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EQUIVARIANT G-STRUCTURE ON VERSAL DEFORMATIONS

BY

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ABSTRACT. Let X_0 be an algebraic variety, and (χ, Σ) its versal deformation. Now let G be an affine algebraic group acting algebraically on X_0. It gives rise to a definite linear G-action on the tangent space of Σ. In this paper we establish that if G is linearly reductive then there is an equivariant G-action on (χ, Σ) which induces given G-action on the special fibre X_0 and its linear G-action on the tangent space of the formal moduli Σ. Furthermore, such equivariant G-structure is shown to be unique up to noncanonical isomorphism.

Let X_0 be an algebraic variety together with an action of an algebraic group G, defined over a fixed field k. A question we pose here is to see if there exists an equivariant G-structure on versal deformation of X_0. To be more precise, we ask if there exists a versal deformation

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \hookrightarrow & s \end{array}$$

where we can provide G-action on X extending the given action on X_0, and G-action on the parameter scheme s, such that all the maps entering in the above diagram are compatible with those G-actions. In the case when X_0 is an affine cone with the obvious G_m-action, an existence theorem was proven by Pinkham in [4] by an elementary technique, and the question for the general case was left open.

The purpose of this paper is to establish an existence theorem and uniqueness for the case of linearly reductive group G, generalizing the case of G_m. Indeed we show that if $H^1(G, -) = 0 = H^2(G, -)$ for a class of G-modules determined by X_0, then an equivariant G-structure exists, and is unique up to equivariant isomorphism.

Our technique is parallel to the original method of M. Schlessinger in proving the existence theorem for versal deformations [8]. A crucial difference is that we have to stay in the category of deformations and not the isomorphism classes of deformations, since we have to deal with a successive extension problem of G-actions and equivariant isomorphisms. The same

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reason was indeed a motivating force in writing an exposé in [5], though it may have been too heavy with the language of categories for some readers.

1. Fibred category with G-action. We denote by k a fixed field, and set $\mathfrak{B}$ to be the category of "fat point" over k i.e., Spec(R) where R is an artinian local k-algebra with the residue field k. If $\mathbf{M}^\nu \neq 0$ but $\mathbf{M}^{\nu+1} = 0$, where $\mathbf{M}$ is the maximal ideal of R, then Spec(R) is called a fat point of order ν. If V is a finite-dimensional vector space over k, $A(V)$ stands for an affine space (with the origin as a base point) i.e., $A(V) = \operatorname{Spec}(\operatorname{Symm}(V^*))$. We set $A_\nu(V) = \operatorname{Spec}(\operatorname{Symm}(V^*)/M^{\nu+1})$ where M is the maximal ideal of Symm(V^*) generated by the linear forms. We shall further adopt the following notations: $\cdot = \operatorname{Spec}(k)$, $\varepsilon = \operatorname{Spec}(k[\varepsilon])$, (1) = category with a single object without automorphism, $\mathfrak{B}_\nu$ = the subcategory of $\mathfrak{B}$ consisting of fat points of order $\leqslant \nu$.

Let X_0 be an algebraic variety over k. A deformation of X_0 over a fat point s in $\mathfrak{B}$ is a diagram

$$\begin{tikzcd} X_0 \arrow[r, hook] \arrow[d] & X \arrow[d] \\ \cdot \arrow[r, hook] & s \end{tikzcd}$$

where $X \to s$ is flat. A morphism of deformation of X_0 is a pair (ϕ, α) which yield a commutative diagram

$$\begin{tikzcd}[row sep=small] & X \arrow[rr, "\phi"] \arrow[dd] & & X' \arrow[dd] \\ X_0 \arrow[ur] \arrow[urrr] \arrow[dd] & & & \\ & s \arrow[rr, "\alpha"] & & s' \\ \cdot \arrow[ur] \arrow[urrr] & & & \end{tikzcd}$$

Thus the deformations of X_0 form a category $\mathfrak{D}$, and the assignment of base scheme to each deformation yields a functor γ: $\mathfrak{D} \to \mathfrak{B}$. We note the following facts:

(1) If $X \to s$ is a deformation of X_0, then for any base change $s' \to s$, $X \times_s s' \to s'$ is a deformation of X_0 over s'.

