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ON THE EMBEDDING PROBLEM FOR 1-CONVEX SPACES

BY

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ABSTRACT. In this paper we provide a necessary and sufficient condition for 1-convex spaces (i.e., strongly pseudoconvex spaces) which can be realized as closed analytic subvarieties in some $\mathbf{C}^N \times \mathbf{P}_M$. A construction of some normal 3-dimensional 1-convex space which cannot be embedded in any $\mathbf{C}^N \times \mathbf{P}_M$ is given. Furthermore, we construct explicitly a non-kählerian 3-dimensional 1-convex manifold which answers a question posed by Grauert.

Unless otherwise specified, all C-analytic spaces considered here will be noncompact, countable at infinity, reduced C-analytic spaces of bounded Zariski dimension. Furthermore, the category of analytic coherent sheaves on a C-analytic space X will be denoted by Coh(X).

0. Introduction. On the one hand, it is well known that all Stein spaces can be embedded in $\mathbf{C}^N$ and any compact C-analytic space carrying a positive line bundle is embeddable into a complex projective space $\mathbf{P}_M$ for arbitrary large integers N and M. On the other hand, topologically any 1-convex space is obtained by "welding" some compact analytic space onto some Stein space. Therefore, one naturally raises the question of embedding 1-convex spaces into $\mathbf{C}^N \times \mathbf{P}_M$. From now on, such 1-convex spaces will be called "embeddable 1-convex spaces".

Our purpose here is twofold. First of all, we will provide a necessary and sufficient condition for embedding 1-convex spaces. Second, we will construct some 1-convex spaces (resp., 1-convex manifolds) which are not embeddable, therefore providing us some peculiar aspect of 1-convex spaces (resp., 1-convex manifolds).

The organization of this paper is as follows: In §I, all the basic definitions will be given. With those notions in hand, the statement of our problem will be formulated. Next, §II is devoted to the study of 1-convex spaces. The

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construction of some nonembeddable 1-convex space (resp., nonembeddable 1-convex manifold) will be taken up in §III. We will end our discussion with some open problems.

I. The context of the problem.

DEFINITION 1 [2]. Let S be a compact analytic subvariety in a C-analytic space X. Then S is said to be *exceptional* if

(i) $\dim S_x > 0$ for all $x \in S$,

(ii) there exist a C-analytic space Y and a proper, surjective and holomorphic map ¶: $X \to Y$, inducing a biholomorphism $X \setminus S \simeq Y \setminus T$, where T consists of finitely many points and

(iii) $¶_* O_X \simeq O_Y$.

EXAMPLE 1. Let X be the blowing up of $\mathbf{C}^2$ at the origin and let S be the proper transform of the origin. Clearly, $S \simeq \mathbf{P}_1$ is exceptional in X.

DEFINITION 2 [6a]. Let X be a given C-analytic space with its exceptional subvariety S. X is said to be 1-*convex* if Y is Stein. (Y is called the *Remmert reduction* of X. Sometimes we will use the notation (X, S) to denote 1-convex spaces.)

In Example 1, (X, S) is a 1-convex manifold, since $Y \simeq \mathbf{C}^2$. In fact, this example is the prototype for our next investigation.

REMARK. Notice that the definition of 1-convex spaces given here is not a standard one. However, it is known that, in fact, it is equivalent to the usual definitions (see [2] or [6a]). We adopt such a definition here because, as we will see, it is very convenient in our context.

DEFINITION 3 [2]. Let S be a compact C-analytic space and let L be a holomorphic line bundle on S. Let us identify the zero section Σ of L with S. Then L is said to be *weakly negative* if Σ (as a compact analytic subvariety in L) is exceptional. L is said to be *weakly positive* if L^* is weakly negative.

EXAMPLE. The hyperplane section bundle on $\mathbf{P}_M$ is weakly positive.

DEFINITION 3′ [0]. Let X be a C-analytic space, let L be a holomorphic line bundle on X and let (U_i, e_{ij}) be a system of 1-cocycles determining L. Then L is said to be *positive* if there exists a system $\{h_i\}$ of smooth and positive functions on (U_i) such that on $U_i \cap U_j$,

$$h_j = |e_{ij}|^2 h_i$$

and such that the functions $g_i := -\log h_i$ are strongly pseudoconvex on U_i.

REMARK. For X compact, it has been proved that the notions of weakly positive and positive are equivalent (see [2]). In [6b] the relationship between these two concepts on 1-convex spaces is studied.

