THE THEOREMS OF BETH AND CRAIG IN ABSTRACT MODEL THEORY. I. THE ABSTRACT SETTING

BY

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שנינו ביחד וכל אחד לחוד [1]

ABSTRACT. In the context of abstract model theory various definability properties, their interrelations and their relation to compactness are investigated.

Introduction. This is the first of three papers on the analogues of the theorems of Beth and Craig of first order logic in abstract model theory. They grew out of an unpublished preprint [MS][2] which was revised and extended several times by results of both authors as well as other people. They unify results due to Badger, Ebbinghaus, Friedman, Gostanian, Gregory, Hrbacek, Hutchinson, Kaufmann, Magidor, Malitz, Makkai, Makowsky, Paulos, Shelah and Stavi.

In this paper we present the abstract setting; we suppose that the reader is familiar with a standard text on model theory such as [BS] and [CK], with Barwise's [Ba1] and [MSS]. The main results here are that:

Beth's theorem together with a Feferman-Vaught theorem for tree-like sums implies a weak form of Robinson's consistency lemma (5.4) and the Robinson consistency lemma together with the Feferman-Vaught theorem for pairs implies full compactness (6.2).

In [MS2] and [MS3], the continuations of the present paper, we present applications of the general theory to particular logics. [MS2] is devoted to compact logics and some new logics are introduced, and [MS3] is devoted to infinitary logics and Δ-logics. Some complicated constructions are presented in detail there.

There are three aspects of Craig's interpolation theorem and its corollary, Beth's definability theorem:

(A) *Philosophical.* Every implicit definition is equivalent to an explicit definition.

(B) *Mathematical.* Although mathematicians never really cared whether they use explicit or implicit definitions, Beth's theorem tries to explain why we always have an explicit definition of a concept whenever it is uniquely describable. The ordering in real closed fields, e.g., is unique and the proof of this gives us an explicit definition of the ordering. Or formally real fields, where every element or its inverse can be represented as a sum of squares, can be uniquely ordered. Again the proof gives us an explicit definition. Another example: Uniquely orderable groups can be characterized by an $L_{\omega_1 \omega}$-sentence (cf. Bludov [B1]) so again there must exist an explicit definition of this ordering in $L_{\omega_1 \omega}$. Since most proofs of Craig's theorem are effective (in the finitary case, in the infinitary case this is more complicated) the explicit definition can also be obtained on a purely syntactical level. But we do not know of any example where this observation gives us anything deeply mathematical.

The main motivation for this paper, however, stems from the third aspect:

(C) *Metamathematical.* Craig's theorem holds for predicate logic and second order logic as well as for $L_{\omega_1 \omega}$ and L_A where A is an increasing union of countable admissible sets. First order logic can be characterized in terms of maximality with respect to some model theoretic properties. Now most of the proofs of such maximality theorems also give a proof of Craig's theorem. So "maximality" implies "Craig's theorem" sounds like a nice motto. Unfortunately L_A does not seem to fit into this picture. More generally we do not know if there is a maximal logic satisfying the Löwenheim-Skolem theorem down to ω and Craig's theorem; in particular Harrington and Kunen have shown independently that $L_{\omega_1 \omega}$ is not maximal in a very strong sense. Kunen uses CH and gets explicit extensions of $L_{\omega_1 \omega}$ adding propositional connectives. Harrington gives an existence proof of 2^{2^ω}-many other connectives P such that $L_{\omega_1 \omega}(P)$ shares most of the properties of $L_{\omega_1 \omega}$ without CH. Both results are presented in [Ha].

Other aspects of maximality in connection with definability properties are discussed in Feferman [Fe1] and Stavi [St1].

The aim of these papers is twofold: To provide the reader with a reasonably complete picture of what logics do or do not satisfy what kind of definability theorems, providing proofs, counterexamples or references, and to discuss definability theorems in the setting of abstract model theory. The two cannot be separated completely. Some abstract theorems give us hints of how to find counterexamples. Some constructions of counterexamples split into two different parts one of which can be easily captured as an abstract theorem. What we hope to show, too, is how this interplay works. In detail this paper is organized as follows.

In §1 compactness and Löwenheim-Skolem properties are defined as well

as the Karp-property. Some theorems relating these properties are studied or quoted for later use.

In §2 the interpolation and definability properties are introduced for pairs of logics: Craig's interpolation theorem, Beth's definability theorem, a weakened version of Beth's theorem. Δ-interpolation and variations around Robinson's consistency theorem. Their mutual logical interdependence is exhibited, and a theorem of Barwise is improved (Theorem I.2.4 and its corollaries). Also a new characterization of predicate calculus is given as the maximal Karp-logic which satisfies Robinson's consistency theorem (Theorem 2.14).

In §3 Feferman-Vaught-type theorems are discussed for pairs of logics and various sum-like operations on many-sorted structures. Another characterization of predicate calculus is given involving the Löwenheim-Skolem theorem, the weakened form of Beth's theorem and a Feferman-Vaught theorem for pairs of structures (Theorem 3.3 and Corollary 3.2).

In §4 we review briefly some results due to Feferman and Makowsky on uniform reduction which relates Feferman-Vaught-type theorems with definability theorems.

In §5 we describe a construction whose main motivation is to prove not Beth when not Craig. For credit see there. Variations of this construction will appear also in [MS2] and [MS3]. Its main content is a way to get implicit definitions out of counterexamples for definability theorems. It also yields an abstract theorem: Beth's theorem and a strong form of a Feferman-Vaught theorem imply the weak Robinson consistency theorem (Theorem 5.4).

In §6 we prove that under some weak assumption on set theory Robinson's consistency theorem[3] implies full compactness. Although not surprising, this is a highly nontrivial theorem of abstract model theory and shows that with more effort more abstract theorems should be provable.

In §7 we finally describe a generalization of an old trick to show that second order logic does not satisfy Beth's theorem, which has been worked out by Paulos and Burgess.

We conclude the paper with a survey of what logics do or do not satisfy what kind of definability property and an extensive bibliography.

There are many open problems left in the field: How can one construct explicitly logics with prescribed properties? We are still in the state of many, but scattered, examples, sometimes with rather well developed model theory (cf. [MS2], [Ke1], [Ma3] and [BKM] for $L(Q)$ and its extensions and [Ke2], [Ba2] for infinitary logic), but no coherent theory is in sight. Abstract model theory had two great impulses from Lindström and Barwise, but the more

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[3]Together with a Feferman-Vaught theorem for pairs of structures.

218 J. A. MAKOWSKY AND S. SHELAH

intriguing questions, the hard technical results, which give a field its living, seem to develop slowly. We hope to go an important step in this direction.

Here are some problems:

Problem 1. Is there any fully compact logic which satisfies Craig's theorem or is there any countably generated logic which satisfies Robinson's consistency theorem and which is different from (extending) predicate calculus?

Problem 2. What are the properties compact logics have in common? What conditions are needed to prove that the union of two compact logics is again compact (or similar for other properties than compactness)?

Kueker [**Kue**] has developed a general theory around the Löwenheim-Skolem theorem. This is what we have in mind for an answer to Problem 2. A first step in such a direction has been made by Stavi [**St2**]. More explicit problems may be found in [**Sh2**] or in the remainder of this paper and in [**MS2**], [**MS3**].

Problem 3. What kind of partial isomorphisms generate reasonable logics? Is there a partial isomorphism relation (game characterization) for any elementary equivalence relation of an abstract logic? How do the partial isomorphisms reflect the properties of logics?

Stavi and Nadel [**NSt**], [**Na**] have some interesting work in this direction. An interesting test-case are the various Henkin-quantifiers discussed in [**Wa**] and [**Ba4**]. Very recently Caicedo [**Ca**] got very general results concerning Problem 3, part 1.

1. Karp-logics, compactness and cardinality of models. All the definitions of abstract model theory are taken from [**Ba1**] unless otherwise stated. Note that our languages are mostly finite and all logics have the *finite occurence property*. Infinite languages occur in connection with diagrams. Universes are many sorted and PC-classes may admit extra universes, so reducts are relativized reducts. We let L vary over languages and L^*, $L^\dagger$, etc. over logics. We use also L^* for the L^*-formulas (sentences).[4] Satisfaction is denoted by $\models_{L^*}$ or, when no confusion is possible, by $\models$. L-structures are denoted by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{M}$, $\mathfrak{N}$, etc., their universes by A, B, M, N (or $A_1, \ldots, A_n$, etc., for the many-sorted case).

We denote by $\mathrm{Th}_{L^*}(\mathfrak{A}) = \{\varphi \in L^* | \mathfrak{A} \models \varphi\}$ and $\mathfrak{A} \equiv \mathfrak{B}(L^*)$ if $\mathrm{Th}_{L^*}(\mathfrak{A}) = \mathrm{Th}_{L^*}(\mathfrak{B})$. If L^* and $L^\dagger$ are two logics, we write $L^* <_{\mathrm{Th}} L^\dagger$ if whenever $\mathfrak{A} \equiv \mathfrak{B}(L^\dagger)$ then also $\mathfrak{A} \equiv \mathfrak{B}(L^*)$.

Let L^* be a logic. L^* is said to have the *Karp-property* (is a *Karp-logic*) if for every L and two L-structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{A} \cong_p \mathfrak{B}$ implies that $\mathfrak{A} \equiv \mathfrak{B}(L^*)$.

Here $\mathfrak{A} \cong_p \mathfrak{B}$ means that there is a set of partial isomorphisms with the Back and Forth Property (cf. [**Ba1**]). By a theorem of Karp [**Ba1**] this is equivalent to $\mathfrak{A} \equiv \mathfrak{B}(L_{\infty\omega})$.

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[4]Note that L^* is not always a set.

THEOREM 1.1 (BARWISE). *If L^* is a Karp-logic then $L^* <_{\text{Th}} L_{\infty\omega}$.*

A proof may be found in [**Ba3**].

THEOREM 1.2 (LINDSTRÖM/BARWISE). *If L^* is a Karp-logic in which $\langle \omega, < \rangle$ is not characterizable by a single sentence even with additional predicates, then $L^* = L_{\omega\omega}$.*

A proof may be found in [**Ba2**] or [**Li1**], [**Li2**].

A logic L is said to be (κ, λ)-*compact* $(\kappa, \lambda$ cardinals, $\kappa \geqslant \lambda)$ whenever given a set of sentences Σ of L^* of cardinality κ such that each subset Σ_0 of Σ of cardinality $< \lambda$ has a model, then Σ has a model. Given two logics L^* and $L^{\sharp}$ we write $L^* < L^{\sharp}$ if for every $\varphi \in L^*$ there is a $\varphi' \in L^{\sharp}$ with the same models, $L^* = L^{\sharp}$ if $L^* < L^{\sharp}$ and $L^{\sharp} < L^*$.

Note that if $\kappa_1 \leqslant \kappa$ and $\lambda_1 \geqslant \lambda$ then (κ, λ)-compactness implies (κ_1, λ_1)-compactness.

