for every $t \in]0, 1]$:

$$f(x + th) - f(x) = f(t(x + h) + (1 - t)x) - f(x) \leqslant t[f(x + h) - f(x)],$$

$$\begin{aligned} f(x) - f(x + th) &= f(t(x - h + th) + (1 - t)(x + th)) - f(x + th) \\ &\leqslant t[f(x - h + th) - f(x + th)]. \end{aligned}$$

Hence, for $x \in A$, $h \in U$, $t \in]0, 1]$, we have

$$-2M \leqslant f(x - h + th) - f(x + th) \leqslant f_t(x)(h) \leqslant f(x + h) - f(x) \leqslant 2M$$

and, therefore, for every $x \in A$, $\sigma_f(x)(U) \subset [-2M, 2M]$. So we have shown that, for every $a \in \Omega$, there exist a neighbourhood A of a in Ω and a circled neighbourhood U of 0 in E such that $\bigcup_{x \in A} \sigma_f(x)(U)$ is a bounded subset of R, and hence the desired result. Now let $x \in \Omega$ and u be a subgradient of f at

x; for every $t \in]0, 1]$, and $h \in t^{-1}(\Omega - x)$, $u(h) \leqslant f_t(x)(h)$, whence

$$u(h) \leqslant \limsup_{t \downarrow 0} f_t(x)(h) \leqslant \limsup_{\substack{y \to x \\ t \downarrow 0}} f_t(y)(h) = \sigma_f(x)(h).$$

Therefore, $u \in \partial f(x)$. This shows that the set of the subgradients of f at x is contained in $\partial f(x)$. Conversely, the set-valued mapping which maps every $x \in \Omega$ into the set of the subgradients of f at x is an upper-semi-continuous w^*-compact convex valued mapping from Ω into E (endowed with the weak*-topology) and has the mean value property with respect to f. The theorem is thus a consequence of Theorem (1.8).

2. Generic differentiability of locally Lipschitzian functions and their generalized gradient. Let f be a real-valued function defined on an open subset Ω of a topological vector space E and $\mathcal{S}$ any family of subsets of E; we say that f is "$\mathcal{S}$-differentiable" at some $x \in \Omega$ if and only if $f_t(x)(h)$ converges to $u(h)$ for some continuous linear functional u on E, uniformly in h on every element S of $\mathcal{S}$. When this occurs, u is the $\mathcal{S}$-derivative of f at x and denoted by $f'(x)$. Among all the involved notions of a derivative, we are mainly concerned with the weakest and the strongest ones:

(i) If $\mathcal{S}$ is the family of all finite subsets of E, f is "*Gâteaux-differentiable*" at x.

(ii) When E is a normed space and $\mathcal{S}$ only contains the unit ball of E, the notion reduces to the well-known "*Fréchet-differentiability*".

Given a family $\mathcal{S}$ of bounded subsets in E, we shall consider on the topological dual E^* of E the $\mathcal{S}$-*topology* of the uniform convergence on the subsets $S \in \mathcal{S}$. If $\bigcup_{S \in \mathcal{S}} S$ is total in E, E^* endowed with the $\mathcal{S}$-topology is a locally convex vector space.

PROPOSITION (2.1) *Let f be any locally Lipschitzian function defined on an open subset Ω of E; if $\partial f(a)$ is reduced to a singleton for some $a \in \Omega$ and the mapping $x \to \partial f(x)$ from Ω into E^* continuous for the $\mathcal{S}$-topology on E^*, then f is $\mathcal{S}$-differentiable at a and $\partial f(a) = \{f'(a)\}$.*

PROOF. For every $t \in]0, 1]$ and $h \in E$ such that $[a, a + th] \subset \Omega$, there exist $x \in]a, a + th[$ and $v \in \partial f(x)$ such that $f_t(a)(h) = v(h)$ (cf. Theorem (1.7)). Now, if $\partial f(a) = \{u\}$, we derive from the continuity of the generalized gradient at a, that, for every $S \in \mathcal{S}$ and $\varepsilon > 0$, there exists a neighbourhood A of a in Ω, such that, for all $x \in A$, $v \in \partial f(x)$,

$$\sup_{h \in S} |(v - u)(h)| < \varepsilon.$$

Thus, for t small enough, this yields

$$\sup_{h \in S} |f_t(a)(h) - u(h)| < \varepsilon$$

which shows that u is the $\mathcal{S}$-derivative of f at a. $\square$

We have already noticed in the previous section that the converse of Proposition (2.1) is not true since f may be $\mathcal{S}$-differentiable at some $a \in \Omega$ whereas ∂f is not even reduced to a singleton. However, as expressed in Theorem (2.2), such a converse holds at least "almost everywhere"; here, "almost everywhere" means on a dense G_δ subset of Ω. More precisely, we shall make use, throughout the following, of the terms "generic" and "generically" to refer to properties, depending on the points of an open subset Ω of a topological vector space which is a Baire space, which hold on a dense G_δ subset of Ω.