It is well known (see [7] and the references there) that any Stein space can be embedded in $\mathbf{C}^N$. Also, it is known (see [2]) that any compact C-analytic space carrying a weakly positive line bundle can be embedded biholomorphically into some $\mathbf{P}_M$. Therefore, one is led to:

Problem. When is it possible to embed a given 1-convex space into $\mathbf{C}^N \times \mathbf{P}_M$?

Notice that in our Example 1, in view of the definition of the blow-up, the 2-dimensional 1-convex manifold

$$X = \{(x_0, x_1; z_0, z_1) \in \mathbf{C}^2 \times \mathbf{P}_1 | x_0 z_1 - x_1 z_0 = 0\}$$

is actually a closed submanifold in $\mathbf{C}^2 \times \mathbf{P}_1$.

II. The embeddable 1-convex spaces. The first result in this direction in the nonsingular case was established in [**1**].

THEOREM 1 [**1**]. *Let X be a* 1*-convex manifold and let us assume that there exists a positive line bundle L on X. Then X is embeddable.*

By slightly modifying the proof in [**1**] and by using some standard techniques, we shall generalize Theorem 1 in two directions. First of all we do not require X to be nonsingular, and secondly the line bundle L does not need to be positive on the whole space X.

As a common philosophy in this kind of business, any embedding theorem is preceded, in general, by a vanishing theorem. So to begin, let us mention the following version.

THEOREM 2 [**0**]. *Let X be a* 1*-convex space, let L be a positive line bundle on X and let $\mathfrak{F} \in \mathrm{Coh}(X)$. Then there exists an integer $k_0 = k_0(L, \mathfrak{F})$ such that*

$$H^i(X, \mathfrak{F} \otimes L^k) = 0$$

for all $k \geqslant k_0$ and all $i \geqslant 1$.

Actually in [**0**] (see also [**1**]), the previous result was proved for q-convex spaces, for any $q \geqslant 1$. However, in the special case of 1-convex spaces, Theorem 2 can be sharpened as follows.

THEOREM I. *Let (X, S) be a* 1*-convex space and let us assume that there exists a holomorphic line bundle L on X such that $L|S$ (sheaf restriction) is positive. Then, for any $\mathfrak{F} \in \mathrm{Coh}(X)$, there exists an integer $k_0 := k_0(L, \mathfrak{F})$ such that*

$$H^i(X, L^k \otimes \mathfrak{F}) = 0$$

for all $k \geqslant k_0$ and all $i \geqslant 1$.

The proof of Theorem I is based on the following important result.

EXTENSION LEMMA. *Let (X, S) and L be as in the hypothesis of Theorem* I. *Then, after modifying the metric of L, one can find a* 1*-convex neighborhood Ξ of S, with $\Xi \subset X$, such that $L|\Xi$ is positive.*

PROOF. We shall denote the Zariski tangent space of X at a point $x \in X$ by $T_{X,x}$. Let (h_i, U_i) be the metric associated to the line bundle L. Since by hypothesis, $L|S$ is positive, i.e. for all $x \in V_i := U_i \cap S$,

$$-\partial\bar{\partial}\log h_i(x) > 0 \quad \text{on } T_{V_i,x}. \tag{1}$$

In view of Definition 2 for 1-convex spaces, one can find a smooth function Ψ on X such that

$$\partial\bar{\partial}\Psi(x) \geqslant 0 \quad \text{on } T_{X,x} \quad \text{for } x \in X, \tag{2}$$

$$\partial\bar{\partial}\Psi(x) > 0 \quad \text{on } T_{X,x} \quad \text{for } x \in X \setminus S, \tag{3}$$

$$\partial\bar{\partial}\Psi(x) > 0 \quad \text{on} N_x \quad \text{for } x \in S, \tag{4}$$

where N_x is the complementary space of $T_{S,x}$ in $T_{X,x}$.

Now on each open covering U_i with $U_i \cap S \neq \varnothing$ and for any integer k_i the smooth function

$$A_i := h_i e^{-k_i\Psi}: U_i \to R^+$$

is well defined.