PROPOSITION 1.3. *A logic L^* is (ω, ω)-compact iff $\langle \omega, < \rangle$ is not characterizable by a countable set of sentences of L_1^* for some $L_1 \supset L$.*

A proof may be found in [**Ba2**] or [**F1**].

Let λ, κ, μ be cardinals and L^* a logic. We say that L^* has the (λ, κ)-*Löwenheim-Skolem property for sets of sentences of cardinality* μ and write $\text{LS}_\mu(\lambda, \kappa)$ if whenever Σ is a set of sentences of L^* of cardinality μ which has a model of cardinality λ then it has a model of cardinality κ. If μ is finite we omit it. $\text{LS}_\mu(\kappa)$ stands "for all $\lambda \geqslant \kappa \text{LS}_\mu(\lambda, \kappa)$".

PROPOSITION 1.4. *Let L^* be a fixed logic.*
(i) *If $\mu_0 \leqslant \mu$ and L^* has $\text{LS}_\mu(\kappa, \lambda)$ then L^* has $\text{LS}_{\mu_0}(\kappa, \lambda)$.*
(ii) *If $L^* < L^{\dagger}$ and $L^{\dagger}$ is (κ, λ)-compact (has $\text{LS}_\mu(\kappa, \lambda)$) so L^* is (κ, λ)-compact (has $\text{LS}_\mu(\kappa, \lambda)$).*

THEOREM 1.5. (i) *(Barwise) If L^* has $\text{LS}(\kappa, \omega)$ for all $\kappa \geqslant \omega$ then L^* is a Karp-logic, provided L is finite.*
(ii) *If L^* has $\text{LS}_\omega(\omega, \kappa)$ for some $\kappa > \omega$ then L is (ω, ω)-compact.*
(iii) *(Lindström) If L^* has $\text{LS}(\omega, \kappa)$ for some $\kappa > \omega$ and $\text{LS}(\omega)$ then $L^* = L_{\omega\omega}$.*
(iv) *(Lindström) If L^* is (ω, ω)-compact and has $\text{LS}(\omega)$ then $L = L_{\omega\omega}$.*

Proofs of (i) may be found in [**Ba1**] as well as for (ii), (iii) and (iv) follow using Theorem 1.2.

Shelah showed that $\text{LS}(\kappa, \omega)$ does not imply $\text{LS}_\omega(\kappa, \omega)$. A detailed study of $\text{LS}(\omega)$ may be found in [**Kue**].

The following is an easy exercise and will be used later.

PROPOSITION 1.6. *If a logic L^* staisfies* $\mathrm{LS}(\lambda)$ *then it has up to logical equivalence at most* 2^{2^λ}-*many sentences and therefore almost* $2^{2^{2^\lambda}}$-*many complete theories.*

2. Definability and interpolation properties. Let L^* be a logic, K a class of L-structures, and $\varphi \in L^*$. We define

$$\mathrm{Mod}(\varphi) = \{\mathfrak{A} | \mathfrak{A} \text{ is an } L\text{-structure and } \mathfrak{A} \vDash \varphi\}.$$

If $L_0 \subset L$,

$$\mathrm{Mod}(\varphi) \upharpoonright_{L_0} = \{\mathfrak{A} | \mathfrak{A} \text{ is an } L_0\text{-structure which has an expansion}$$
$$\text{to an } L\text{-structure } \mathfrak{A}' \vDash \varphi\},$$

$$K \in \mathrm{EC}(L^*) \text{ iff } K = \mathrm{Mod}(\varphi) \text{ for some } \varphi \in L^*,$$

$$K \in \mathrm{PC}(L^*) \text{ iff } K = \mathrm{Mod}(\varphi) \upharpoonright_L \text{ for some } \varphi \in L_1^*, L_1 \supset L.$$

Interpolation properties. Let L^* and $L^\dagger$ be two logics, $L^* < L^\dagger$. We say $\mathrm{CRAIG}(L^*, L^\dagger)$ *holds* if whenever $K_1, K_2 \in \mathrm{PC}(L^*)$ and $K_1 \cap K_2 = \varnothing$ then there is $K_3 \in \mathrm{EC}(L^\dagger)$ such that $K_1 \subseteq K_3$ and $K_3 \subseteq \bar{K}_2$ ($\bar{K}_2$ the complement of K_2 with respect to L-structures).

We say $\Delta\text{-Int}(L^*, L^\dagger)$ holds if whenever $K_1, K_2 \in \mathrm{PC}(L^*)$, $K_1 \cap K_2 = \varnothing$ and $K_1 \cup K_2 = \mathrm{str}(L)$ (= all L-structures) then $K_1 = \bar{K}_2 \in \mathrm{EC}(L^\dagger)$. $\Delta(L^*)$ is the smallest extension $L^\dagger$ of L^* such that $\Delta\text{-Int}(L^*, L^\dagger)$ holds. (For its existence and construction, cf. [MSS].)

Definability properties. We say that $\mathrm{BETH}(L^*, L^\dagger)$ ($\mathrm{WBETH}(L^*, L^\dagger)$) *holds*, if given $\varphi \in L_1^*$, $L_1 = L \cup \{P\}$, P an n-ary predicate symbol, every L-structure has at most (exactly one) expansion $\langle \mathfrak{A}, P \rangle \vDash \varphi$; then

$$\{\langle \mathfrak{A}, \bar{a} \rangle | \mathfrak{A} \text{ has an expansion } \langle \mathfrak{A}, P \rangle \vDash \varphi \text{ and } \bar{a} \in P^{\mathfrak{A}}\} \in \mathrm{EC}(L^\dagger).$$

Joint consistency. $T \subseteq L^*$ is L^*-*complete* if whenever T is a set, $\mathfrak{A} \vDash T$ and $\mathfrak{B} \vDash T$ then $\mathfrak{A} \equiv \mathfrak{B}(L^*)$. We say that $\mathrm{ROB}(L^*, L^\dagger)$ ($\mathrm{WROB}(L^*, L^\dagger)$) holds if whenever $T \subseteq L^\dagger$ is $L^\dagger$-complete, $L_1 = L \cup \{P\}$, P an n-ary predicate symbol, and $\varphi, \psi \in L_1^*$ are such that $T \cup \{\varphi(P)\}$ and $T \cup \{\psi(P)\}$ have a model, then $T \cup (\{\varphi(P), \psi(P')\})$ ($\{\varphi(P), \psi(P')\}$) has a model (where P' is a new n-ary predicate symbol not in L_1 and $\psi(P')$ is the result of substituting P' for P in ψ).

PROPOSITION 2.1. *Assume* $|L| < \mu$, $|L^\dagger| < \mu$ *for some infinite* μ *and that* $L^\dagger$ *is* (μ, ω)-*compact; then the following are equivalent*:
   (i) $\mathrm{CRAIG}(L^\dagger, L^\dagger)$,
   (ii) $\mathrm{ROB}(L^\dagger, L^\dagger)$,
   (iii) $\mathrm{WROB}(L^\dagger, L^\dagger)$.

The proof is left as an exercise.

Let DEF be any of the prefixes CRAIG, BETH, WBETH, ROB, WROB or ΔInt. The following is immediate:

PROPOSITION 2.2. *If L^*, $L^\dagger$, L^{**}, $L^{\dagger\dagger}$ are logics, $L^{**} < L^*$ and $L^\dagger < L^{\dagger\dagger}$, then* DEF($L^*$, $L^\dagger$) *implies* DEF(L^{**}, $L^{\dagger\dagger}$).

THEOREM 2.3. *The following diagrams are true*:

$$\begin{array}{ccc} & \text{CRAIG}(L^*, L^\dagger) & \\ \swarrow & & \searrow \\ \Delta\text{Int}(L^*, L^\dagger) & & \text{BETH}(L^*, L^\dagger) \\ \searrow & & \swarrow \\ & \text{WBETH}(L^*, L^\dagger) & \end{array} \qquad \text{(i)}$$

$$\begin{array}{ccc} \text{CRAIG}(L^*, L^\dagger) & & \text{ROB}(L^*, L^\dagger) \\ \searrow & & \swarrow \\ & \text{WROB}(L^*, L^\dagger) & \end{array} \qquad \text{(ii)}$$

The proofs are left as an exercise.

Before we can discuss other implications we need some facts from [MSS]:

Every logic L^* has a least extension $\Delta(L^*)$ which satisfies

$$\Delta\text{Int}(\Delta(L^*), \Delta(L^*)).$$

Similarly for BETH($B(L^*)$, $B(L^*)$) and WBETH(WB(L^*), WB(L^*)). The following is taken from [MSS].

THEOREM 2.4. *Assume that L^* is a logic.*

(i) *If L^* is (κ, λ)-compact, so are* WB(L^*) *and* $\Delta(L^*)$.

(ii) *If L^* satisfies* $\text{LS}_\mu(\kappa, \lambda)$ *so do* $\Delta(L^*)$ *and* WB(L^*).

(iii) WB(L^*) *does not preserve the Karp-property (neither do $B(L^*)$ nor $\Delta(L^*)$).*

It is an open question, whether a similar theorem holds for $B(L^*)$. There are many ways to define extensions $L^\dagger$ (minimal extensions) of a given logic L^* which satisfy CRAIG($L^\dagger$, $L^\dagger$). The problem is to define one which is in some reasonable sense canonical and satisfies an analogue of Theorem 2.4.

THEOREM 2.5. (i) BETH(L^*, $L^\dagger$) $\not\Rightarrow$ $\Delta\text{Int}(L^*, L^\dagger)$.

(ii) $\Delta\text{Int}(L^*, L^\dagger)$ $\not\Rightarrow$ BETH(L^*, $L^\dagger$).

So no other implication in the diagram of 2.2 does hold.

PROOF. To prove 2.5 we shall construct two new logics and calculate their BETH-closure and Δ-closure:

(i) Let $L^{wo} = L_{\omega\omega}[Q^{wo}]$ be the logic obtained from $L_{\omega\omega}$ by adjunction of a binary quantifier $Q^{wo}xy\varphi(x, y)$ which says that $\varphi(x, y)$ well-orders its field. L^{wo} satisfies LS(ω) since it is a special case of a securable quantifier discussed in [Ma1]. Note that by Theorem 6.1 below WBETH(L^{wo}, L^{wo}) does not hold. To calculate $B(L^{wo})$ we prove:

LEMMA 2.6. *If L contains only unary predicate symbols and equality, then $L_{\omega\omega} = L^{wo} = B(L^{wo})$.*

OUTLINE OF PROOF. We prove by induction on the explicit construction of $B(L^{wo})$ that for such L as in the hypothesis there are no formulas satisfying the hypothesis of the Beth-property in the nontrivial way.

To calculate $\Delta(L^{wo})$ we observe:

LEMMA 2.7. $L(Q_0) \subseteq \Delta(L^{wo})$.