THEOREM (2.2) *Let E be a topological vector space which is a Baire space and $\mathcal{S}$ a countable family of bounded subsets of E such that $\bigcup_{S \in \mathcal{S}} S$ is total in E. A locally Lipschitzian function defined on an open subset $\Omega \subset E$ is generically $\mathcal{S}$-differentiable on Ω if and only if for generically every point $x \in \Omega$, $\partial f(x)$ is reduced to a singleton and ∂f is continuous at x (for the $\mathcal{S}$-topology on E^*).*

The "if" part follows from Proposition (2.1). The "only if" will be deduced from a series of lemmas; recall that, for given x and $t \in]0, 1]$, $f_t(x)$ is the real-valued function defined on $t^{-1}(\Omega - x)$ by $f_t(x)(h) = t^{-1}[f(x + th) - f(x)]$.

PROPOSITION (2.3) *For every nonempty open subset A of Ω and every $S \in \mathcal{S}$, there exist a nonempty open subset $B \subset A$ and $s > 0$ in R such that, for every $t \in]0, s[$, $\{x \to f_t(x)(h) | h \in S\}$ is an equicontinuous set of continuous functions on B.*

PROOF. Since f is locally Lipschitzian on Ω, we can always assume that there exists a circled neighbourhood U of 0 in E such that $\bigcup_{x \in A} \sigma_f(x)(U)$ is bounded in R (even by considering such an open subset contained in A instead of A); let $M = \sup \bigcup_{x \in A} \sigma_f(x)(U)$.

Since S is bounded in E, there exist a nonempty open subset $B \subset A$ and $s > 0$ in R such that $B + [0, s[S \subset A$. Let b be any point in B, then $C = \{b' \in B | [b, b'] \subset B\}$ is a neighbourhood of b in B. For every $b' \in C$, $t \in [0, s]$, and $h \in S$, we have

$$[b + th, b' + th] \subset [b, b'] + th \subset B + [0, s[S \subset A.$$

Now, for fixed $t \in]0, s[$, we have, for every $b' \in C$,

$$f_t(b)(h) - f_t(b')(h) = t^{-1}[f(b + th) - f(b' + th)] + t^{-1}[f(b') - f(b)].$$

By applying twice the mean value theorem (cf. (1.7)) on the closed intervals $[b + th, b' + th]$ and $[b, b']$ contained in A, we obtain

$$|f_t(b)(h) - f_t(b')(h)| \leq t^{-1}|u(b - b')| + t^{-1}|v(b - b')|$$

where u (resp., v) belongs to the generalized gradient of f at some point of the closed interval $[b + th, b' + th]$ (resp., $[b, b']$).

Finally, if moreover $b' \in b + (\varepsilon t/2M)U$, we derive

$$|f_t(b)(h) - f_t(b')(h)| < 2t^{-1} \cdot \frac{\varepsilon t}{2M} \sup \bigcup_{x \in A} \sigma_f(x)(U) = \varepsilon.$$

Thus, for every $t \in]0, s[$, $b \in B$, and $\varepsilon > 0$, there exists a neighbourhood $D = C \cap (b + (\varepsilon t/2M)U)$ of b in B such that

$$b' \in D, h \in S \Rightarrow |f_t(b)(h) - f_t(b')(h)| < \varepsilon$$

which gives the expected result. □

Let us now introduce the set $\operatorname{dom} f'$ of the points x where the $\mathcal{S}$-derivative $f'(x)$ of f at x is well-defined; by assumption, $\operatorname{dom} f'$ contains a dense G_δ subset of Ω, hence it is a Baire space in the induced topology.

PROPOSITION (2.4) *For every nonempty open subset A of Ω, every $S \in \mathcal{S}$ and every $\varepsilon > 0$, there exist a nonempty open subset $C \subset A$ and $r > 0$ in R such that, for every $x \in \operatorname{dom} f' \cap C$, $h \in S$, and every $t \in]0, r[$,*

$$\sup_{h \in S} |f'(x)(h) - f_t(x)(h)| < \varepsilon.$$

PROOF. By Proposition (2.3), there exist a nonempty open subset $B \subset A$ and $s > 0$ in R such that, for every $t \in]0, s[$ and $h \in S$, the function $x \to f_t(x)(h)$ is continuous on B. Therefore the set

$$F_n = \left\{ x \in B | t, t' \in]0, 2^{-n}s[\Rightarrow \sup_{h \in S} |f_{t'}(x)(h) - f_t(x)(h)| < \varepsilon \right\}$$

is closed in B (for the induced topology). Since, by assumption, $f_t(x)(h)$ converges uniformly in h on S to $f'(x)(h)$ when t converges to 0 in R at every point x of $\operatorname{dom} f'$, then

$$\operatorname{dom} f' \cap B = \left(\bigcup_{n \in N} F_n \right) \cap \operatorname{dom} f' = \bigcup_{n \in N} (F_n \cap \operatorname{dom} f').$$