In view of (1), (2) and (4), one can choose a $k_i \gg 0$, such that

$$-\partial\bar{\partial}\log A_i(x) > 0 \quad \text{on } T_{U_i,x} \quad \text{for all } x \in V_i. \tag{5}$$

Meanwhile, in view of (3), again with a suitable $k_i \gg 0$, one has

$$-\partial\bar{\partial}\log A_i(x) > 0 \quad \text{on } T_{U_i,x} \quad \text{for } x \in U_i \setminus V_i. \tag{6}$$

In view of the compactness of S, (5) and (6), an integer $k := \max_i k_i$ can be selected such that, on the relative compact neighborhood $N := \bigcup_i U_i$ of S in X,

$$-\partial\bar{\partial}\log(h_i e^{-k\Psi}) > 0.$$

In other words, with the new metric $g_i := h_i e^{-k\Psi}$, $L|N$ is positive. Now, since S is exceptional, it admits a fundamental system of 1-convex neighborhoods; let Ξ be one of them such that $\Xi \subset N$. Clearly, $L|\Xi$ is positive. Q.E.D.

Proof of Theorem I. In view of the Extension Lemma and Theorem 2 above, it suffices to prove that the restriction map

$$\lambda_i: H^i(X, \mathfrak{F}) \to H^i(\Xi, \mathfrak{F}) \tag{$*$}$$

is injective for all $i \geqslant 1$ and $\mathfrak{F} \in \mathrm{Coh}(X)$.

In fact our present situation can be summarized by the following diagram

$$\begin{array}{ccccc} S & \hookrightarrow & \Xi & \hookrightarrow & X \\ \downarrow & & \downarrow & & \downarrow\P \\ T & \hookrightarrow & V & \hookrightarrow & Y \end{array}$$

where Y, T and $\P$ are as in Definition 2 and V is an arbitrary, small Stein neighborhood (disconnected) of T in Y. In fact without loss of generality, one can choose V such that $\Xi = \P^{-1}(V)$.

Now notice that $X = (X \setminus S) \cup \Xi$ and $(X \setminus S) \cap \Xi = \Xi \setminus S$. Similarly $Y = (Y \setminus T) \cup V$ and $(Y \setminus T) \cap V = V \setminus T$. Hence we can apply the standard technique of Mayer-Vietoris exact sequences in this context; namely let us look at the following commutative diagrams with exact rows.

$$\begin{array}{ccccc}
\cdots \longrightarrow & H^i(\Xi, \mathfrak{F}) \oplus H^i(X\setminus S, \mathfrak{F}) & \xrightarrow{\delta_i} & H^i(\Xi \setminus S, \mathfrak{F}) & \xrightarrow{\epsilon_i} \\
& \uparrow \alpha_i \qquad \wr\uparrow \beta_i & & \wr\uparrow \gamma_i & \\
\cdots \longrightarrow & H^i(V, \hat{\mathfrak{F}}) \oplus H^i(Y\setminus T, \hat{\mathfrak{F}}) & \xrightarrow[\delta_i']{} & H^i(V\setminus T, \hat{\mathfrak{F}}) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccc}
\xrightarrow{\epsilon_i} & H^{i+1}(X, \mathfrak{F}) & \xrightarrow{\theta_{i+1}} & H^{i+1}(\Xi, \mathfrak{F}) \oplus H^{i+1}(X\setminus S, \mathfrak{F}) & \longrightarrow \\
& & & \uparrow \alpha_{i+1} \qquad \wr\uparrow \beta_{i+1} & \\
\longrightarrow & H^{i+1}(Y, \hat{\mathfrak{F}}) & \longrightarrow & H^{i+1}(V, \hat{\mathfrak{F}}) \oplus H^{i+1}(Y\setminus T, \hat{\mathfrak{F}}) & \longrightarrow
\end{array}$$

Now for $i > 0$, $H^{i+1}(Y, \hat{\mathfrak{F}}) = 0$ since Y is Stein and $\hat{\mathfrak{F}} := \P_*(\mathfrak{F}) \in \mathrm{Coh}(Y)$. Hence δ_i' is surjective. Furthermore, γ_i is bijective, in view of (ii) in Definition 1. Therefore δ_i is also surjective. Consequently ε_i is a zero map and this implies that θ_{i+1} is injective. But the latter map factors via λ_{i+1} which is therefore injective. Hence (∗) is proved. Q.E.D.

REMARKS. (a) The theorem above is often alluded to as the *imprecise vanishing theorem* since it holds in general for some power $k \gg 0$. However, if one is willing to deal only with C-analytic manifold, then the so-called *precise vanishing theorem* can be obtained (see **[6a]** for complete proof and more related results), namely:

PROPOSITION 1. *Let X be a 1-convex manifold and let L be a holomorphic line bundle on X such that $L|S$ is weakly positive (S is singular in general). Then*

$$H^p(X, \Omega^q(L)) = 0 \quad \textit{for all } p + q > \dim X + 1.$$

(b) Actually, there is more than one way to prove Theorem I above. Another proof can be found in **[6b]** where a direct argument is used in order to avoid a detour through Theorem 2.