PROOF. Since $\langle \omega, < \rangle$ is characterizable by a sentence of L^{wo} the class of finite sets is in $PC(L^{wo})$, as well as the class of infinite sets. □

Now Lemma 2.6 shows us that $L(Q_0)$ is not a sublogic of $B(L^{wo})$, but by 2.7 it is a sublogic of $\Delta(L^{wo})$, hence of $\Delta(B(L^{wo})) \neq B(L^{wo})$.

(ii) In [MS3] it is proven that $\Delta(L_{\omega\omega}[Q_1])$ does not satisfy Beth's theorem. □

THEOREM 2.8. (i) $CRAIG(L^*, L^\dagger) \not\Rightarrow ROB(L^*, L^\dagger)$,

(ii) $ROB(L^*, L^\dagger) \not\Rightarrow WBETH(L^*, L^\dagger)$,

(iii) $\Delta Int(L^*, L^\dagger) \Rightarrow WROB(L^*, L^\dagger)$.

PROOF. (i) Put $L^* = L_{\omega_1\omega}$, $L^\dagger = L_{\omega_1\omega}$. We have $CRAIG(L_{\omega_1\omega}, L_{\omega_1\omega})$ by [Ke2, p. 19] and not $ROB(L_{\omega_1\omega}, L_{\omega_1\omega})$ by [Ke2, exercise 4, p. 22] (cf. also 2.14).

(ii) Put

$$L^* = L_{\omega_1\omega_1} \quad \text{and} \quad L^\dagger = L_{\infty\infty}.$$

Malitz showed that $WBETH(L_{\omega_1\omega_1}, L_{\infty\infty})$ fails (cf. [GH, Theorem 2]). To prove $ROB(L_{\omega_1\omega_1}, L_{\infty\infty})$ we note first that a complete theory in $L_{\infty\infty}$ is always categorical, so the Robinson-property follows trivially.

(iii) Hutchinson (cf. [MSS]) showed that $\Delta(L_{\omega\omega}(Q_1))$ (with quantifier "there exist uncountably many") does not satisfy Craig's theorem, in fact not $CRAIG(L_{\omega\omega}(Q), \Delta(L_{\omega\omega}(Q)))$. But $\Delta(L_{\omega\omega}(Q))$ is (ω, ω)-compact, so by Propositions 2.1 and 2.2 the result follows, with $L^* = L^\dagger = \Delta(L_{\omega\omega}(Q))$. □

THEOREM 2.9. (i) $BETH(L^*, L^\dagger) \not\Rightarrow WROB(L^*, L^\dagger)$.

(ii) *All possible relations between two definability properties are exhibited.*

PROOF. (i) Let $L^{cf\,1}$ be the logic obtained from $L_{\omega\omega}$ by adjunction of a binary quantifier $Q^{cf\,1}xy\varphi(x, y)$ which says that $\varphi(x, y)$ orders its field in an ω_1-like way. As in 2.4(i) we need a lemma.

LEMMA 2.10. *If L contains only unary predicate symbols and equality then* $B(L^{cf\,1}) = L^{cf\,1} = L_{\omega\omega}$.

The proof is as for Lemma 2.6. To end the proof of 2.9 we need:

PROPOSITION 2.11. $L^{cf\,1}$ *does not satisfy* $WROB(L^{cf\,1}, L^{cf\,1})$.

PROOF. By Lemma 2.10 and first order model theory the L-theory T_{inf} of infinite sets for $L = \{=\}$ is complete in $L^{cf\,1}$. Let φ_1 say that an ordering $<_1$ of a model is ω_1-like and let φ_2 say that an ordering $<_2$ is a proper initial segment of an ω_1-like ordering (using additional predicates). Clearly $T_{inf} \cup \{\varphi_i\}$ has a model for each $i = 1,2$, but $T_{inf} \cup \{\varphi_1, \varphi_2\}$ has no model. $\square$

To prove (ii) of 2.9 one uses a finitary checking argument. $\square$

For the rest of this section we look at Karp-logics, modifying results of Barwise.

THEOREM 2.12. *Let L^* be a Karp-logic (hence $L^* <_{Th} L_{\infty\omega}$).*
(i) *BETH(L^*, $L_{\infty\omega}$) implies WROB(L^*, $L_{\infty\omega}$).*
(ii) *WROB(L^*, $L_{\infty\omega}$) implies that L^* satisfies LS(ω).*

COROLLARY 2.13 [Ba]. *If L^* is a Karp-logic and CRAIG(L^*, L^*) holds then LS(ω) holds.*

The corollary follows from the theorem via Proposition 2.2.
PROOF OF THE THEOREM. (i) will follow from Theorem 5.3 taking $L^* = L^\dagger = L_{\infty\omega}$.
(ii) By 1.2 and 1.5 we may assume that $\langle\omega,<\rangle$ is characterizable by $\varphi \in L^*$. Assume for contradiction that ψ is a formula in L^* with no countable but some uncountable models. Clearly $\{\varphi, \psi\}$ has no model. Let T be the $L_{\infty\omega}$-theory of infinite sets. Since all infinite sets are partially isomorphic, T is $L_{\infty\omega}$-complete. But $T \cup \{\varphi\}$ and $T \cup \{\psi\}$ have models, a contradiction. $\square$

THEOREM 2.14. *Assume L^* is a Karp-logic, ROB(L^*, L^*) holds and for every L-structure $\mathfrak{M}$, $Th_L^*(\mathfrak{M})$ is equivalent to a set. Then $L^* = L_{\omega\omega}$.*

PROOF. Again we can assume that $\langle\omega,<\rangle$ is characterizable by a sentence φ in L^*. Put $\mathfrak{M}_1 = \langle\omega,<,\dots\rangle$ and $\mathfrak{M}_2 = \langle C, c_\alpha : \alpha < \omega_1\rangle$, a pure set with equality and ω_1-many distinct constants.
Put $\mathfrak{M}_3^0 = |\omega_0|$ and $\mathfrak{M}_3^1 = |\omega_1|$. Using the Back and Forth characterization of $\cong_p$ (cf. [Ba1]) we get that $[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^0] \cong_p [\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^1]$. Put

$$T = Th_{L_1}\big[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^0\big] = Th_{L_1}\big[\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3^1\big],$$

so T can be assumed to be a set and T is complete. Let ψ_i say that g_i is a bijection from $|\mathfrak{M}_i|$ to $|\mathfrak{M}_3|$ ($i = 1,2$). Clearly $T \cup \{\psi_i\}$ has a model ($i = 1,2$) but $T \cup \{\psi_1, \psi_2\}$ has no model since $\mathfrak{M}_1 \vDash \varphi$ and therefore $\mathfrak{M}_1$ is countable and $\mathfrak{M}_2$ is uncountable, a contradiction. $\square$

COROLLARY 2.15. *Assume $L^* \leqslant L_{\infty\omega}$ and ROB(L^*, $L_{\infty\omega}$). Then $L^* = L_{\omega\omega}$.*

PROOF. Note that for every structure M, $Th_{L_{\infty\omega}}(M)$ is equivalent to a set, and repeat the proof of 2.14. $\square$

3. Feferman-Vaught-type theorems. One basis construction in abstract model theory is the operation of taking sums of given models. In its simplest form we have the operation of pairing which associates with two structures $\mathfrak{A}, \mathfrak{B}$ the structure $[\mathfrak{A}, \mathfrak{B}]$ for the language L which is the many-sorted disjoint union of the languages of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Similarly one can form triplets, etc., or more complicated objects, of which one will be studied in §5.

DEFINITION. Let L_1, L_2 be languages and L_i^*, $L_i^\dagger$ $(i = 1, 2)$ logics with $L_i^* < L_i^\dagger$. Let $\mathfrak{A}_i$, $\mathfrak{A}_i'$ be L_i-structures. We say that $\mathrm{FV}_p(L^*, L^\dagger)$ holds if, whenever $\mathfrak{A}_i \equiv \mathfrak{A}_i'(L_i^\dagger)$ for $i = 1, 2$ and L_3 is the language of the pair $[\mathfrak{A}_0, \mathfrak{A}_1]$ then $[\mathfrak{A}_1, \mathfrak{A}_2] \equiv [\mathfrak{A}_1', \mathfrak{A}_2'](L_3^*)$. We say that $\mathrm{WFV}_p(L^*, L^\dagger)$ holds whenever $\mathrm{FV}_p(L^*, L^\dagger)$ holds for $L_1 = L_2 = L$ and $\mathfrak{A}_1 \equiv \mathfrak{A}_2(L^\dagger)$ (F stands for Feferman, V for Vaught and W for Weak).

Remember that the pair $[\mathfrak{A}_1, \mathfrak{A}_2]$ is the structure with the universes from $\mathfrak{A}_1$ and from $\mathfrak{A}_2$ and relations acting on the appropriate universes. It is, loosely speaking, the disjoint union of the structures $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with variables ranging over the universes of $\mathfrak{A}_1$ and variables ranging over the universes of $\mathfrak{A}_2$.

Note that if $L^{**} < L^*$ and $L^{\dagger\dagger} > L^\dagger$, then $X(L^*, L^\dagger)$ implies $X(L^{**}, L^{\dagger\dagger})$ where X is one of the properties above.

THEOREM 3.1. *Assume L^* is a logic which satisfies* WBETH(L^*, L^*) *and* WFV$_p(L^*, L^*)$ *and* LS(λ) *for some* $\lambda < \beth_{\omega_1 ck}$. *Then* $\langle \omega, < \rangle$ *is not characterizable by a single sentence.*

PROOF. We give a proof for the case that $\lambda < \beth_\omega$, the general case being essentially the same but more complicated. Assume for contradiction that $\varphi \in L^*$ does characterize $\langle \omega, < \rangle$. By assumption and 1.6 there are at most 2^{2^λ}-many theories over countable types. Consider the structure $\mathfrak{A} = \langle V, \omega, P^n(\omega), \varepsilon \rangle$ where $V = \bigcup_{k \in \omega} P^k(\omega)$, $n \in \omega$, such that $2^{2^\lambda} < \beth_n$, and $P^n(\omega)$ is the nth iteration of the power set operation. Now consider the structure $[\mathfrak{A}, \mathfrak{A}]$ and let ψ be the formula which says "ω is standard" using φ:

"F is a partial map from V of the first sort (i.e., V_1) to V of the second sort (i.e., V_2)";

"F and F^{-1} preserve ε";

"F is hereditary, i.e. if F is defined on x and $y \in x$, it is defined on y";

"The domain of F is maximal".

Clearly ψ defines F implicitly and ψ is in L^*. Since there are at most 2^{2^λ}-many theories we may find two structures $\mathfrak{A}_1 = \langle V_1, \omega, P_1, \varepsilon \rangle$ and $\mathfrak{A}_2 = \langle V_2, \omega, P_2, \varepsilon \rangle$ such that $\mathfrak{A}_1 \equiv \mathfrak{A}_2(L^*)$ and $P_1 \neq P_2$ but $P_1, P_2 \subset P^n(\omega)$.