(2) Let

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & X' \arrow[d] \\ s \arrow[r, "\alpha"] & s' \end{tikzcd}$$

be a morphism of deformations of X_0. If α is an isomorphism, then so is ϕ.

(3) Consider two morphisms (ϕ, α) and (ψ, β):

$$\begin{tikzcd} X' \arrow[d] & X \arrow[l, "\phi"'] \arrow[r, "\psi"] \arrow[d] & X'' \arrow[d] \\ s' & s \arrow[l, "\alpha"] \arrow[r, "\beta"'] & s'' \end{tikzcd}$$

If one of α, β is a closed immersion, then $X' \amalg_X X'' \to s' \amalg_s s''$ is a deformation of X_0.

The above properties (1) and (2) shows that the category $\mathfrak{D}$ of deformations of X_0 is a fibred category in groupoid over $\mathfrak{B}$ [1], [5]. We also note that $\mathfrak{D}(\cdot)$ is a one-point category without automorphisms. Abstracting the crucial property (3) above, we define

Definition. Let $\gamma\colon \mathfrak{F} \to \mathfrak{B}$ be a fibred category in groupoid, such that $\mathfrak{F}(\cdot) = (1)$. It is called "homogeneous" if we have that any diagram

$$\begin{tikzcd} & \xi \\ \eta \arrow[ur, "\phi"] \arrow[dr, "\phi'"'] & \\ & \xi' \end{tikzcd}$$

in $\mathfrak{F}$ admits $\xi \amalg_\eta \xi'$, provided at least one of $\gamma(\phi)$, $\gamma(\phi')$ is a closed immersion.

In other words, for any diagram

$$\begin{tikzcd} & s' \\ s \arrow[ur] \arrow[dr] & \\ & s'' \end{tikzcd}$$

in $\mathfrak{B}$ where at least one of them is a closed immersion,

$$\mathfrak{F}\Big(s' \coprod_s s''\Big) \to \mathfrak{F}(s') \overset{2}{\underset{\mathfrak{F}(s)}{\times}} \mathfrak{F}(s'')$$

is an equivalence of categories, where $\mathfrak{F}(b)$ stands for the subcategory of $\mathfrak{F}$ such that $\xi \in \operatorname{ob} \mathfrak{F}(b) \Leftrightarrow \gamma(\xi) = b$, and $\phi \in \operatorname{Mor} \mathfrak{F}(b) \Leftrightarrow \gamma(\phi) =$ the identity on b, and the right-hand side above stands for fibre-product of categories in the sense of 2-category.

Thus the category of deformations is a good example of a homogeneous fibred category. In case when $s = \cdot$, then it follows from $\mathfrak{F}(\cdot) = (1)$ that

$$|\mathfrak{F}(s' \amalg s'')| \overset{\approx}{\to} |\mathfrak{F}(s')| \times |\mathfrak{F}(s'')|,$$
$$\operatorname{Mor}(\xi' \amalg \xi'', \eta' \amalg \eta'') \overset{\approx}{\to} \operatorname{Mor}(\xi', \eta') \times \operatorname{Mor}(\xi'', \eta''),$$

where we set $|\mathcal{C}| =$ the set of isomorphism classes of objects in the category $\mathcal{C}$. These facts entail that $|\mathfrak{F}(A_1(V))|$ and $\operatorname{Aut}(1_V)$ carries a canonical structure of vector spaces over k where 1_V stands for the trivial object over $A_1(V)$ i.e., the pull-back under $A_1(V) \to \cdot$ of the object 1. Indeed, set

$$\nabla\colon A_1(V) \to A_1(V) \amalg A_1(V)$$

be the map given by the addition map $V^* \oplus V^* \to V^*$. Then $|\mathfrak{F}(A_1(V))|$ becomes a vector space over k via the addition:

$$|\mathfrak{F}(A_1(V))| \times |\mathfrak{F}(A_1(V))| = |\mathfrak{F}(A_1(V) \amalg A_1(V))| \overset{\nabla^*}{\to} |\mathfrak{F}(A_1(V))|.$$