Using the standard technique to derive Theorem A of Cartan from his Theorem B, we can deduce from our previous Theorem I the following:

THEOREM II. *Let (X, S) and L be as in Theorem I. Then, for any $x \in X$, there exists an integer k_x, such that the stalk $(L^k \otimes \mathfrak{F})_x$ is generated by its global sections for any $k > k_x$.*

PROOF. Let $x \in X$ and let I_x be the ideal sheaf of germs of holomorphic functions vanishing at x. Then for any $\mathfrak{F} \in \mathrm{Coh}(X)$, one has the following exact sequence:

$$0 \to L^k \otimes I_x\mathfrak{F} \to L^k \otimes \mathfrak{F} \to L^k \otimes \mathfrak{F}/I_x\mathfrak{F} \to 0,$$

which in turn induces the following exact sequence:

$$\cdots \to H^0(X, L^k \otimes \mathfrak{F}) \xrightarrow{\alpha} H^0(X, L^k \otimes \mathfrak{F}/I_x\mathfrak{F}) \to H^1(X, L^k \otimes I_x\mathfrak{F}).$$

In view of Theorem I, there exists an integer k_x, such that for all $k > k_x$, $H^1(X, L^k \otimes I_x\mathfrak{F}) = 0$. Therefore α is surjective. Nakayama's lemma tells us that $(L^k \otimes \mathfrak{F})_x$ is generated by its global sections. Q.E.D.

REMARK. Theorem II was also proved in [1]. Our proof here is simpler. From Theorem II, we can deduce the following useful result.

COROLLARY 1. *Let (X, S), L and $\mathfrak{F}$ be as in Theorem* I. *Then for any relative compact domain D in X, there exist finitely many global sections $\{f_i\} \in H^0(X, L^k \otimes \mathfrak{F})$, with $k \gg 0$, such that the stalk $(L^k \otimes \mathfrak{F})_x$ is generated by those $\{f_i\}$ for any $x \in D$.*

We are now in a position to state the main result of this section.

THEOREM III. *Let (X, S) be a given* 1-*convex space. There exists a line bundle L on X such that $L|S$ is positive if and only if X is embeddable.*

The following result will be needed later.

LEMMA 1 [1]. *Let X be a Stein space and let L be a holomorphic line bundle on X. Then there exist finitely many global sections $f_1, \ldots, f_m \in H^0(X, L)$ such that the set $\{x \in X | f_1(x) = \cdots = f_m(x) = 0\}$ is empty.*

PROOF OF THEOREM III. (i) *Sufficiency.*

(a) Let D be a relative compact domain in X such that $S \subset D$. For any point $x \in X$ and any integer r, one has the following exact sequence:

$$0 \to I_x^2 \otimes L^r \to I_x \otimes L^r \to I_x/I_x^2 \otimes L^r \to 0.$$

In view of the compactness of D, Theorem I tells us that there exists an integer r_D such that the differential map

$$H^0(X, I_x \otimes L^r) \to I_x/I_x^2 \otimes L_x^r \qquad (*)$$

is surjective for any point $x \in D$ and any integer $r > r_D$.

Similarly, by considering the following exact sequence:

$$0 \to I_{xy} \otimes L^s \to L^s \to L_x^s \oplus L_y^s \to 0,$$

one can prove that there exist an integer s_D such that the restriction map

$$H^0(X, L^s) \to L_x^s \oplus L_y^s \qquad (**)$$

is surjective for any points $x \neq y \in D$ and any integer $s > s_D$.

Now let $t_D := r_D \cdot s_D$. Corollary 1 tells us that there exists an integer p which is a multiple of t_D and sections $f_0, \ldots, f_q \in H^0(X, L^p)$ which give rise to the well-defined holomorphic map

$$\Sigma := (f_0, \ldots, f_q)\colon D \to \mathbf{P}_q.$$

In view of (*) and (**), Σ is clearly regular and injective.