So by WFV$_p(L^*, L^*)$ we have $[\mathfrak{A}_1, \mathfrak{A}_2] \equiv [\mathfrak{A}_1, \mathfrak{A}_1](L^*)$. But assuming that $\theta(x, y)$ defines F explicitly in L^* we get that $\theta(x, y)$ defines a map with domain P_1 on $[\mathfrak{A}_1, \mathfrak{A}_1]$, but not on $[\mathfrak{A}_1, \mathfrak{A}_2]$, a contradiction. So $\langle \omega, < \rangle$ is not characterizable in L. In fact with more coding it suffices to assume that $\lambda < \beth_{\omega_1 ck}$. $\square$

COROLLARY 3.2. *The following properties characterize $L_{\omega\omega}$:*
(i) LS(ω), WFV$_P(L_{\omega\omega}, L_{\omega\omega})$ *and* WBETH($L_{\omega\omega}, L_{\omega\omega}$).
(ii) *The Karp-property*, WFV$_P(L_{\omega\omega}, L_{\omega\omega})$ *and* WBETH($L_{\omega\omega}, L_{\omega\omega}$).

Compare 3.2 also with 4.2.
The following is an abstract version of a result of Malitz [MI] cf. also [GH].

THEOREM 3.3. *Let L^* be a set for all languages L and assume that $L^\square \subset L^\dagger \subset L^*$ are three logics such that* WBETH($L^\square, L^\dagger$) *and* WFV$_P(L^\dagger, L^*)$. *Then the class of well-orderings is not* EC($L^\square$).

PROOF. Let $\alpha < \beta$ be two infinite ordinals such that $\langle \alpha, \varepsilon \rangle \equiv \langle \beta, \varepsilon \rangle(L^*)$. They exist for L^* is a set. Now look at two-sorted structures of the form $[\mathfrak{A}, \mathfrak{B}]$ where $\mathfrak{A}$ and $\mathfrak{B}$ are well-orderings and let F be a binary function symbol with domain $\mathfrak{A}$ and range $\mathfrak{B}$. Assume for contradiction that the class of well-orderings is EC($L^\square$).Let θ be a sentence expressing that:
(i) $\mathfrak{A}$ and $\mathfrak{B}$ are well-orderings,
(ii) F is an order-preserving map which is one-to-one,
(iii) either F is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$ or of $\mathfrak{B}$ into $\mathfrak{A}$.
Now θ satisfies the hypothesis of WBETH($L^\square, L'$). Therefore there is $\varphi(x, y)$ in $L^\dagger$, L the language of $[\mathfrak{A}, \mathfrak{B}]$ without F, defining F. Now put $\mathfrak{A} = \langle \alpha, \varepsilon \rangle$, $\mathfrak{B} = \langle \beta, \varepsilon \rangle$. By WFV$_P(L^\dagger, L^*)$ we have $[\mathfrak{A}, \mathfrak{B}] \equiv [\mathfrak{B}, \mathfrak{A}](L^\dagger)$. Put φ_1 to express that $\varphi(x, y)$ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$ and φ_2 that $\varphi(x, y)$ is an embedding of $\mathfrak{B}$ into $\mathfrak{A}$.
So we have $[\mathfrak{A}, \mathfrak{B}] \vDash \varphi_1 \wedge \neg \varphi_2$ and $[\mathfrak{B}, \mathfrak{A}] \vDash \varphi_2 \wedge \neg \varphi_1$, a contradiction.
Theorem 3.1 can be easily generalized to:

THEOREM 3.4. *Let $L^* \subset L^\dagger \subset L^\square$ be three logics, such that* WBETH($L^*, L^\dagger$) *and* WFV$_P(L^\dagger, L^\square)$ *and* L *satisfies* LS(λ) *for some* $\lambda < \beth_{\omega_1}$. *Then $\langle \omega, < \rangle$ is not characterizable by a single sentence of L^*.*

In many cases, where FV$_P$ does not hold, it may still hold for specially chosen structures. How this is used is shown in [MS2], [MS3].

4. Uniform reduction. In Feferman [Fe2] and Makowsky [Ma2] a property of logics is studied which has two advantages: It comes nearer to the original Feferman-Vaught theorem and it is directly related to Craig's theorem.
Let $\mathfrak{A}_i$ be an L_i-structure ($i = 1,2$) and let L_3 be a language for $[\mathfrak{A}_1, \mathfrak{A}_2] = P(\mathfrak{A}_1, \mathfrak{A}_2)$. For a logic L^* we say that UR$_P(L^*)$ holds if for every sentence φ of L_3^* there exists a pair of sequences of formulae $\psi_1^1, \ldots, \psi_{n_1}^1$ of L_1^* and $\psi_1^2, \ldots, \psi_{n_2}^2$ of L_2^* and a Boolean function $B \in 2^{n_1+n_2}$ such that

$$\left[\mathfrak{A}_1, \mathfrak{A}_2\right] \vDash \varphi \text{ iff } B\left(a_1^1, \ldots, a_{n_1}^1, a_1^2, \ldots, a_{n_2}^2\right) = 1$$

where

$$a_j^i = \begin{cases} 0 & \text{if } \mathfrak{A}_i \vDash \neg \psi_j^i, \\ 1 & \text{if } \mathfrak{A}_i \vDash \psi_j^i. \end{cases}$$

THEOREM 4.1. *Assume* $\mathrm{UR}_P(L^*)$ *holds for a logic* L^*. *Then* $\mathrm{FV}_P(L^*, L^*)$ *holds*.

The proof is left to the reader. We now want to generalize UR_P (uniform reduction) to a more general class of operations. Let L_i ($i = 1, \ldots, n \in \omega$) be languages, $\mathfrak{A}_i$ be L_i-structures and R an n-ary relation on structures such that the ith argument is an L_i-structure. Let L_0 be the language for a structure $[\mathfrak{A}_1, \ldots, \mathfrak{A}_n]$. R is said to be *projective* in the logic L^* if there is a PC'-class K for L^* such that $R(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ iff $[\mathfrak{A}_1, \ldots, \mathfrak{A}_n] \in K$. A sentence $\varphi \in L_n^*$ is said to be *invariant on the range of* R if for all $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n, \mathfrak{A}'_n$ with $R(\mathfrak{A}_1, \mathfrak{A}_{n-1}, \mathfrak{A}_n)$ and $R(\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}'_n)$ we have $\mathfrak{A}_n \vDash \varphi$ iff $\mathfrak{A}'_n \vDash \varphi$.

An n-tuple of sequences of formulas $\bar{\psi}_1, \ldots, \bar{\psi}_{n-1}$ with $\bar{\psi}_k = \psi_1^k, \ldots, \psi_{m_k}^k$ and ψ_i^k in L_k^* together with a Boolean function $B \in 2^m$, $m = \sum_{k=0}^{n-1} m_k$ is called *an associate pair for* φ *on the domain of* R if for all $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1}, \mathfrak{A}_n$ we have that $R(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$ implies $\mathfrak{A}_n \vDash \varphi$ iff $B(a_1^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_{n-1}}^{n-1}) = 1$ where

$$a_j^i = \begin{cases} 1 & \mathfrak{A}_i \vDash \psi_j^i, \\ 0 & \text{otherwise.} \end{cases}$$

A logic L^* is said to satisfy $\mathrm{UR}_n(L^*)$ (*uniform reduction for n-ary relations*) if for every n-ary relation R which is projective in L^* and for every φ in L_n^* which is invariant on the range of R there is an associate pair for φ on the domain of R.

THEOREM 4.2. (i) $\mathrm{UR}_1(L^*)$ *iff* $\mathrm{CRAIG}(L^*, L^*)$.
(ii) $\mathrm{UR}_2(L^*)$ *implies* $\mathrm{UR}_P(L^*)$.
(iii) $\mathrm{UR}_2(L^*)$ *implies* $\mathrm{UR}_n(L^*)$.

The proofs of (i)–(iii) may be found in [Fe2] and [Ma1].

COROLLARY 4.3. *If* L^* *is a logic with* $\mathrm{UR}_2(L^*)$ *and the Karp-property and* L^* *is a set, then* $L^* = L_{\omega\omega}$.

PROOF. Use Theorems 4.2 and 2.14. □

5. Constructing implicit definitions. In this section we study a construction which enables us to obtain implicit definitions. The construction originates in the ideas of Friedman [Fr] and Gregory [Gr]. Shelah realized that the construction can be formulated in an abstract setting.

Let $L = L_0 \cup \{P\}$ be a language with a distinguished predicate symbol P

(unary for simplicity). Let $M = L \cup \{f, C\}$ where f is a unary function symbol and C a constant, both not in L. Let $\mathfrak{A}$, $\mathfrak{B}$ be L-structures.

We define now $S_P^i(\mathfrak{A}, \mathfrak{B})$ $(i = 0,1)$.

(i) $S_P^i(\mathfrak{A}, \mathfrak{B})$ is an M-structure with universe N of cardinality $\text{card}(\mathfrak{A}) + \text{card}(\mathfrak{B}) + \aleph_0$.

(ii) f is a function on N such that

(a) $f(x) = x$ iff $C = x$,

(b) for all $a \in N$ there is $n \in \omega$ with $f^n(a) = C$,

(c) f is onto.

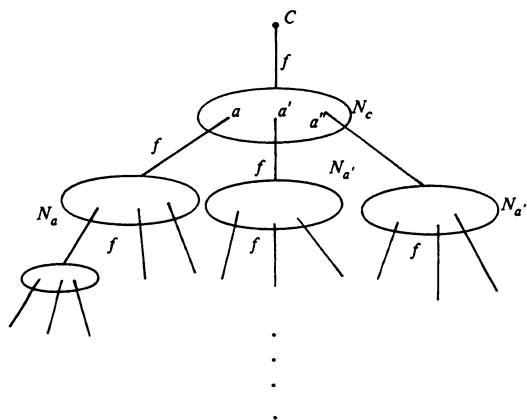Denote $f^{-1}(a) - \{a\}$ by N_a (see Figure 1).

FIGURE 1

(iii) There is a bijection $i_a \colon \mathcal{C} \to N_a$ $(i = 0,1)$ where $\mathcal{C}$ is either $\mathfrak{A}$ or $\mathfrak{B}$.

i_a makes N_a naturally into an $L_0 \cup \{P\}$-structure which we shall denote also by N_a.

(iv) for each relation symbol $R \in L$ (or function symbol) put R_a to be its interpretation on N_a and put $R_N = \bigcup_{a \in N} R_a$ its interpretation on N.

(v) P_N is defined by $a \in P_N \Rightarrow N_a \cong \mathfrak{A}$, $a \notin P_N \Rightarrow N_a \cong \mathfrak{B}$.

(vi) $C \in P_N$ if $i = 1$, $C \notin P_N$ if $i = 0$.