We further note that the canonical map $V^* \otimes |\mathfrak{F}(\varepsilon)| \to |\mathfrak{F}(A_1(V))|$ is an isomorphism of vector spaces. For any $\xi \in \mathrm{ob}(\mathfrak{F}(A_1(V)))$, the same will hold for $\mathrm{Aut}(\xi)$, if we define the addition via $\nabla^*(\sigma, \tau)$ for σ, τ in $\mathrm{Aut}(\xi)$. However, $\nabla^*(\sigma, \tau)$ is equal to the ordinary composition of automorphisms, since

$$\nabla^*(\sigma, \tau) = \nabla^*((\sigma, 1) \cdot (1, \tau)) = \nabla^*(\sigma, 1)\nabla^*(1, \tau) = \sigma\tau.$$

We also note that the canonical map $\mathrm{Aut}(1_V) \to V^* \otimes \mathrm{Aut}(1_\varepsilon)$ is an isomorphism of vector spaces.

In this categorical language, a versal object for $\mathfrak{F} \to \mathfrak{B}$ can be explained as follows:

Definition. A versal object for $\gamma\colon \mathfrak{F} \to \mathfrak{B}$ is an object $\xi \in \mathrm{ob}(\hat{\mathfrak{F}})$ such that every $x \in \mathrm{ob}(\mathfrak{F})$ is induced by ξ i.e., there exists a morphism $x \to^{\phi_x} \xi$, and furthermore it is minimal in the sense that the canonical map $\tilde{\xi}\colon \mathrm{Mor}(\varepsilon, \gamma(\xi)) \to |\mathfrak{F}(\varepsilon)|$ given by pull-back is an isomorphism.

One may note that this property entails that the canonical map $\tilde{\xi}$: $\mathrm{Mor}(s, \gamma(\xi)) \to |\mathfrak{F}(s)|$ is an isomorphism for every fat point s of order 1.

It is a theorem of M. Schlessinger that a versal object for $\mathfrak{F}$ exists provided $|\mathfrak{F}(\varepsilon)|$ is finite-dimensional. One important remark is in order. If $|\mathfrak{F}(\varepsilon)|$ is finite-dimensional, then infinitesimal neighborhoods of the origin in the affine space $|\mathfrak{F}(\varepsilon)|$ form a subcategory $\mathfrak{B}'$ of $\mathfrak{B}$. Then a versal object for $\mathfrak{F}' = \mathfrak{F}|\mathfrak{B}'$ is indeed a versal object for $\mathfrak{F}$. This fact will be used in the sequel, and throughout the rest we assume that $|\mathfrak{F}(\varepsilon)|$ is finite-dimensional.

Now assume that an algebraic group G acts algebraically on the variety X_0. Let $\sigma \in G$. If

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & X \\ \downarrow & & \downarrow j \\ \cdot & \hookrightarrow & s \end{array}$$

is a deformation of X_0, then

$$\begin{array}{ccc} X_0 & \overset{i\sigma^{-1}}{\hookrightarrow} & X \\ \downarrow & & \downarrow j \\ \cdot & \hookrightarrow & s \end{array}$$

is also a deformation of X_0, provided $s \neq \cdot$. Consequently, G acts on the category $\mathfrak{D}(s)$ and in particular on $\mathfrak{D}(\varepsilon)$. Since this G-action commutes with amalgamated sum, it follows that the vector space $|\mathfrak{D}(\varepsilon)|$ receives a linear representation of G. Consequently, if we set $\mathfrak{B}'$ to be the subcategory of $\mathfrak{B}$ consisting of infinitesimal neighborhoods of the origin in the affine space $|\mathfrak{D}(\varepsilon)|$, then G acts on the category $\mathfrak{B}'$. We now define the action of G on $\mathfrak{D}' = \mathfrak{D}|\mathfrak{B}'$ via

$$\left(\begin{array}{ccc} X_0 & \xrightarrow{\quad i \quad} & X \\ \downarrow & & \downarrow j \\ \cdot & \longrightarrow & s \end{array}\right) \longmapsto \left(\begin{array}{ccc} X_0 & \xrightarrow{\quad i\sigma^{-1} \quad} & X \\ \downarrow & & \downarrow \sigma j \\ \cdot & \longrightarrow & \sigma(s) \end{array}\right)$$