Finally, since $\operatorname{dom} f' \cap B$ is a Baire space in the induced topology [4, cf. 5, $n°$ 3, Propositions 3 and 5] and $F_n \cap \operatorname{dom} f'$ is closed in $\operatorname{dom} f'$ for every $n \in N$, we derive from the Baire category theorem that there exists a nonempty open subset $C \subset B$, which is contained in one of the F_n. Thus there exists $r > 0$ in R such that, for every $x \in C \cap \operatorname{dom} f'$, we have

$$t, t' \in]0, r[\Rightarrow \sup_{h \in S} |f_{t'}(x)(h) - f_t(x)(h)| < \varepsilon.$$

Letting t' go to zero, this yields: for every $x \in C \cap \operatorname{dom} f'$, $t \in]0, r[$,

$$\sup_{h \in S} |f'(x)(h) - f_t(x)(h)| < \varepsilon$$

which gives the desired result. □

PROPOSITION (2.5) *For every* $\varepsilon > 0$, *and* $S \in \mathcal{S}$, *let* $G(\varepsilon, S)$ *be the set of the points* $a \in \operatorname{dom} f'$ *having the following property*:

there exist a neighbourhood A *of* a *in* Ω *and* $s > 0$ *in* R *such that, for every* $x \in A$ *and* $t \in]0, s[$,

$$\sup_{h \in S} |f'(a)(h) - f_t(x)(h)| < 4\varepsilon.$$

Then $G(\varepsilon, S)$ *contains a dense open subset of* $\operatorname{dom} f'$ (*for the induced topology*).

PROOF. It suffices to show that, for every nonempty open subset A of Ω, $G(\varepsilon, S) \cap A$ contains a nonempty open subset of $\operatorname{dom} f'$. So let A be any nonempty open subset of Ω and take C and $r > 0$ defined as in Proposition (2.4). Then, applying Proposition (2.3) to the nonempty open subset $C \subset \Omega$, there exist a nonempty subset $B \subset C$ and $s > 0$ in R such that, for every $t \in]0, s[$, $\{x \to f_t(x)(h) | h \in S\}$ is an equicontinuous set of continuous functions on B. We claim that $B \cap \operatorname{dom} f' \subset G(\varepsilon, S)$; to prove the claim, take any $a \in B \cap \operatorname{dom} f'$ and $t \in]0, \min(r, s)[$. From the choice of B and s, it follows that, for every fixed t in R which satisfies $0 < t < \min(r, s)$, there exists a neighbourhood D of a in B such that

$$h \in S, x \in D \Rightarrow |f_t(x)(h) - f_t(a)(h)| < \varepsilon.$$

Thus, for $x \in D \cap \operatorname{dom} f'$, and $h \in S$, we get

$$\begin{aligned} |f'(x)(h) - f'(a)(h)| &< |f'(x)(h) - f_t(x)(h)| + |f_t(x)(h) - f_t(a)(h)| \\ &\quad + |f_t(a)(h) - f'(a)(h)| \\ &< |f'(x)(h) - f_t(x)(h)| + \varepsilon + |f_t(a)(h) - f'(a)(h)| \end{aligned}$$

and finally, it follows from the choice of C and r that

$$|f'(x)(h) - f'(a)(h)| < 3\varepsilon,$$

for every $x \in \operatorname{dom} f' \cap D$, and $h \in S$.

Now, using again the definition of C and r (cf. Proposition (2.4)) we compute

$$\begin{aligned} |f'(a)(h) - f_t(x)(h)| &< |f'(a)(h) - f'(x)(h)| + |f'(x)(h) - f_t(x)(h)| \\ &< 3\varepsilon + \varepsilon = 4\varepsilon. \end{aligned}$$

Since $\operatorname{dom} f'$ is dense in D and the function $x \to f_t(x)(h)$ is continuous on D for fixed $t > 0$ and $h \in S$, we can easily extend the previous inequality to

$$x \in D, h \in S, t \in]0, r[\Rightarrow |f'(a)(h) - f_t(x)(h)| < 4\varepsilon.$$

Therefore, we have shown that $B \cap \operatorname{dom} f' \subset G(\varepsilon, S)$, and hence the desired result. □

PROPOSITION (2.6) *For generically every point* $x \in \Omega$, $\partial f(x)$ *is reduced to a singleton, namely the* $\mathcal{S}$*-derivative of* f *at* x, *and* ∂f *is continuous at* x (*for the* $\mathcal{S}$*-topology on* E^*).