$S_P^0(\mathfrak{A}, \mathfrak{B})$ and $S_P^1(\mathfrak{A}, \mathfrak{B})$ differ only with respect to N_C. $S_P^i(\mathfrak{A}, \mathfrak{B})$ is a tree with root C and nodes N_a. At the node N_a it splits $|N_a|$-many times. Each branch has length ω. $N_a \cong \mathfrak{A}$ iff $f(a) \in P$ and $N_a \cong \mathfrak{B}$ iff $f(a) \notin P$ (see Figure 2).

Now we put $T_P^i(\mathfrak{A}, \mathfrak{B}) := S_P^i(\mathfrak{A}, \mathfrak{B}) \setminus L_0 \cup \{f, c\}$. We say that WFVT($L^*$, $L^\dagger$) holds if $\mathfrak{A} \equiv \mathfrak{B}(L^\dagger)$ implies that $T_P^0(\mathfrak{A}, \mathfrak{B}) \equiv T_P^1(\mathfrak{A}, \mathfrak{B})(L^*)$.

PROPOSITION 5.1. *If* $\langle \omega, < \rangle$ *is* PC(L^*). *Then* UR$_2$(L^*) *implies* WFVT(L^*, L^*).
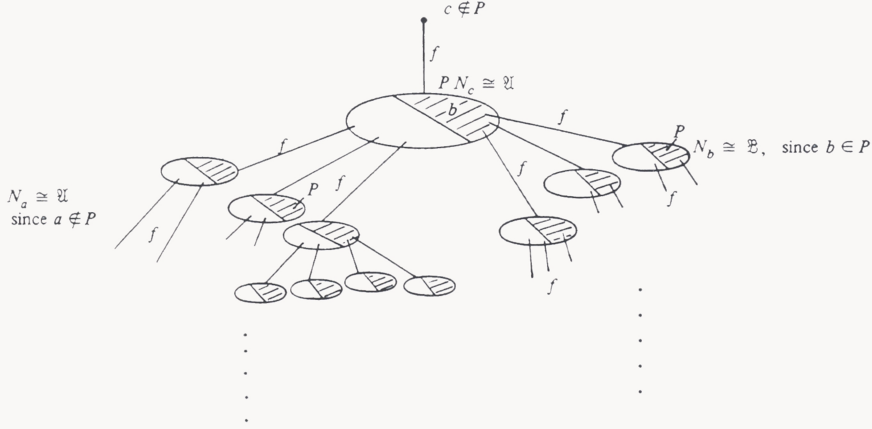
FIGURE 2

PROOF. $S_P^i(\mathfrak{A}, \mathfrak{B})$ is a projective operation via the characterization of $\langle \omega, < \rangle$, if we replace it by a ternary operation $S_P(\mathfrak{A}', \mathfrak{A}, \mathfrak{B})$ restricted to the cases $\mathfrak{A}' \cong \mathfrak{A}$ or $\mathfrak{A}' \cong \mathfrak{B}$, so that the first argument describes N_C and the second argument describes N_a for $a \in P$, the third N_a for $a \notin P$. $\square$

LEMMA 5.2. *Let* L^* *be a logic,* L *a language and* P_i ($i = 0, 1, 2$) *be unary predicate symbols not in* L.

Let φ_i ($i = 1,2$) *be sentences having a model but* $\{\varphi_1, \varphi_2\}$ *has no model. Then there is a sentence* $\psi \in (L \cup \{f, c, P_0\})^*$ (f, c *not in* L) *such that*:

(i) *Every* $L \cup \{f, c\}$-*structure* $\mathfrak{A}$ *has at most one expansion* $\mathfrak{A}^* \vDash \psi$.

(ii) $S_{P_0}^i(\mathfrak{A}, \mathfrak{B}) \vDash \psi$ *provided* $\mathfrak{A} \vDash \varphi_1$ *and* $\mathfrak{B} \vDash \varphi_2$ *where* P_1 *and* P_2, *respectively, are substituted for* P_0.

PROOF. (i) Let ψ be $\psi_1 \wedge \psi_2$ with

$$\psi_1 \colon f(c) = c \wedge f(x) = x \Leftrightarrow x = C \wedge \text{``} f \text{ is onto''};$$

$$\psi_2 \colon a \in P_0 \Rightarrow N_a = \left\langle f^{-1}(a) - \{a\}, L \cup \{f, c, P_0\} \right\rangle \vDash \varphi_1,$$

$$a \notin P_0 \Rightarrow N_a = \left\langle f^{-1}(a) - \{a\}, L \cup \{f, c, P_0\} \right\rangle \vDash \varphi_2.$$

Note that $\psi_1 \in L_{\omega\omega}$ and $\psi_2 \in L^*$. Now let $\mathfrak{A}$ be an $L \cup \{f, c\}$-structure and $\langle \mathfrak{A}, P_i \rangle \vDash \psi$ ($i = 1,2$) (ignoring substitutions of P_i for P_0 in ψ). We have to show that $P_1 = P_2$. If $a \in P_1$ then $N_a \vDash \varphi_1$, if $a \notin P_2$ then $N_a \vDash \varphi_2$ so

$$\left\langle f^{-1}(a) - \{a\}, L \cup \{f, c, P_1, P_2\} \right\rangle \vDash \varphi_1 \wedge \varphi_2,$$

a contradiction.

(ii) is obvious by the construction of $S_P^i(\mathfrak{A}, \mathfrak{B})$. $\square$

THEOREM 5.3. *Let* $L^* < L^\dagger$ *be logics* φ_i ($i = 1,2$) *as in Lemma 5.2 and assume that there are* $\mathfrak{A}_i \vDash \varphi_i$ *such that* $S_P^0(\mathfrak{A}_0, \mathfrak{A}_1) \equiv S_P^1(\mathfrak{A}_0, \mathfrak{A}_1)(L^\dagger)$. *Then* BETH($L^*$, $L^\dagger$) *fails*.

PROOF. By Lemma 5.2, ψ defines P implicitly. So assume for contradiction there is θ in $L^\dagger$, an $L \cup \{f, c\}$-formula, such that, for all $L \cup \{f, c\}$-structures $\mathfrak{A}$, $\mathfrak{A} \vDash \theta(a)$ iff there is an expansion $\mathfrak{A}^* \vDash \psi \wedge P(a)$.

Then we have $S_P^0(\mathfrak{A}_0, \mathfrak{A}_1) \vDash \neg \theta(c)$ and $S_P^1(\mathfrak{A}_0, \mathfrak{A}_1) \vDash \theta(c)$ which contradicts $S_P^0(\mathfrak{A}_0, \mathfrak{A}_1) \equiv S_P^1(\mathfrak{A}_0, \mathfrak{A}_1)(L^\dagger)$. □

The abstract version of 5.3 is the following:

THEOREM 5.4. *Assume $L^0 < L^* < L^\dagger$ are logics such that*
(i) BETH(L^0, L^*),
(ii) WFVT($L^*, L^\dagger$).
Then WROB($L^0, L^\dagger$) *holds.*

PROOF. Assume WROB($L^0, L^\dagger$) does not hold then we have φ_i ($i = 0,1$) as in Lemma 5.2 (if the P_i are not unary we have the obvious modification of the definition of $S_P^i(\mathfrak{A}, \mathfrak{B})$). Also there are $\mathfrak{A}_i \vDash \varphi_i$ such that $\mathfrak{A}_0 \equiv \mathfrak{A}_1(L^\dagger)$ and $S_P^0(\mathfrak{A}_0, \mathfrak{B}_1) \equiv S_P^1(\mathfrak{A}_0, \mathfrak{B}_1)(L^*)$, so we can apply Theorem 5.3 and BETH(L^0, L^*) fails. □

Note that we have BETH($L_{\omega_1\omega}, L_{\infty\omega}$) and WFVT($L_{\infty\omega}, L_{\infty\omega}$) but not ROB($L_{\omega_1\omega}, L_{\infty\omega}$) (by 2.15), so the theorem cannot be improved (cf. §8).

6. Compactness and definability. The aim of this section is to find sufficient conditions for a logic to be compact, more precisely to study definability criteria which imply compactness. It was Lindström who observed that:

THEOREM 6.1. *If $L^* = L_{\omega\omega}[Q^i]_{i<n<\omega}$ is finitely generated and* WBETH(L^*, L^*) *holds, then $\langle \omega, < \rangle$ is not characterizable by a single sentence.*

This implies that L^* is recursively compact, i.e. if Σ is a recursive set of sentences, such that every finite subset of Σ has a model, then Σ has a model.

PROOF OF 6.1. Assume for contradiction that $\psi \in L^*$ characterizes $\langle \omega, <, \overline{R} \rangle$ with $L = \{<, R_1, \ldots, R_m\}$ including enough number theory. Let $L_1 = L \cup \{c\}$ and $\{\psi_n | n \in \omega\}$ a recursive enumeration of L_1 where c is a constant-symbol. (Here we use the fact that L^* is finitely generated.)

Consider the set of formulas of L_2^* ($L_2 = L_1 \cup \{P\}$): $\Sigma = \{P(n) \Leftrightarrow \psi_n \wedge c = n | n \in \omega\}$ where P is an unary predicate symbol not in L_1. Σ is recursive; therefore it is replaceable by some formula $\Phi(P, n)$ using a trick due to Kleene which has been generalized by Lindström for finitely generated logics ([Kl], [MSS]). Now $\Phi \wedge \psi$ defines P implicitly, since the only model of Σ is $\langle \omega, <, \overline{R} \rangle$ and we have enough number theory.

By assumption there is $\theta(c') \in L_1^*$ such that for every L_2-structure $\mathfrak{A}$ we have $\mathfrak{A} \vDash P(c) \Leftrightarrow \theta(c')$. But $\neg \theta(c')$ is ψ_m for some $m \in \omega$. So we have $\mathfrak{A} \vDash P(m)$ iff $\mathfrak{A} \vDash \theta(m)$ iff $\mathfrak{A} \vDash \neg \psi_m(m)$ iff $\mathfrak{A} \vDash \neg P(m)$, a contradiction. □

Väänänen has observed that this can be generalized to an extended version

of BETH(L^*, L^*) where sentences are replaced by countable sets of sentences in the hypotheses only. The conclusion of 6.1 then changes to L^* is (ω, ω)-compact.

The main result of this section is:

THEOREM 6.2. *Assume* $L^* = L_{\omega\omega}[Q^i]_{i\in\omega}$ *and* $2^{\aleph_0} < 2^{\aleph_n}$ *for some* $n \in \omega$. *If* L^* *satisfies* ROB(L^*, L^*) *and* FV$_P$(L^*, L^*) *then* L^* *is* (κ, ω)-*compact for every* κ.

Theorem 6.2 is due to Shelah and we shall see in Theorem 6.8 and following that the precise set-theoretic conditions. Here the hypothesis that L^* is countably generated is for simplicity only. The proof of 6.2 involves two aspects: a set-theoretic one and a model-theoretic one. Instead of proving compactness we study a related notion.