i.e., if we denote a deformation by a triple then

$$\sigma(j, X, i) = (\sigma j, X, i\sigma^{-1}).$$

The action of G on $\mathfrak{D}'$ so defined is now compatible with that on $\mathfrak{B}'$, i.e., γ: $\mathfrak{D}' \to \mathfrak{B}'$ is a G-equivariant fibration. We note that, for any G-invariant fat point s of order 1 in $\mathfrak{B}'$, the canonical isomorphism $\mathrm{Mor}(s, \varepsilon) \otimes |\mathfrak{D}(\varepsilon)| \xrightarrow{\sim} |\mathfrak{D}(s)|$ is G-linear where G-action on $\mathrm{Mor}(s, \varepsilon) \otimes |\mathfrak{D}(s)|$ is given by $\sigma(h \otimes \eta) = h\sigma^{-1} \otimes \sigma\eta$.

Definition. We say that $\xi = (j, X, i)$ in $\mathfrak{D}'$ admits a data for G-equivariance if $s = \gamma(\xi)$ is G-invariant, and for each $\sigma \in G$ there exists an isomorphism ϕ_σ: $\sigma(\xi) \to \xi$, identity on s i.e., we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "\phi_\sigma"] \arrow[dd] & & X \arrow[dd] \\
& X_0 \arrow[ul, "i\sigma^{-1}"'] \arrow[ur, "i"'] \arrow[dd] & \\
s \arrow[rr, equal] & & s \\
& \cdot \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

or, what is the same, a commutative diagram

$$\begin{tikzcd}
& X \arrow[rr, "\phi_\sigma"] \arrow[dd, "j"] & & X \arrow[dd, "j"] \\
X_0 \arrow[ur, "i"] \arrow[rr, "\sigma"] \arrow[dd] & & X_0 \arrow[ur, "i"] \arrow[dd] & \\
& s \arrow[rr, "\sigma"] & & s \\
\cdot \arrow[ur] \arrow[rr, equal] & & \cdot \arrow[ur] &
\end{tikzcd}$$

This means alternatively that $s = \gamma(\xi)$ is G-invariant and the isomorphism class ξ is a G-invariant element in $|\mathfrak{D}(\varepsilon)|$. We say that $\xi = (j, X, i)$ is G-equivariant if it admits a data $\{\phi_\sigma\colon \sigma(\xi) \to \xi \mid \sigma \in G\}$ for G-equivariance in such a way that $\phi_\sigma \sigma(\phi_\tau) = \phi_{\sigma\tau}$ for all σ, τ in G where $\sigma(\phi_\tau)$ stands for the isomorphism $\sigma(\tau(\xi) \xrightarrow{\phi_\tau} \xi) = (\sigma\tau(\xi) \xrightarrow{\sigma(\phi_\tau)} \sigma(\xi))$. In case when ξ is G-equivariant we simply use σ instead of ϕ_σ in case when there is no possible confusion.

Our objective is to establish an existence of G-equivariant versal deformation under a suitable condition on linear representations of G on $|\mathfrak{D}(\varepsilon)|$ and $\mathrm{Aut}(1_\varepsilon)$. A remark is in order concerning G-action on $\mathrm{Aut}(1_\varepsilon)$. Let η be a G-equivariant object in $\mathfrak{D}$. Then for any $u \in \mathrm{Aut}(\eta)$, we set $u^\sigma = \sigma \cdot \sigma(u) \cdot \sigma^{-1}$. Now for any G-invariant vector space W, 1_W (= the pull-back of $1 \in \mathrm{ob}\, \mathfrak{D}(\cdot)$ under $A_1(W) \to \cdot$) is G-equivariant, and G-action on $\mathrm{Aut}(1_W)$ is k-linear. Furthermore, the canonical map $W^* \otimes \mathrm{Aut}(1_\varepsilon) \to \mathrm{Aut}(1_W)$ is an isomorphism of $k[G]$-modules.