PROOF. Since, by assumption, E is a Baire space, $\operatorname{dom} f'$ is still a Baire space and we deduce from the Baire category theorem and the countability assumption on $\mathcal{S}$ that

$$G = \bigcap_{\substack{S \in \mathcal{S} \\ \varepsilon > 0}} G(\varepsilon, S) = \bigcap_{\substack{S \in \mathcal{S} \\ n \in N}} G(1/n, S)$$

contains a dense G_δ subset of $\operatorname{dom} f'$, hence a dense G_δ subset of Ω (where $G(\varepsilon, S)$ is defined as in (2.5)). Now, take any $a \in G$; since $G \subset \operatorname{dom} f'$, $f'(a)$ is well-defined and we claim that $\partial f(a) = \{f'(a)\}$ and the multi-application $x \to \partial f(x)$ continuous at a for the $\mathcal{S}$-topology on E^*. To prove the claim it suffices to prove that for every $S \in \mathcal{S}$ and every $\varepsilon > 0$ there exists a neighbourhood A of a in Ω such that

$$x \in A, u \in \partial f(x) \Rightarrow \sup_{h \in S} |u(h) - f'(a)(h)| < \varepsilon.$$

Since $a \in G \subset G(\varepsilon, S)$, there exist a neighbourhood A of a in Ω and $s > 0$ in R such that, for every $x \in A$ and $t \in]0, s[$,

$$\sup_{h \in S} |f'(a)(h) - f_t(x)(h)| < \varepsilon.$$

Since S is bounded in E, there exist a neighbourhood A' of a in A and $s' > 0$ in R such that $A' -]0, s'[S \subset A$. Now take $x \in A'$, $h \in S$, and $t \in]0, \min(s, s')[$: obviously

$$f_t(x)(h) < f'(a)(h) + \varepsilon.$$

But, since $x - th$ still belongs to A,

$$f_t(x)(-h) = -f_t(x - th)(h) < -f'(a)(h) + \varepsilon.$$

Finally, for every $x \in A'$, $h \in S \cup -S$, and $t \in]0, \min(s, s')[$,

$$f_t(x)(h) < f'(a)(h) + \varepsilon$$

and therefore $\sigma_f(x)(h) < f'(a)(h) + \varepsilon$. Thus $u \in \partial f(x)$ for some $x \in A'$ implies

$$\sup_{h \in S \cup -S} [u(h) - f'(a)(h)] = \sup_{h \in S} |u(h) - f'(a)(h)| < \varepsilon. \quad \square$$

From Theorem (2.2) we easily derive the two following fundamental corollaries.

COROLLARY (2.7) *Let E be a separable topological vector space which is a Baire space and Ω an open subset of E; a locally Lipschitzian function defined on Ω is generically Gâteaux-differentiable on Ω if and only if its generalized gradient is generically reduced to a singleton.*

PROOF. Take $\mathcal{S}$ to be the family of the subsets $\{h_n\}$ where $\{h_n | n \in N\}$ is dense in E. If f is generically Gâteaux-differentiable on Ω, then it is generically $\mathcal{S}$-differentiable on Ω and, by Theorem (2.2), its generalized gradient is

generically reduced to a singleton; conversely, if the generalized gradient of f is generically reduced to a singleton on Ω, then by Proposition (1.5) it is generically continuous on Ω for the weak*-topology on E^*, thus, by Proposition (2.1), f is generically Gâteaux-differentiable on Ω. □

COROLLARY (2.8) *Let E be any Banach space and Ω an open subset of E; a locally Lipschitzian function defined on Ω is generically Fréchet-differentiable on Ω if and only if for generically every point $x \in \Omega$, $\partial f(x)$ is reduced to a singleton and ∂f is continuous at x (for the dual norm topology on E^*).*

PROOF. Take $\mathcal{S}$ to be the family which reduces to the only unit ball of E and apply Theorem (2.2). □

The previous results lead to the following definition.

DEFINITION (2.9) A real-valued function f defined on an open subset Ω of a topological vector space E is said to be "smooth" at a if it is locally Lipschitzian on some neighbourhood of a in Ω and $\partial f(a)$ reduces to a singleton.

In separable Baire spaces, a locally Lipschitzian function is generically smooth on Ω if and only if it is generically Gâteaux-differentiable on Ω (cf. Corollary (2.7)). It is not known if the equivalence is still valid when the separability assumption is dropped.

The main interest of this notion is that generic differentiability properties of generically smooth functions seem to depend only on the topological regularity of their generalized gradient; this is true at least under some nice countability assumptions on the involved topologies (cf. Theorem (2.2)). Supported by the analogy with continuous convex functions **[14]**, one might expect that these generic differentiability properties of generically smooth functions depend ultimately on the structural properties of the spaces on which they are defined.

An immediate illustration is the following proposition.