DEFINITION. L^* is μ-rc (μ-*relatively compact*) if for any two sets of sentences $\Sigma, \Sigma_1 \in L^*$ with $|\Sigma| = \mu$ and $|\Sigma_1|$ arbitrary, if for every $\Sigma_0 \subset \Sigma$ with $|\Sigma_0| < \mu$, $\Sigma_0 \cup \Sigma_1$ has a model then $\Sigma \cup \Sigma_1$ has a model.

The following collects some simple facts about μ-rc logics.

LEMMA 6.3. (i) *If* μ *is regular,* L^* *is* μ-rc *and* T_i ($i < \mu$) *an increasing family of theories of* L^* *such that each* T_i *has a model then* $\bigcup_{i<\mu}T_i = T$ *has a model.*
(ii) *If* L^* *is* (cf μ)-rc *then* L^* *is* μ-rc.
(iii) *If* L^* *is* μ-rc *then* L^* *is* (μ, μ)-*compact.*
(iv) L^* *is* (λ, μ)-*compact iff for all* κ *with* $\lambda > \kappa \geqslant \mu$ L^* *is* (κ, κ)-*compact.*
(v) *If* L^* *is* κ-rc *for all* κ $\lambda \geqslant \kappa \geqslant \mu$ *then* L^* *is* (λ, μ)-*compact.*

PROOF. (i) Let P_i ($i < \mu$) be unary predicates not in T and ψ_i be $\exists x P_i(x)$. Put $\Sigma_1 = \{\psi_i \to \varphi | i < \mu$ and $\varphi \in T_i\}$ and $\Sigma = \{\psi_i | i < \mu\}$. Since L^* is μ-rc we verify that Σ and Σ_1 satisfy the hypothesis of μ-rc. $|\Sigma| = \mu$ by definition of Σ. Now let $\Sigma_0 \subset \Sigma |\Sigma_0| < \mu$. W.l.o.g. Σ is well-ordered of order type μ and Σ_0 is an initial segment say $\Sigma_0 = \Sigma \restriction_\lambda$. We have to show that $\Sigma_0 \cup \Sigma_1$ has a model. Let M be a model of T_λ and expand to M' such that $P_i^M \neq \varnothing$ for $i < \lambda$. Obviously $M' \vDash \Sigma_0 \cup \Sigma_1$. Using μ-rc we conclude that $\Sigma_1 \cup \Sigma$ has a model, hence T has a model.

(ii) Let Σ, Σ_1 satisfy the hypothesis required for μ-rc and L^* be cf μ-rc. There is $\Sigma'_\alpha \alpha < \mu$ such that $\bigcup_{\alpha<\mathrm{cf}\mu}\Sigma'_\alpha = \Sigma$ and $|\Sigma'_\alpha| < \mu$. Now put $\Sigma_\alpha = \Sigma'_\alpha \cup \Sigma_1$. Each Σ_α ($\alpha < \mu$) has a model, so, by (i), $\bigcup_{\alpha<\mathrm{cf}\mu}\Sigma_\alpha$ has a model.

(iii) is obvious, taking $\Sigma_1 = \varnothing$.
(iv) easy.
(v) Use (iii) and (iv). □

DEFINITION. A cardinal κ is *weakly characterizable* in a logic L^* if there is a language L_κ containing $\{E\}$ and a theory Γ_κ in L_κ^* such that:
(i) there is an expansion $\mathfrak{A}_\kappa$ of $\langle \kappa, E \rangle$ satisfying Γ_κ.
Let a_i be the name of $i \in \kappa$;

(ii) whenever $\mathfrak{B} \vDash \Gamma_\kappa \cup \{a_i E a_j : i < j < \kappa\}$ then $\{a_i : i < \kappa\}$ is unbounded in κ.

We say that such a Γ_κ weakly characterizes κ.

LEMMA 6.4. (i) *If μ is regular then L^* is μ-rc iff $\langle \mu, < \rangle$ is not weakly characterizable in L^*.*

(ii) *If L^* is μ-rc but not λ-rc, λ regular and $\lambda < \mu$ then there is a uniform ultrafilter D over μ which is λ-descendingly complete.*

PROOF. (i) Assume L^* is μ-rc, μ-regular and let $\mathfrak{A} = \langle \mu, <, \ldots \rangle$ be any expansion of $\langle \mu, < \rangle$. Put Σ_1 to be any subset of the L^*-diagram of $\mathfrak{A}$ and $\Sigma = \{\xi > \alpha | \alpha \in \mu\}$ where ξ is a new constant symbol. Clearly Σ has cardinality μ and satisfies the hypothesis of μ-rc. So $\Sigma_1 \cup \Sigma$ has a model which is not μ-like.

Now assume L^* is not μ-rc and let Σ_1, Σ be a counterexample, $\Sigma = \{\varphi_\alpha : \alpha \in \mu\}$, $\Sigma_\beta = \{\varphi_\alpha : \alpha < \beta\}$. Put U to be a new unary predicate and E a binary relation, say

(a) "E is a linear order on U"; now let R be another new binary relation, say

(b) "$R(x, y) \to U(x)$".

Denote by $\varphi^{R(\xi, x)}$ the relativisation of φ to the set $R(\xi, x)$, where ξ acts as a parameter. Say

(c)$_\gamma$ "$\varphi_\alpha^{R(\alpha, x)} \to \varphi_\beta^{R(\alpha, x)}$" for each $\beta < \alpha < \mu$ and $\varphi_\alpha, \varphi_\beta \in \Sigma_\gamma$.

The set Σ_γ' defined by (a), (b), (c)$_\gamma$ has the same cardinality as γ. Put $\Sigma_1' = \{\varphi^{R(\alpha, x)} : \varphi \in \Sigma_1, \alpha \in \mu\}$.

CLAIM. $\Sigma_1' \cup \Sigma_\mu'$ weakly characterizes μ.

First we construct a model. Let $\mathfrak{B}_\gamma$ be a model of $\Sigma_1 \cup \Sigma_\gamma$, which exists by assumption. Put $B = \mu \cup \bigcup_{\gamma < \mu} B_\gamma$ where the unions are disjoint. Now put $\mathfrak{B} = \langle B, U, E, R, \ldots \rangle$ where $U^{\mathfrak{B}} = \mu$, E is the membership of μ and $R < \bigcup x_{\gamma < \mu} B_\gamma$ such that $R(\gamma, b)$ iff $b \in B_\gamma$. The other relations are naturally taken from the $\mathfrak{B}_\gamma$. Now let $\mathfrak{N} \vDash \Sigma' \cup \Sigma_\mu' \cup \{a_i E a_j : i < j < \mu\}$. Then $\{a_i : i < \kappa\}$ is unbounded in $U^{\mathfrak{N}}$, for $\Sigma_1 \cup \Sigma$ has no model.

(ii) Let $\mathfrak{M}$ be a rich and sufficiently large structure such that $H(\mu^+)$ is contained in $\mathfrak{M}$. Let Σ_1 be a subset of the L^*-diagram of $\mathfrak{M}$ weakly characterizing $\langle \lambda^M, < \rangle$ and put $\Sigma = \{\alpha < \xi < \mu | \alpha \in \mu\}$ where ξ is a new constant symbol. Again Σ and Σ_1 satisfy the hypothesis of μ-rc. So there is a model $\mathfrak{N}$ such that $\mathfrak{M} < \mathfrak{N}(L_{\omega\omega})$ and $\mathfrak{N} = \Sigma_1 \cup \Sigma$. Furthermore we can assume that λ^M is cofinal in λ^N, using (i). Now define D on μ by $A \in D$ iff $A \in M$ (A is a set of M), $A \subseteq \mu$, $\bar{A} = \mu$ and $N \vDash \xi \in A^N$. Now assume $(A_i)_{i \in \lambda}$ is a descending sequence of sets of D, $A_0 = \mu$ and $\bigcap_{i < \lambda} A_i \notin D$ (i.e., w.l.o.g. $\bigcap_{i < \lambda} A_i = \varnothing$). Let f be a function in $\mathfrak{M}$ from μ into λ defined by $f(\alpha) = i$ iff $\alpha \in A_i - A_{i+1}$.

Now we have $\mathfrak{N} \vDash \forall \alpha \in \mu \; (\alpha < \partial \Rightarrow f(\xi) > \alpha)$ since $\mathfrak{N} \vDash \xi \in \bigcap A_i$, hence

we have $\mathfrak{N} \models \exists x \forall \alpha \in \mu \, (\alpha < \partial \Rightarrow f(x) > \alpha)$ and since $\mathfrak{M} < \mathfrak{N} \; \mathfrak{M} \models \exists x \forall \alpha \in \mu$ $(\alpha < \partial \Rightarrow f(x) > \alpha)$. But since λ^M is cofinal in λ^N, we conclude that $\bigcap A_i \neq \varnothing$, a contradiction. $\square$

PROPOSITION 6.5 [KP], [ČČ]. *Let μ be regular and D be a uniform ultrafilter on μ^+. Then*

(i) *D is μ-descendingly incomplete,*

(ii) *if $\lambda < \mu$, λ regular and D is λ^+-descendingly incomplete, then D is λ-descendingly incomplete.*

(iii) [JK] *Assume $V = L$ (or $\neg 0^\#$ or $\neg L^\mu$). Let κ be regular and D be a uniform ultrafilter on κ. Then D is λ-descendingly incomplete for all infinite λ.*

Now we can characterize μ-rc.

THEOREM 6.6. (i) *L^* is (κ, μ)-compact for all κ and given μ iff L^* is κ-rc for all $\kappa > \mu$.*

(ii) *If L^* is μ^+-rc and μ is regular, then L^* is μ-rc.*

(iii) *If $\aleph_\alpha$ is regular, $n \in \omega$, and L^* is $\aleph_{\alpha+n}$-rc, so L^* is $(\aleph_\alpha, \aleph_\alpha)$-compact.*

(iv) *($V = L$ or $\neg 0^\#$ or $\neg L^\mu$). If L^* is μ-rc then, for all $\lambda < \mathrm{cf}(\mu)$, L^* is λ-rc.*

PROOF. (i) One direction is trivial. Now assume L^* is κ-rc for all $\kappa > \mu$. We show for each $\lambda > \mu$ that L^* is (λ, μ)-compact. $\lambda = \mu$ follows from 6.3 (iii), $\lambda > \mu$ from 6.3(iv).

(ii) Assume not, then by 6.4(ii), there is a uniform ultrafilter D on μ^+ which is μ-descendingly complete, but this contradicts 6.5(i).

(iii) Again, assume not; so, by 6.3(iii), L^* is not $\aleph_\alpha$-rc, so, by 6.6(ii), L^* is not $\aleph_{\alpha+n}$-rc for each $n \in \omega$.

(iv) is proved with 6.5(iii) and 6.4(ii). $\square$

After this set-theoretic digression we go back to model theory. We shall need an old result of Rabin.