(b) Now let $Z := \{x \in X \mid f_0(x) = \cdots = f_q(x) = 0\}$. From the construction of the f_i, clearly $Z \cap S = \varnothing$, i.e., Z is a Stein space. Let J be the ideal sheaf determined by Z, then Theorem I tells us that there exists an integer $k_0(L, J)$ such that $H^1(X, L^k \otimes J) = 0$ for all $k > k_0$. Therefore the map

$$\Lambda\colon H^0(X, L^k) \to H^0(Z, O_Z \otimes L^k)$$

is surjective. In view of Lemma 1, there exist global sections $g_1, \ldots, g_r \in H^0(Z, O_Z \otimes L^k)$ which do not have any common zeroes in Z. From the surjectivity of Λ one can assume that $g_i \in H^0(X, L^k)$. Clearly

$$\Phi := \left(f_0^k, \ldots, f_q^k; g_1^p, \ldots, g_r^p\right)\colon X \to \mathbf{P}_M$$

is a well-defined holomorphic map which embedded D biholomorphically into $\mathbf{P}_M$ as a locally closed subspace where $M := q + r$.

(c) Let Y be the Remmert reduction of X and let ρ be the map which embeds Y into some $\mathbf{C}^n$ [7]. Let us look at the following diagram.

$$\begin{tikzcd}
 & X \arrow[r, "\Phi"] \arrow[dl, dashed, "\pi"'] \arrow[dd, "\Psi"] \arrow[ddr, "\Upsilon"] & \mathbf{P}_M \\
Y \arrow[dr, hook, dashed, "\rho"'] & & \\
 & \mathbf{C}^N & \mathbf{C}^N \times \mathbf{P}_M \arrow[l, dashed, "\Pi_1"'] \arrow[uu, dotted, "\Pi_2"']
\end{tikzcd}$$

Clearly the composed map $\Psi := \rho \circ \P$ is a proper holomorphic map which is biholomorphic outside of S. Consequently the map $\Upsilon := \Psi \times \Phi\colon X \to \mathbf{C}^N \times \mathbf{P}_M$ is proper since Ψ is and since $\Psi = \Pi_1 \circ \Upsilon$. Moreover, Ψ (resp., Φ) is injective and regular on $X \setminus S$ (resp., D); one can then easily check, using the commutativity of the diagrams, that Υ is also injective and regular on $X = (X \setminus S) \cup D$.

(ii) *Necessity*. Let us look at the following diagram

$$\begin{tikzcd}
X \arrow[r, hook, "\iota"] & \mathbf{C}^N \times \mathbf{P}_M \arrow[d, "\Pi_2"'] & \\
S \arrow[u, hook] \arrow[r, hook] & \mathbf{P}_M & \mathbf{H} \arrow[l, dashed]
\end{tikzcd}$$

where the maps (except Π_2) are embedding maps. Let $\mathbf{H}$ be the hyperplane section bundle on $\mathbf{P}_M$ and let us lift $\mathbf{H}$, via Π_2, to a holomorphic line bundle

$E := \Pi_2^*(\mathbf{H})$ on $\mathbf{C}^N \times \mathbf{P}_M$. Now let us pull back E to obtain a line bundle $L := \iota^* E$ on X. By construction, clearly, $L|S \simeq \mathbf{H}|S$ is positive. Q.E.D.

REMARKS. (a) With some further work, the Extension Lemma can be strengthened as follows (see **[6b]** for complete proof).

THEOREM IV. *Let (X, S) be a given* 1-*convex space and let L be a line bundle on X such that $L|S$ is weakly positive. By modifying the metric, L is actually positive on all of X.*

(b) In order to illustrate the argument in Part (i) (a) and (b) above, let us consider the following example.

EXAMPLE 2. Let $F \simeq \mathbf{P}_1$ be a line in $\mathbf{P}_2$ and let us take a point $x \notin F$. By blowing up $\mathbf{P}^2$ at x, one obtains a 2-dimensional projective manifold W. Let F' (resp., S) be the proper transform of F (resp., x). Then clearly $X := W \setminus F'$ is a 2-dimensional 1-convex manifold with its exceptional subvariety S. It is well known that one can embed W biholomorphically into $\mathbf{P}_5$. Therefore X is biholomorphic to a Zariski open submanifold in $\mathbf{P}_5$.

III. The nonembeddable 1-convex spaces. Our previous Theorem III suggests the following:

Problem 0. Let (X, S) be a given 1-convex space. Is it always possible to find a holomorphic line bundle L on X such that $L|S$ is positive?

We are going to tackle this problem following two simple observations:

(A) Let (X, S) be an embeddable 1-convex space. Then necessarily S is projective algebraic.

(B) Let X be an embeddable 1-convex manifold. Then necessarily X is kählerian.

Hence these two facts lead us to the following:

Question A. Do there exist 1-convex spaces (X, S) such that the exceptional subvariety S is not projective algebraic?