PROPOSITION (2.10) *Let f be a locally Lipschitzian real-valued function defined on some open subset Ω of a topological vector space E which is a Baire space. If f is generically smooth on Ω, then f is generically $\mathcal{S}$-differentiable on Ω, where $\mathcal{S}$ is the family of all precompact subsets of E.*

PROOF. By Proposition (1.5), the generalized gradient of f is locally contained in some w^*-compact convex subset of E^* on which the $\mathcal{S}$-topology coincides with the weak*-topology **[15**, Chapter III, §4, Proposition 4.5, p. 85]. The result is thus a consequence of Theorem (2.1). □

A much deeper result would be analogous to Proposition (2.10) when E is a Banach space and $\mathcal{S}$ contains only the unit ball of E. Unfortunately, even in the easiest case of a separable Banach space with separable dual where such a

result holds for continuous convex functions, we have been unsuccessful in trying to get a similar statement.

3. Generic smoothness in separable Baire spaces. The natural question which arises is then: what are the generically smooth functions defined on some open subset Ω of a topological vector space E? In the separable case, a satisfying answer is found in Theorem (3.8) below. Unfortunately, the methods used greatly depend on the separability assumption, and no extension can be carried out to the nonseparable case (the problem of such an extension is largely open).

We begin by recalling the two classical notions of directional derivatives and tangent cones. Let f be a real-valued function defined on some open subset Ω in a topological vector space E:

$$\overline{df}(x)(h) = \limsup_{t \downarrow 0} f_t(x)(h),$$

$$\underline{df}(x)(h) = \liminf_{t \downarrow 0} f_t(x)(h),$$

always exist but may be different and even infinite. When both are finite and $\underline{df}(x)(h) = \overline{df}(x)(h)$, we say that *$f$ has a directional derivative in the direction h.* If this occurs for all $h \in E$, the real-valued mapping $h \to \underline{df}(x)(h) = \overline{df}(x)(h)$ is merely denoted by $df(x)$.

Let P be any subset of a topological vector space F; the "inner tangent cone" to P at some point $z \in F$ is the set $I_p(z)$ of the points $z' \in F$ such that there exist some $\varepsilon > 0$ and some neighbourhood V of z' in F satisfying $z +]0, \varepsilon[V \subset P$. Similarly, the set $C_P(z)$ of the points $z' \in F$ such that, for every $\varepsilon > 0$, and every neighbourhood V of z' in F, $z +]0, \varepsilon[V \cap P \neq \varnothing$ is called the "outer tangent cone" to P at z.

Now take F to be the product space $E \times R$ and

$$P = \operatorname{epi} f = \{(x, \mu) \in E \times R \mid f(x) < \mu\}$$

(the epigraph of f on Ω). Epi f can be geometrically viewed as the part of the graph which is above the graph (and contains the graph) of f.

The following are immediate.

PROPOSITION (3.1) *Let f be a locally Lipschitzian function defined on an open subset $\Omega \subset E$; for every $x \in \Omega$ and $h \in E$, we have*

(a) $\overline{df}(x)(h) = \limsup_{\substack{k \to h; \\ t \downarrow 0}} f_t(x)(k)$ *and* $\underline{df}(x)(h) = \liminf_{\substack{k \to h; \\ t \downarrow 0}} f_t(x)(k)$,

(b) *the mappings $h \to \overline{df}(x)(h)$ and $h \to \underline{df}(x)(h)$ are continuous on E.*

PROOF. Take any $a \in \Omega$; by assumption, there exist a neighbourhood A of a in Ω and a circled neighbourhood U of 0 in E such that $\bigcup_{x \in A} \sigma_f(x)(U)$ is bounded in R; let $M = \sup \bigcup_{x \in A} \sigma_f(x)(U)$ and V a circled neighbourhood

of 0 in E such that $a + V + V \subset A$. For given $h \in E$, choose $t \in]0, 1[$ such that $]0, t[h \subset V$; then, for every $s \in]0, t[$ and $k \in h + (V \cap \varepsilon M^{-1} \cdot U) = W$,

$$[a + sh, a + sk] \subset a + sh +]0, s[(k - h) \subset a +]0, t[h +]0, 1[(k - h) \subset a + V + V \subset A.$$

Thus we can apply Theorem (1.7) on the closed interval $[a + sh, a + sk]$:

$$f_s(a)(k) - f_s(a)(h) = s^{-1}[f(a + sk) - f(a + sh)] = u(k - h)$$

where u belongs to the generalized gradient of f at some point of the interval $[a + sh, a + sk]$. Finally, for every $s \in]0, t[$ and $k \in W$,

$$|f_s(a)(k) - f_s(a)(h)| < \varepsilon M^{-1} \sup \bigcup_{x \in A} \sigma_f(x)(U) = \varepsilon;$$

therefore: $f_s(a)(k) - \varepsilon < f_s(a)(h) < f_s(a)(k) + \varepsilon$. Taking respectively the lim sup and lim inf of each side of these inequalities when $k \to h$ and $s \to 0$, we easily derive (a). Similarly, taking the lim sup and lim inf when $s \to 0$ only, this yields, for $h \in W$,

$$\overline{df}(a)(k) - \varepsilon < \overline{df}(a)(h) < \overline{df}(a)(k) + \varepsilon,$$
$$\underline{df}(a)(k) - \varepsilon < \underline{df}(a)(h) < \underline{df}(a)(k) + \varepsilon,$$

which gives the continuity of the mappings $h \to \overline{df}(x)(h)$ and $h \to \underline{df}(x)(h)$. □