Needless to say, the above definition can be carried over verbatim to an abstract homogeneous fibred category with equivariant G-action.

2. Existence of equivariant G-structure on versal deformation. Abstracting the properties on the category of deformations of the variety X_0 with G-action, we put ourselves in the following situation:

(i) V is a fixed finite-dimensional vector space together with a linear representation of G on it, and $\mathfrak{B}$ is the category of infinitesimal neighborhoods of the origin in the affine space V.

(ii) $\gamma: \mathfrak{F} \to \mathfrak{B}$ is a homogeneous fibred category in groupoid, provided with a G-action on $\mathfrak{F}$ compatible with that on $\mathfrak{B}$, and we are given a definite G-linear identification $|\mathfrak{F}(\varepsilon)| = V$. A first step is to choose ξ_1 in $\mathfrak{F}(A_1(V))$ such that $\tilde{\xi}_1$: $\mathrm{Mor}(s, A_1(V)) \to (\mathfrak{F}(\varepsilon))$ is an isomorphism and ξ_1 carries a data of G-equivariance.

Step 0. We choose ξ_1 as follows. We have the canonical G-linear isomorphism $\mathrm{Mor}(A_1(V), \varepsilon) \otimes |\mathfrak{F}(\varepsilon)| \to |\mathfrak{F}(A_1(V))|$ i.e., we have the canonical G-linear isomorphism $V^* \otimes V \xrightarrow{\sim} |\mathfrak{F}(A_1(V))|$. Choose $\xi_1 \in \mathrm{ob}(\mathfrak{F}(A_1(V)))$ such that its isomorphism class $[\xi_1]$ corresponds to $\Sigma_{i=1}^n \lambda_i \otimes x_i$ in $V^* \otimes V$ where $x_1, x_2, \ldots, x_n$ is a basis of V and $\lambda_1, \lambda_2, \ldots, \lambda_n$ is its dual basis. Since $\sigma(\Sigma\lambda_i \otimes x_i) = \Sigma\lambda_i\sigma^{-1} \otimes \sigma x_i = \Sigma\lambda_i \otimes x_i$, $[\xi_1]$ is a G-invariant element in $|\mathfrak{F}(A_1(V))|$, and secondly $\tilde{\xi}_1$: $\mathrm{Mor}(\varepsilon, \gamma(\xi_1)) \to |\mathfrak{F}(\varepsilon)|$ is an isomorphism since it corresponds, under the canonical identification $V = \mathrm{Mor}(\varepsilon, \gamma(\xi_1))$, to the identity map $V \to V$. The fact that $[\xi_1]$ is G-invariant means that ξ_1 admits a data of G-equivariance.

Our construction of G-equivariant versal object for $\mathfrak{F}$ is stepwise, and the following is a key lemma.

LEMMA. *Let $s \to s'$ be G-invariant fat neighborhoods of the origin in V corresponding to the exact sequence* $0 \to W \to R' \to R \to 0$ *such that* $\mathbf{m}W = 0$ *where* $\mathbf{m}$ *is the maximal idea of R'. Let ξ be a G-invariant object in $\mathfrak{F}(s)$, and let η be an extension of ξ to s' with a data of G-equivariance extending the given G-structure on ξ. If* $H^2(G, W \otimes \mathrm{Aut}(1_\varepsilon)) = 0$, *then η can be provided with an equivariant G-structure extending that on ξ.*

PROOF. Let $\{\phi_\sigma: \sigma(\eta) \to \eta | \sigma \in G\}$ be a data of G-equivariance extending the given G-structure on ξ i.e., we have, for each $\sigma \in G$, a commutative diagram

$$\begin{array}{ccc} \sigma\eta & \xrightarrow{\phi_\sigma} & \eta \\ \cup & & \cup \\ \sigma\xi & \xrightarrow{\sigma} & \xi \end{array}$$