PROPOSITION 6.7 [Ra]. *Let μ_0 be the first measurable cardinal. For every $\kappa < \mu_0$ there is a structure $\mathfrak{A}_\kappa = \langle A, < \rangle$ and a language L^κ such that*

(i) *$P \in L^\kappa$, $\langle P^{\mathfrak{A}_\kappa}, <^{\mathfrak{A}_\kappa} \rangle \cong \langle \omega, < \rangle$,*

(ii) *whenever $\mathfrak{B} > \mathfrak{A}_\kappa(L^\kappa_{\omega\omega})$ then $\langle P^{\mathfrak{B}}, <^{\mathfrak{B}} \rangle$ is nonstandard.*

Now we are in a position to prove:

THEOREM 6.8. *Let L^* be a logic such that $|L^*| < \mu_0$ for each L with $|L| < \mu_0$ and such that $\mathrm{ROB}(L^*, L^*)$ and $\mathrm{FV}_P(L^*, L^*)$ hold. Then L^* is (ω, ω)-compact.*

PROOF. Let B_1, B_2 be two infinite sets of different cardinality, β_1, β_2, respectively, such that, for $L = \{=\}$, $\langle B_1 \rangle \equiv \langle B_2 \rangle (L^*)$. Since $|L^*| < \mu_0$ we may assume that $\beta_1, \beta_2 < \mu_0$ using a Hanf-type argument. Now we fix $\kappa > \max\{\beta_1, \beta_2\}$, $\kappa < \mu_0$. Put $\mathfrak{A}'_\kappa = \langle \mathfrak{A}_\kappa, P, Q, a \rangle_{a \in \mathfrak{A}_\kappa}$

where $\mathfrak{A}_\kappa$ is from 6.7, P,Q are unary and $|P| = \beta_1$, $|Q| = \beta_2$. Now assume for contradiction L^* is not (ω, ω)-compact. Then $\mathfrak{A}'_\kappa$ can be expanded to $\mathfrak{A}''_\kappa$ such that $\mathrm{Th}_{L^*}(\mathfrak{A}''_\kappa)$ is categorical (using 6.7 and 1.3). Now by $\mathrm{FV}_P(L^*, L^*)$ we have $[\mathfrak{A}''_\kappa, B_1] \equiv [\mathfrak{A}''_\kappa, B_2](L^*)$. We put now $T = \mathrm{Th}_{L^*}[\mathfrak{A}''_\kappa, B_1]$. Let φ denote the sentence which says that "F_1 is a bijection from $P^{\mathfrak{A}''_\kappa}$ into B (the universe of the second sort)".

Let ψ say "F_2 is a bijection from $Q^{\mathfrak{A}''_\kappa}$ into B". Obviously $T \cup \{\varphi\}$, $T \cup \{\psi\}$ have models $[\mathfrak{A}''_\kappa, B_1]$, $[\mathfrak{A}''_\kappa, B_2]$, respectively, but $T \cup \{\varphi, \psi\}$ has none since $\beta_1 \neq \beta_2$ and $\mathrm{Th}_{L^*}(\mathfrak{A}''_\kappa)$ is categorical. $\square$

The same proof works under the following hypothesis (cf. 2.15):

COROLLARY 6.9. *If L^* is a Karp-logic and $\mathrm{ROB}(L_{\omega\omega}, L^*)$ hold then L^* is (ω, ω)-compact hence $L^* = L_{\omega\omega}$ (by 1.2).*

PROOF. $[\mathfrak{A}''_\kappa, B_1] \cong_P [\mathfrak{A}''_\kappa, B_2]$ since $\langle B_1 \rangle \cong_P \langle B_2 \rangle$ and since L^* is a Karp-logic we have $[\mathfrak{A}''_\kappa, B_1] \equiv [\mathfrak{A}''_\kappa, B_2](L^*)$. Obviously φ_1, φ_2 are in $L_{\omega\omega}$.

THEOREM 6.10. *Let L^* be a logic such that $2^{|L^*|} < 2^{\aleph_{\alpha+n}}$ whenever L is finite (for some fixed regular $\aleph_\alpha$ and L^* satisfies $\mathrm{ROB}(L^*, L^*)$ and $\mathrm{FV}_P(L^*, L^*)$). Then L^* is κ-rc for each $\kappa > \aleph_\alpha$ with $\mathrm{cf}(\kappa) > \aleph_\alpha$.*

PROOF. Assume first κ is regular and for contradiction L^* is not κ-rc. So by 6.6(ii) L^* is not κ^+-rc. For any $S \subseteq \{\beta | \beta \in \kappa^+$ and $\mathrm{cf}(\beta) = \kappa\} = C_\kappa$ put $\mathfrak{M}_S = \langle \kappa^+, \varepsilon, S \rangle$. Assume $\kappa > \aleph_{\alpha+n}$. For $\aleph_\alpha < \kappa < \aleph_{\alpha+n}$ we argue with 6.6(ii) iteratedly. Since $2^{|L^*|} < 2^{\aleph_{\alpha+n}}$ there are stationary sets $S_1, S_2 \in C_\kappa$ with $(S_1 - S_2) \cup (S_2 - S_1) = S_3$ stationary and $\mathfrak{M}_{S_1} \equiv \mathfrak{M}_{S_2}(L^*)$. Put now $\mathfrak{A} = \langle \kappa^+, E, S_1, S_2, S_3, \underline{\mathrm{cf}} \rangle$ with $\underline{\mathrm{cf}}(\alpha) = \mathrm{cf}(\alpha)$. Since L^* is neither κ-rc nor κ^+-rc we can expand $\mathfrak{A}$ to a structure $\mathfrak{B}_0$ and find a theory Γ in the expanded language M such that

(i) $\mathfrak{B}_0 \vDash \Gamma$,
(ii) Γ weakly characterizes κ^+,
(iii) Γ weakly characterizes κ,
(iv) for every $\alpha \in \kappa^+$ with $\mathrm{cf}(\alpha) = \kappa$ Γ weakly characterizes α.

Put $\mathfrak{B}'_0 = \langle \mathfrak{B}_0, b \in B \rangle$. By FV_P, $[\mathfrak{B}'_0, \mathfrak{M}_{S_1}] \equiv [\mathfrak{B}'_0, \mathfrak{M}_{S_2}](M_0^*)$ where M_0 is the appropriate language. Now put T to be the M_0^*-theory of $[\mathfrak{B}'_0, \mathfrak{M}_{S_1}]$ and let φ_i ($i = 1,2$) be the sentence which says "F_i is an isomorphism between $\langle (\kappa^+)^{\mathfrak{B}_0}, E^{\mathfrak{B}_0}, S_i^{\mathfrak{B}_0} \rangle$ and $\mathfrak{M}_{S_i}$" where $F_1 \neq F_2$ both not in M_0. Obviously $T \cup \{\varphi_i\}$ is satisfiable in $[\mathfrak{B}'_0, \mathfrak{M}_{S_i}]$.

CLAIM 1. $T \cup \{\varphi_1, \varphi_2\}$ has no model. Assume, for contradiction, $T \cup \{\varphi_1, \varphi_2\}$ has a model $[\mathfrak{B}, \mathfrak{M}]$. Now

$$\langle (\kappa^+), E, S_1 \rangle \cong_{F_1} \mathfrak{M} \cong_{F_2} \langle (\kappa^+), E, S_2 \rangle,$$

put $f = F_1 \circ F_2$. Without loss of generality f is a monotone map $f: (\kappa^+) \to (\kappa^+)$, for any such isomorphism g is piecewise monotone and a monotone

map can be constructed from g. Now put

$$A_0 = \left\{ \alpha \in (\kappa^+)^{\mathfrak{B}} : \alpha \in \kappa^+ \right\} \quad \text{and} \quad A_{n+1} = f(A_n).$$

$A = \bigcup_{n<\omega} A_n.$

CLAIM 2. A is f-closed, $\langle A, E \rangle$ is a κ^+-like ordering and A_0 is cofinal in A.

Call $X \subseteq A$ c.u.b. if $X \cap A_0$ is c.u.b. in A_0 and $X \leq A$ stationary if X meets every c.u.b. set in A.

CLAIM 3. $S_i \cap A$ $(i = 1,2,3)$ are stationary in A. Now we distinguish 2 cases:

Case 1. There is a c.u.b set $C \subseteq A$ on which f is the identity. Then $S_1 \cap A = S_2 \cap A$ but A_0 is c.u.b in A, hence $A_0 \cap S_3 \neq \varnothing$, a contradiction.

Case 2. There is a stationary set $S \subseteq A$ such that $f: S \to A$ is regressive (otherwise take f^{-1}). We shall prove an analogue of Fodor's theorem.

CLAIM 4. There is $\xi \in A$ with $f^{-1}(\xi)$ stationary in A. Assume there is no such ξ. Then for each $\xi \in A$ there is a c.u.b $C_\xi \subseteq A$ such that, for all $\eta \in C_\xi \cap S, f(\eta) \neq \xi$. Let

$$D = \left\{ \eta \in A : \forall \xi E \eta \; (\eta \in C_\xi) \right\}.$$

Then $D \cap S = \varnothing$, so we shall obtain a contradiction if we show that D is c.u.b in A. D is clearly closed (i.e., $D \cap A_0$ is closed in A_0). Now fix $\gamma \in A_0$. Let $\gamma_0 = \gamma$ and $\gamma_{\alpha^+} \in A_0$ be some element of A_0 such that $\gamma_\alpha E \gamma_{\alpha^+}$ and $\gamma_{\alpha^+} \in \bigcap \{ C_\xi : \xi E \gamma_\alpha \}$ (which is c.u.b in A). For limits similarly, if $\mathrm{cf}(\alpha) < \kappa$. So we obtain a sequence $\{ \gamma_{\alpha^+} : \alpha \in \kappa \} \leq A_0$, which has a supremum δ in A_0. By property (iv) of Γ δ is also a supremum in A. So we conclude that $\delta \in D$.

This proves Claim 4 and shows that f is not injective, which ends the proof of Claim 1.

If κ is singular and $\mathrm{cf}(\kappa) \geqslant \aleph_\alpha$, L^* is $\mathrm{cf}(\kappa)$-rc and by Lemma 6.4(ii) L^* is κ-rc. $\square$

Piecing this together we now prove Theorem 6.2: The set-theoretic conditions fit into the framework of 6.8 and 6.10. So L^* is ω-rc and since we can take $\alpha = 0$ in 6.10 L^* is κ-rc for each $\kappa \geqslant \omega$. With 6.6(i) we get the conclusion. $\square$

Using 1.6 the condition on the size of L^* can be replaced by a suitable Löwenheim-Skolem property. Also Theorems 6.8 and 6.10 can be rephrased with several logics; e.g.:

THEOREM 6.10′ (GCH). *Assume $L^\square \subset L^*$ are two logics s.t.*

(i) L^* *satisfies* $LS(\aleph_n)$ *for some* $n \in \omega$,

(ii) $FV_P(L^\square, L^*)$ *holds, and*

(iii) $ROB(L_{\omega\omega}, L^\square)$ *holds.*

Then $L^\square$ is (κ, ω)-compact for all $\kappa \geqslant \omega$.