PROPOSITION (3.2) *Let f be any locally Lipschitzian function defined on an open subset $\Omega \subset E$; for every $x \in \Omega$,*

$$C_{\operatorname{epi} f}(x, f(x)) = \{(h, \mu) \in E \in R \mid \underline{df}(x)(h) < \mu\},$$
$$I_{\operatorname{epi} f}(x, f(x)) = \{(h, \mu) \in E \times R \mid \overline{df}(x)(h) < \mu\}.$$

PROOF. The following are equivalent:

(i) $(h, \mu) \notin C_{\operatorname{epi} f}(x, f(x))$,

(ii) there exist $t > 0$, $\eta > 0$ in R, and a neighbourhood V of h in E, such that

$$\{(x, f(x)) +]0, t[(V \times]\mu - \eta, \mu + \eta[)\} \cap \operatorname{epi} f = \varnothing$$

or, equivalently, $s \in]0, t[$, $k \in V \Rightarrow f(x + sk) > f(x) + s(\mu + \eta)$:

(i) there exists $\eta > 0$ such that

$$\liminf_{\substack{k \to h \\ s \downarrow 0}} f_s(x)(k) > \mu + \eta,$$

(ii) $\underline{df}(x)(h) > \mu$ (using Proposition (3.1)(a)),

and,

(iii) $(h, \mu) \in I_{\operatorname{epi} f}(x, f(x))$,

(iv) there exists $t > 0$, $\eta > 0$ in R, and a neighbourhood V of h in E such that: $(x, f(x)) +]0, t[(V \times]\mu - \eta, \mu + \eta[) \subset \operatorname{epi} f$ or, equivalently, $s \in]0, t[$, $k \in V \Rightarrow f(x + sk) < f(x) + s(\mu - \eta)$:

(v) there exists $\eta > 0$ such that $\limsup_{k \to h; s \downarrow 0} f_s(x)(k) < \mu - \eta$,
(vi) $\overline{df}(x)(h) < \mu$ (using Proposition (3.1)(a)). □

PROPOSITION (3.3) *If f is a locally Lipschitzian function defined on an open subset $\Omega \subset E$, for every $x \in \Omega$, $I_{\mathrm{epi}\, f}(x, f(x))$ is an open cone in $E \times R$, contained in the interior of the closed cone $C_{\mathrm{epi}\, f}(x, f(x))$.*

This is easily derived from Propositions (3.1) and (3.2).

The following result is much deeper.

THEOREM (3.4) *Let E be a separable topological vector space which is a Baire space and f a locally Lipschitzian function defined on some open subset $\Omega \subset E$; all the following properties are generic on Ω:*

(a) *$h \to \overline{df}(x)(h)$ (resp., $h \to \underline{df}(x)(h)$) is a continuous convex (resp., concave) function on E;*

(b) *for every $h \in E$, $\overline{df}(x)(h) = -\underline{df}(x)(-h)$;*

(c) *the inner tangent cone to the epigraph of f on Ω at the point $(x, f(x))$ is a nonempty convex open cone;*

(d) *$I_{\mathrm{epi}\, f}(x, f(x)) = \{(h, \mu) \in E \times R \mid (-h, -\mu) \in E \times R \setminus C_{\mathrm{epi}\, f}(x, f(x))\}$.*

PROOF. We shall deduce this theorem from a series of technical lemmas:

LEMMA (3.5) *For every $h \in E$, there exists a dense G_δ subset G of Ω such that the restriction of the mapping $x \to \overline{df}(x)(h)$ to G is continuous.*

PROOF. For every real number μ,

$$\{x \in \Omega \mid \overline{df}(x)(h) < \mu\} = \bigcup_{n \in N} \left\{x \in \Omega \mid \sup_{0<t<1/n} f_t(x)(h) < \mu - 1/n\right\}$$

is the countable union of closed subsets in Ω since, for fixed $t \in]0, 1[$ and $h \in E$, the mapping $x \to f_t(x)(h)$ is continuous on Ω; thus the lemma follows from [**16**, §8, Theorem 8.1].