Question B. Do there exist non-kählerian 1-convex manifolds?

(Questions A and B were first raised to the author by H. Grauert.) In this section, we shall provide satisfactory answers for both Questions A and B as well as for Problem 0.

($\hat{\text{A}}$) Construction of 3-dimensional normal 1-convex spaces (X, S) such that S is not projective.

First we shall need the following result:

LEMMA 2. *Let L be a holomorphic line bundle over a* $\mathbf{C}$-*analytic manifold X and let us identify X with the zero section of L. Then there exists a canonical isomorphism* (*holomorphic*) *from the normal bundle of X in L onto the line bundle L.*

The proof of Lemma 2 is purely topologic so we leave it to the reader. Now we are in a position to begin our construction.

Step 1. Let us consider the following special projective algebraic 2-fold exhibited by Hironaka (see [4, Chapter V]). Let C be a nonsingular cubic curve in $\mathbf{P}_2$. It is known that C also acquires a group structure on its set of points. Fix an inflection point $p_0 \in C$ to be the origin of the group law on C. Since the torsion-free part of the abelian group C has infinite rank, therefore one can select ten points, say $p_1, \ldots, p_{10} \in C$ which are linearly independent over $\mathbf{Z}$ in the group law. Now blow up $p_1, \ldots, p_{10}$ in $\mathbf{P}_2$ successively, let Z be the resulting manifold and let W be the strict transform of C. Since $C^2 = 9$, we have $W^2 = 9 - 10 = -1$. Following [2], W is exceptional in Z.

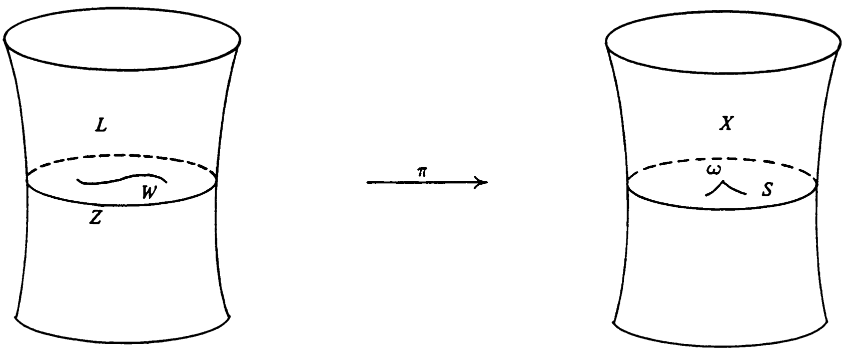

FIGURE 1

Step 2. Let us put on Z a weakly negative line bundle, say L, which always exists since Z is projective. In view of Lemma 2, $N_{Z/L}$, the normal bundle of Z in L, is also weakly negative. Following [2, Satz 5] L is a 1-convex manifold admitting Z as its exceptional subvariety. Furthermore, since W is exceptional in Z and $\dim W = 1$, a result in [2, Satz 9] tells us that the normal bundle $N_{W/Z}$ of W in Z is actually weakly negative (see Figure 1).

Step 3. It is known that the line bundle L is weakly negative iff L^* is ample in the sense of Grothendieck [3]. Now, let us look at the following exact sequence of bundles on W:

$$0 \to N_{W/Z} \to N_{W/L} \to N_{Z/L|W} \to 0 \tag{†}$$

where $N_{W/L}$ is the normal bundle of W in L. By dualizing (†), we obtain an exact sequence of bundles where the extreme terms $N^*_{Z/L}$ and $N^*_{W/Z}$ are ample. A result in [3] tells us that $N^*_{W/L}$ is also ample, i.e., $N_{W/L}$ is weakly negative.

Step 4. Since $N_{W/L}$ is weakly negative, following [2], W is exceptional in L. From Definition 1, this implies the existence of a C-analytic space X and a birational morphism $\P: L \to X$ inducing a biholomorphism

$$L \setminus W \simeq X \setminus \{\omega\} \tag{‡}$$

where $\{\omega\}$ is a point in X.

Clearly X is a 3-dimensional normal C-analytic space with only one isolated singular point $\{\omega\}$. Consequently, Riemann's extension theorem tells us that X is a holomorphically convex space since L is the one. Furthermore, in view of (‡), $S := \P(Z)$ is the maximal compact subvariety in X, in the sense of [2]. From there, one can check that (X, S) is actually 1-convex in the sense of Definition 2.

Step 5. S is not projective (see [4]).