THEOREM 6.10''. ($V = L$ or $\neg 0^{\#}$). *Assume* $L^{\square} \subset L^*$ *are two logics s.t.* (ii) *and* (iii) *from* 6.10' *hold, then* $L^{\square}$ *is* (κ, ω)-*compact for all* $\kappa > \omega$.

PROOF. Use 6.6(iv). $\square$

COROLLARY 6.11. *Assume* $L^* = L_{\omega\omega}[Q^i]_{i \in \omega}$ *and* $2^{\aleph_0} < 2^{\aleph_n}$ *for some* $n \in \omega$. *Then if* ROB(L^*, L^*) *and* FV$_P(L^*, L^*)$ *hold then also* CRAIG(L^*, L^*) *holds*.

Using the following theorem of Lindström [Li3] we get another characterization of $L_{\omega\omega}$.

THEOREM 6.12 (LINDSTRÖM). *Assume* L^* *is* (κ, ω)-*compact for each* κ *and* $T(L)$ (*the union of any* L-*elementary chain is an* L^*-*elementary extension of each member of the chain*) *holds then* $L^* = L_{\omega\omega}$.

COROLLARY 6.13. $L_{\omega\omega}$ *is characterized by*
(i) $T(L)$,
(ii) ROB(L, L),
(iii) $VP(L, L)$,
(iv) $L^* = L_{\omega\omega}[Q^1]_{i \in n}$ *provided* $2^{\aleph_0} < 2^{\aleph_n}$ *for some* $n \in \omega$.

One might ask whether there are proper extensions of $L_{\omega\omega}$ satisfying ROB and FV$_P$. Under the hypothesis of 6.10 such a logic is fully compact and hence satisfies Craig's theorem. In fact we conjecture

CONJECTURE 6.14. Let $L^* = L_{\omega\omega}(Q^i)_{i \in \omega}$ be a logic which satisfies ROB and FV$_P$. Then $L^* = L_{\omega\omega}$.

If the Q^i are all monadic quantifiers, i.e. quantifiers of the form

$$Q^i x_1, \ldots, x_{n_i}\big(\varphi_1(x_1), \varphi_2(x_2), \ldots, \varphi_{n_i}(x_{n_i})\big)$$

then we have:

THEOREM 6.15. *Let* $L^* = L_{\omega\omega}(Q^i)_{i \in \aleph_\omega}$ *be a logic with* ROB, FV$_P$, *each* Q^i *monadic and* $\aleph_\omega$ *strong limit. Then* $L^* = L_{\omega\omega}$.

PROOF. By 6.9 and 6.7 L^* is (κ, ω)-compact for each $\kappa > \omega$. Now let K be a class of monadic structures, i.e. $\mathfrak{A} \in K$ is of the form

$$\mathfrak{A} = \langle A_1, \ldots, A_n; P_1, \ldots, P_\kappa \rangle$$

where A_i are universes and P_j are unary predicates. Using FV$_P$ we can restrict ourselves to one sorted structures. W.l.o.g. K is a class of cardinals. It suffices to show that $K \in \mathrm{EC}(L^*)$ implies $K \in \mathrm{EC}(L_{\omega\omega})$. Now assume there are infinite cardinals κ, μ with $\kappa \in K$, $\mu \notin K$ and $K \in \mathrm{EC}(L^*)$. Put $T = \mathrm{Th}_{L^*}([\kappa, \mu])$, $T_1 = \mathrm{Th}_{L^*}(\langle \kappa \rangle)$ and $T_2 = \mathrm{Th}_{L^*}(\langle \mu \rangle)$. By compactness T_1 and T_2 have arbitrary large models. Let $\varphi_1(f)$ say that "f is an injection of the first into the second sort" and $\varphi_2(g)$ say that "g is an injection of the second into the first sort". Clearly $T \cup \{\varphi_i\}$ ($i = 1,2$) has a model. For let κ' be such that

$\langle \kappa' \rangle \vDash T_1$ and $\kappa' > \mu$, so $[\kappa', \mu]$ can be expanded to satisfy φ_1 and, by FV_P, $[\kappa', \mu] \vDash T$ and similarly for φ_2. But then by ROB $T \cup \{\varphi_1, \varphi_2\}$ has a model $[\kappa'', \mu'']$ so by the Cantor-Bernstein Theorem $\kappa'' = \mu''$. But by FV_P, $\kappa'' \in K$, $\mu'' \notin K$ a contradiction. So w.l.o.g. K contains only finite cardinals, hence, by compactness, the cardinals in K are bounded by some natural number $n \in \omega$ and $K \in EC(L_{\omega\omega})$. $\square$

The argument above can still be extended (cf. [Ma5]). Let K be a class of structures $\mathfrak{A}$ of the form $\langle A, E, P_1, \ldots, P_n \rangle$ where A is the universe, E is an equivalence relation and P_i are monadic predicates. If K is closed under isomorphism and $K \in EC(L^*)$ for $L^* = L_{\omega\omega}(Q^i)_{i < \aleph_\omega}$, the Q^i arbitrary quantifiers, L^* satisfies ROB and FV_P and $\aleph_\omega$ strong limit, then $K \in EC(L_{\omega\omega})$. Here we use that two equivalence relations with κ equivalence classes, each of cardinarlity κ, are isomorphic.

This proves Conjecture 6.14 for some special classes of quantifiers. The problem remains of how to extend the above proof to arbitrary quantifiers.

Note that Caicedo [Ca] got a weak version of 6.15 without set-theoretic assumptions and FV_P but using (ω, ω)-compactness (cf. [Ma5]).

7. The definability of the syntax-structure. In this section we present without proofs two abstract theorems due to Burgess and Paulos on the weak Beth-property. We follow [Fe2] for terminology.

THEOREM 7.1 (BURGESS). *Let L^* be an absolute, normal logic, $L^* \subseteq H(\omega_1)$ such that $\mathrm{WBETH}(L^*, L^*)$ holds. Then the class of countable well-orderings is not $PC(L^*)$.*

THEOREM 7.2 (PAULOS). *Let $L^{\boxplus} \subset L^{\square}$ be two logics such that $L^{\boxplus}$ is adequate to truth for $L^{\square}$ and $\mathrm{WBETH}(L^{\boxplus}, L^{\square})$ holds, then no syntax-structure of $L^{\square}$ is $L^{\boxplus}$-definable.*

Theorem 7.1 is a slight generalization of Paulos' theorem [Pa]. The main reason we quote these theorems here is that combining the methods of proofs together with 6.1 Gostanian and Hrbacek [GH] produced uniformly rather tricky counterexamples to $\mathrm{WBETH}(L^*, L^{\square})$ for

$L^* = L^G_{\omega,\omega}, L^{\square} = L^G_{\infty\infty}$,

$L^* = L_{\omega,\omega}, L^{\square} = L_{\infty\infty}$,

$L^* = L_{\kappa^+\omega}, L^{\square} = L_{\lambda\lambda}$ for all regular λ which are $L_{\kappa^+\omega}$ accessible.

8. Some direct applications. In this section we present some applications of the abstract theory. Less direct applications may be found in [MS2], [MS3]. The table below shows how much can be drawn from the abstract approach.

Most of the notation of the table is self-evident. $L^G_{\omega_1\omega}$ is the logic obtained from $L_{\omega_1\omega}$ adding the game quantifier. L^H is the logic obtained from $L_{\omega\omega}$ adding the Henkin quantifier. In fact any nonlinear partially ordered quantifier will lead to the same situation (cf. [Wa]).

	CRAIG	ROB	BETH	WBETH	WFV$_P$
$L_{\omega\omega}$	yes [Ke2]	yes [Ke2]	yes	yes	yes
$L_{\omega_1\omega}(L_A)$ A countable	yes [Ke2] [LE]	no 2.14 (but yes for countable T)	yes	yes	no 3.2
$L_{\omega_1\omega}^G$	no [GH]	no 2.14 yes for countable T	no [GH]	no [GH]	no 3.4
$L_{\kappa^+\omega}$ $\kappa \geqslant \omega_1$	no 2.12	no 2.14 or 6	no 2.12	no 3.4 [GH] cf(κ) $>\omega$	no 3.4
$L(Q_\kappa)$ $\kappa \geqslant \omega_1$	no [Fe]	no (compactness)	no 5.4 and [Wo]	?	yes [Wo]
$L(Q_\kappa)$ $\kappa = \omega$	no 3.2 2.3	no 2.15	no 3.2 2.3	no 3.2	yes [Wo]
L^H	no 6.1 2.3	no 1.5 2.14 or 6.1	no 6.1 2.3	no 6.1	no 8.1
L^P, $L(aa)$	no [MS2]	no (comp)	no [MS2]	?	yes L^P no $L(aa)$
$L(Q^{<n})$	no [Bd]	no (comp)	no 5.4	?	yes [Bd]
L^n	no 5.4	no 6.2	no 5.4	no 6.1	yes [MS2]
L^{II}	no 7.2	no 6.2	no 7.2	no 7.2	no 8.1

Finally L^P is positive logic and $L(aa)$ stationary logic (cf. [BKM], [Ma4] and [MS2]). $L(Q^{<n})$ is the Magidor-Malitz logic from [MM], L^n is negative logic (cf. [BKM], [MS2]) and L^{II} is second order logic.

To make the table below more complete let us finish with a last observation.

Let $Exy(\phi(x), \psi(y))$ be a binary quantifier binding two formulas and expressing that ϕ and ψ define sets of the same cardinality.

PROPOSITION 8.1. *Let L^* be a logic which is a set for each L and which extends $L_{\omega\omega}(E)$. Then* $\mathrm{WFV}_P(L_{\omega\omega}(E), L^*)$ *fails.*

Examples for such L are L^H or L^{II} (cf. [KL]).

PROOF. Let A,B be two sets of different cardinality such that $A \equiv B(L^*)$. Such sets exist by a Hanf type argument. But $[A, B] \not\equiv [A, A](L^*)$ using the quantifier E. $\square$

ADDED IN PROOF (June 25, 1979). If we redefine ROB by allowing arbitrary theories T_1, T_2 over arbitrary sets of predicates $\bar{P}$ instead of $\phi(P)$, $\psi(P)$, the results of this section can be improved.

THEOREM. *Under the same set-theoretic hypothesis as in* 6.10 *if L^* satisfies* ROB, *then L^* is κ-rc for each $\kappa \geqslant \aleph_\alpha$.*

In the forthcoming paper by the same authors the amalgamation property of logics will be studied. We shall show that under much weaker assumptions than in 6.10 the amalgamation property implies full compactness. This will

give us also some answers to Problem 2 in the introduction to this paper. The above version of ROB was suggested to us by a preprint of D. Mundici.

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