LEMMA (3.6) *For every $h \in E$, we have generically $\sigma_f(x)(h) = \overline{df}(x)(h)$.*

PROOF. For given h in E, let G be defined as in Lemma (3.5); we claim that the set $G(\varepsilon)$ of the points $a \in \Omega$ having the following property:

> "there exist a neighbourhood A of a in Ω and $s > 0$ such that
> $x \in A \cap G$, $0 < t < s$, $f_t(x)(h) < \overline{df}(x)(h) + \varepsilon$"

contains a dense open subset of Ω. To prove the claim, it suffices to show that every nonempty open subset U of Ω contains a nonempty open subset V contained in $G(\varepsilon)$. Let U be such a nonempty open subset of Ω; since the restriction of the mapping $x \to \overline{df}(x)(h)$ to G is continuous, $U \cap G$ can be represented as a countable union of closed sets (with respect to the induced topology on G):

$$U \cap G = \bigcup_{n \in N} \{x \in U \cap G | t < 1/n \Rightarrow f_t(x)(h) < \overline{df}(x)(h) + \varepsilon\}.$$

Since E is a Baire space, $U \cap G$ is still a Baire space in the induced topology [4, cf. 5, $n°$ 3, Propositions 3 and 5] and we derive from the Baire category theorem that there exist an open subset V in U and $n \in N$ such that

$$V \cap G = \{x \in V \cap G | t < 1/n \Rightarrow f_t(x)(h) < \overline{df}(x)(h) + \varepsilon\}.$$

Therefore $V \subset G(\varepsilon)$, which ends the proof.

LEMMA (3.7) *For every $x \in \Omega$ and $h \in E$, $\sigma_f(x)(h) = \sigma_{-f}(x)(-h)$.*

PROOF.

$$\begin{aligned}\sigma_{-f}(x)(-h) &= \limsup_{\substack{y \to x \\ t \downarrow 0}} (-f)_t(y)(-h) = \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}[f(y) - f(y - th)] \\ &= \limsup_{\substack{y \to x \\ t \downarrow 0}} t^{-1}[f(y - th + th) - f(y - th)] \\ &= \limsup_{\substack{z \to x \\ t \downarrow 0}} t^{-1}[f(z + th) - f(z)] = \sigma_f(x)(h).\end{aligned}$$

We are now able to derive Theorem (3.4) from these lemmas.

Let $\{h_n | n \in N\}$ be a dense sequence in E; by Lemma (3.6) and the Baire category theorem, we have generically, $\forall n \in N$,

$$\sigma_f(x)(h_n) = \overline{df}(x)(h_n).$$

Now by Propositions (1.2) and (3.1)(b) this equality holds generically for every $h \in E$. Thus (a) follows from Lemma (1.1) and Proposition (1.2); (b) is a consequence of Lemma (3.7) since, applying Lemma (3.6) to f and $-f$, we obtain, generically:

$$\sigma_f(x)(h) = \overline{df}(x)(h),$$

$$\sigma_{-f}(x)(-h) = \overline{d(-f)}(x)(-h) = -\underline{df}(x)(-h).$$

(c) and (d) are the geometrical equivalent of (a) and (b) which are easily derived from Proposition (3.2). □

Theorem (3.4) enables us to state the following characterization of generically smooth functions defined on an open subset of a separable Baire space.

THEOREM (3.8) *Let E be a separable topological vector space which is a Baire space and f any locally Lipschitzian function defined on some open subset $\Omega \subset E$, the following are equivalent:*

(a) *for every $h \in E$, f admits generically a directional derivative in the direction h;*

(b) *the outer tangent cone to the epigraph of f on Ω is generically a closed convex cone;*

(c) *f is generically Gâteaux-differentiable on Ω;*

(d) *f is generically smooth on Ω.*

PROOF. (a) $\Rightarrow$ (b). Let $\{h_n | n \in N\}$ be any dense sequence in E; for every $n \in N$, there exists a dense G_δ subset G_n in Ω, at every point of which $\overline{df}(x)(h_n) = \underline{df}(x)(h_n)$ and thus there exists a dense G_δ subset $G = \bigcap_{n \in N} G_n$ in Ω such that

$$x \in G \Rightarrow \forall n \in N, \quad \overline{df}(x)(h_n) = \underline{df}(x)(h_n).$$

Using Proposition (3.1)(b), we have

$$x \in G \Rightarrow \forall h \in E, \quad \overline{df}(x)(h) = \underline{df}(x)(h),$$

which means exactly that df is well-defined on G.

It follows from Propositions (3.2) and (3.1)(b) that, for every $x \in G$, $I_{\text{epi } f}(x, f(x))$ is the interior of $C_{\text{epi } f}(x, f(x))$. Combining this result with Theorem (3.4)(c) we obtain (b).

(b) $\Rightarrow$ (c): by Proposition (3.2) $C_{\text{epi } f}(x, f(x))$ is the epigraph of the function $h \to \underline{df}(x)(h)$, thus (b) means that generically, $\underline{df}(x)$ is a continuous convex function. Combining this result with Theorem (3.4)(a) and (b), we derive that $df(x)$ is generically well-defined and $h \to df(x)(h)$ is a continuous linear functional on E, which means exactly that f is generically Gâteaux-differentiable on Ω.