If it were, there would exist a curve, say D, on S with $\omega \notin D$. Consequently $\P^{-1}(D) \subset Z$ would be a curve not intersecting W and the image $D_0 := \theta(\P^{-1}(D))$ would be a curve in $\mathbf{P}_2$ which does not meet C except at the points $p_1, \ldots, p_{10}$, where $\theta: Z \to \mathbf{P}_2$ is the blowing up map. Let $d := \deg D_0$. In view of Bezout's theorem $D_0 \cdot C = 3d > 0$. So one can write

$$D_0 \cap C = \sum_{i=1}^{10} n_i p_i \quad \text{on } C$$

with $n_i \geqslant 0$ and $\Sigma n_i = 3d$. But $D_0 \sim dL$ (linear equivalence) where L is a line in $\mathbf{P}_2$ and $L \cdot C \sim 3p_0$, so one has $\Sigma n_i p_i = 0$ in the group law on C. But this contradicts the linear independency of the points $p_1, \ldots, p_{10}$.

($\hat{\mathrm{B}}$) Construction of 3-dimensional non-kählerian 1-convex manifolds.[2]

Step 1. Let Z be the blowing up of $\mathbf{C}^3$ at the origin and let $M \simeq \mathbf{P}_2$ be the exceptional subvariety. Let K be a singular cubic curve with only one node ω, embedded in $\mathbf{P}_2$. Certainly $K \setminus \omega$ is smooth and there exists an open neighborhood U of ω in Z such that $K \cap U$ is a union of two smooth irreducible branches, say K_1 and K_2 which intersect at ω with distinct tangents.

Step 2. We are going to use a basic idea which is due to Hironaka (see [4, Appendix B]). Let $(\hat{U}, f)$ be the composite of two blowings up over U in which the first is the blowing up with center K_1 and the second is the blowing up with center K_2' where K_2' denotes the proper transform of K_2 by the first blowing up. Let $(\hat{V}, g)$ be the blowing up over $V := Z \setminus \omega$ with center $K \setminus \omega$. Now let $\hat{S}_1 := f^{-1}(K_1 \cup K_2)$ and let $\hat{S}_2 := g^{-1}(K \setminus \omega)$.

Step 3. Notice that $(\hat{U}, f)$ and $(\hat{V}, g)$ agree on the inverse image $W := f^{-1}(U \setminus \omega)$. Glue $(\hat{U}, f)$ and $(\hat{V}, g)$ along W to obtain a 3-dimensional C-analytic manifold, say X and a proper morphism $\Pi: X \to Z$ which induces

[2]The author would like to thank Professor Mumford for his penetrating remark which greatly simplified the construction.

the blowing up $(\hat{U}, f)$ (resp., $(\hat{V}, g)$) in the open subset $U \subset Z$ (resp., $V \subset Z$). Let S be the total transform of M by Π. One has $S = \hat{M} \cup \hat{S}$ where $\hat{M}$ (resp., $\hat{S}$) is the proper transform of $\mathbf{P}_2$ (resp., K) by Π. Notice that $\hat{S}$ is obtained by glueing $\hat{S}_1$ and $\hat{S}_2$. Furthermore, the inverse image of ω by Π is the union of two compact 1-cycles, say $\bar{\alpha}$ and $\bar{\beta}$ which intersect transversally (see Figure 2). Clearly (X, S) is a 3-dimensional 1-convex manifold.

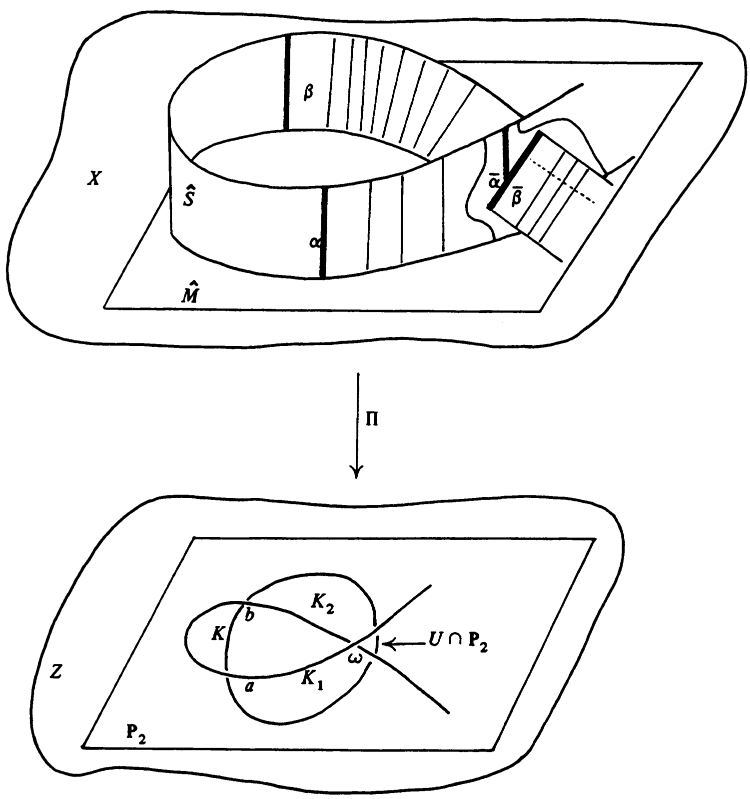

FIGURE 2

Step 4. Let a (resp., b) be a general point on $K_1 \setminus \omega$ (resp., $K_2 \setminus \omega$). Their inverse images α (resp., β) are isomorphic to the projective line. Hence $\alpha \sim \beta$ (homological equivalence). Furthermore, in view of the order of blowing up K_1 and K_2 in U, one has

$$\alpha \sim \bar{\alpha} + \bar{\beta}, \qquad \beta \sim \bar{\beta}.$$

This implies that

$$\bar{\alpha} \sim 0 \tag{††}$$

Step 5. X is not kählerian.

If it were, let Ω be the positive, closed (1, 1) form associated to some kähler metric on X. Then one has

$$\int_{\bar{\alpha}} \Omega > 0.$$

But this contradicts the existence of the compact 1-cycle $\bar{\alpha}$ satisfying (††).

REMARKS. (a) Dimensionwise, example ($\hat{\mathrm{B}}$) is sharp. In fact, in a forthcoming paper, the following result will be proved.

THEOREM V. *Let X be a given* 1-*convex manifold with its exceptional subvariety S (singular, in general). If either*

(i) $\dim X = 3$ *and* $\dim S = 1$, *or*

(ii) $\dim X = 2$,

then X is kählerian.

(b) In both examples ($\hat{\mathrm{A}}$) and ($\hat{\mathrm{B}}$) above, their exceptional subvariety S is Moishezon. This is by no means accidental. This fact has been pointed out in [5, Corollary to Theorem 2] namely:

PROPOSITION 2. *Let (X, S) be a given* 1-*convex space. Then the exceptional subvariety S is Moishezon.*

To round off this discussion, we would like to mention a few problems related to the structure of 1-convex spaces.

Problem 1. Let (X, S) be a 1-convex space such that S is projective algebraic. Is X embeddable?

Problem 2. Let X be a 1-convex kähler manifold. Is X embeddable?

However a weaker version than Problems 1 and 2 seems more interesting.

Problem 3. Let X be a 1-convex manifold with its exceptional subvariety S (singular, in general).

(a) If S is projective, is X then kählerian?

(b) If X is kählerian, is S then projective?

Finally, in correlation with the previous Proposition 2, one would like to raise the following:

Problem 4.[3] Let S be a given Moishezon space. Is it always possible to construct a 1-convex space X, admitting S as its exceptional subvariety?

REFERENCES

0. A. Andreotti and G. Tomassini, *A remark on the vanishing of certain cohomology groups*, Compositio Math. **21** (1969), 417–430.

1. S. Eto, H. Kazama and K. Watanabe, *On strongly q-pseudoconvex spaces with positive vector bundles*, Mem. Fac. Sci. Kyushu Univ. Ser. A **28** (1974), 135–146.

[3] This problem has recently been settled by the author in the affirmative.

2. H. Grauert, *Über Modifikationen und exzeptionell analytische Mengen*, Math. Ann. **146** (1962), 331–368.

3. R. Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math. No. 29 (1966), 63–94.

4. ______, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, Berlin and New York, 1977.

5. B. G. Moishezon, *Modifications of complex varieties and Chow lemma*, Classification of Algebraic Varieties and Compact Complex Manifolds, Lecture Notes in Math., vol. 412, Springer-Verlag, Berlin and New York, 1974, pp. 133–139.

6a. Vo Van Tan, *On the classification of* 1-*convex spaces* (preprint).

6b. ______, *On the positivity of line bundles on* 1-*convex spaces* (in preparation)

7. K. W. Wiegmann, *Einbettungen komplexer Raumer in Zahlenraume*, Invent. Math. 1 (1966), 229–242.

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