Finally, (c) $\Leftrightarrow$ (d) follows from (2.7) and (c) $\Rightarrow$ (a) is trivial. $\square$

Obviously conditions (a) and (b) of Theorem (3.8) are satisfied if f is any continuous convex function, thus we have immediately:

PROPOSITION (3.9) *Every continuous convex function defined on some convex open subset Ω of a separable topological vector space which is a Baire space is generically smooth on Ω.*

Another class of real-valued functions which admit everywhere a directional derivative in each direction is the class of those functions which are defined as suprema of regular functions:

COROLLARY (3.10) *Let F be any continuous function defined on the product space $\Omega \times Y$ where Ω is an open subset of a separable topological vector space E which is a Baire space and Y any compact space. Suppose that:*

(i) *For every $y \in Y$, $x \to F(x, y)$ is locally Lipschitzian on Ω and the pointwise supremum $x \to f(x) = \max_{y \in Y} F(x, y)$ is locally Lipschitzian on Ω.*

(ii) *$(x, y) \to \partial F_x(x, y)$ is an upper-semi-continuous mapping from $\Omega \times Y$ into E^* endowed with the weak*-topology. ($\partial F_x(x, y)$ denotes the generalized gradient of the function $x \to F(x, y)$ at x.)*

(iii) *For every $x \in \Omega$, $h \in E$, and $y \in Y$,*

$$dF_x(x, y)(h) = \sup_{u \in \partial F_x(x,y)} u(h).$$

($dF_x(x, y)(h)$ is the directional derivative of the function $x \to F(x, y)$ in the direction h.)

Then the pointwise supremum f is generically smooth on Ω.

PROOF. For every $x \in \Omega$, $t \in]0, 1[$, $h \in E$, we have

$$f(x + th) - f(x) \geq F(x + th, y) - F(x, y)$$

if $y \in M(x) = \{y \in Y | F(x, y) = f(x) = \max_{y \in Y} F(x, y)\}$. Thus $\underline{df}(x)(h) \geq \sup_{y \in M(x)} dF_x(x, y)(h)$. On the other hand,

$$f(x + th) - f(x) \leq F(x + th, y_t) - F(x, y_t)$$

if $y_t \in M(x + th)$; therefore, applying Theorem (1.7) we have

$$f(x + th) - f(x) \leq t \sup_{u \in \partial F_x(x + \theta th, y_t)} u(h)$$

for some $\theta \in]0, 1[$. Now letting $t \to 0$, this yields

$$\overline{df}(x)(h) \leq \limsup_{t \downarrow 0} \sup_{u \in \partial F_x(x + \theta th, y_t)} u(h).$$

Since Y is compact, and the mapping $(x, y) \to \partial F_x(x, y)$ upper-semi-continuous, the net $(y_t)_{t>0}$ has some cluster point y in Y (which necessarily belongs to $M(x)$) such that

$$\overline{df}(x)(h) \leq \sup_{u \in \partial F_x(x, y)} u(h).$$

By assumption, we derive $\overline{df}(x)(h) \leq dF_x(x, y)$ hence:

$$\overline{df}(x)(h) \leq \sup_{y \in M(x)} dF_x(x, y)(h)$$

which shows that $df(x)(h)$ is well-defined and equal to $\sup_{y \in M(x)} dF_x(x, y)(h)$. Finally we have shown that assertion (a) of Theorem (3.8) is true, hence the desired result. □

Assumption (iii) of the previous corollary is trivially satisfied if every function $x \to F(x, y)$ is continuously Gâteaux-differentiable on Ω (the continuity being taken with respect to the weak*-topology on E^*; the generalized gradient is then everywhere reduced to the Gâteaux-derivative).

The proof of the existence of a directional derivative has been taken from F. H. Clarke [7, Theorem 2.1].

To highlight the importance of this later corollary, we find it useful to include here some trivial application which is closely related to best approximation theory.

PROPOSITION (3.11) *Let $E = R^n$ be the usual n-dimensional Euclidean space and C a closed bounded subset in E, the pointwise infimum*

$$x \to \min_{y \in C} \|x - y\|$$

is generically smooth.

PROOF. For every $y \in C$, $x \to F(x, y) = \|x - y\|$ is continuously differentiable on $\mathbf{R}^n \setminus C$. Thus the pointwise supremum $x \to \max_{Y \in C}(-\|x - y\|)$, which vanishes identically on C, satisfies the assumptions of Corollary (3.10) on the open subset $\Omega = R^n \setminus C$.

We refer the reader interested in smoothness properties of pointwise suprema (or infima) to I. Ekeland and G. Lebourg [10] where a general theorem is given for generic Fréchet-differentiability of such functions defined on infinite dimensional Banach spaces (in particular, the compacity assumption of Corollary (3.10) is dropped).

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