We want to adjust the family $\{\phi_\sigma | \sigma \in G\}$ by the automorphisms of η identity on ξ, so that we can have $\phi_\sigma \cdot \sigma(\phi_\tau) = \phi_{\sigma\tau}$ for all σ, τ in G. For this purpose we have to look into $\mathrm{Aut}_\xi(\eta)$ = the group of automorphisms of η identity on ξ. Now the canonical isomorphism $s' \amalg_s s' \simeq s' \amalg A_1(W^*)$ gives us the canonical isomorphism $\eta \amalg_\xi \eta \simeq \eta \amalg 1_{W^*}$ identity on the first factor, and consequently the canonical identification $\mathrm{Aut}_\xi(\eta) = \mathrm{Aut}(1_{W^*}) = W \otimes \mathrm{Aut}(1_\varepsilon)$. Thus

$\mathrm{Aut}_\xi(\eta)$ is an abelian group and is indeed a vector space over k. Furthermore, the G-action on $\mathrm{Aut}_\xi(\eta)$ carried over from that on $\mathrm{Aut}(1_{W^*})$ corresponds to the following operation: Given $u \in \mathrm{Aut}_\xi(\eta)$, $u^\sigma = \phi_\sigma \sigma(u)\phi_\sigma^{-1}$ (one may note that $u \to \phi_\sigma\sigma(u)\phi_\sigma^{-1}$ does not depend on the choice of $\{\phi_\sigma | \sigma \in G\}$ extending the given data on ξ, since any other choice would differ by elements of $\mathrm{Aut}_\xi(\eta)$ which is abelian). Now let us return to the given data of G-equivariance $\{\phi_\sigma\colon \sigma(\eta) \to \eta\}$ extending that on ξ. Then $f(\sigma, \tau) = \phi_\sigma \cdot \sigma(\phi_\tau) \cdot \phi_{\sigma\tau}^{-1}$ is an element of G-module $\mathrm{Aut}_\xi(\eta)$. Then

$$\begin{aligned} f(\sigma\tau, \rho)f(\sigma, \tau\rho)^{-1}f(\sigma, \tau) &= f(\sigma, \tau)f(\sigma\tau, \rho)f(\sigma, \tau\rho)^{-1} \\ &= \phi_\sigma \cdot \sigma(\phi_\tau) \cdot \sigma\tau(\phi_\rho) \cdot \sigma(\phi_{\tau\rho})^{-1} \cdot \phi_\sigma^{-1} \\ &= \phi_\sigma \cdot \sigma(f(\tau, \rho))\phi_\sigma^{-1} = f(\tau, \rho)^\sigma \end{aligned}$$

i.e., $f\colon G \times G \to \mathrm{Aut}_\xi(\eta) = W \otimes \mathrm{Aut}(1_\varepsilon)$ is a 2-cocycle. Therefore, if we have $H^2(G, W \otimes \mathrm{Aut}(1_\varepsilon)) = 0$, then there exists $u\colon G \to \mathrm{Aut}_\xi(\eta)$ such that $f(\sigma, \tau) = u_\tau^\sigma u_{\sigma\tau}^{-1} u_\sigma$ for all σ, τ. In other words, $\phi_\sigma\sigma(\phi_\tau)\phi_{\sigma\tau}^{-1} = u_\sigma u_\tau^\sigma u_{\sigma\tau}^{-1} = \phi_\sigma\sigma(u_\tau)\phi_\sigma^{-1}u_\sigma u_{\sigma\tau}^{-1}$ i.e., $\sigma(\phi_\tau)\phi_{\sigma\tau}^{-1} = \sigma(u_\tau)\phi_\sigma^{-1}u_\sigma u_{\sigma\tau}^{-1}$ i.e., $(u_\sigma^{-1}\phi_\sigma) \cdot \sigma(u_\tau^{-1}\phi_\tau) = u_{\sigma\tau}^{-1}\phi_{\sigma\tau}$. Therefore if we replace the data $\{\phi_\sigma\colon \sigma(\eta) \to \eta\}$ by $\{\psi_\sigma = u_\sigma^{-1}\phi_\sigma\colon \sigma(\eta) \to \eta\}$ then $\psi_\sigma\sigma(\psi_\tau) = \psi_{\sigma\tau}$ for all σ, τ in G i.e., η is a G-equivariant extension of ξ.

A consequence of the above lemma is that our choice of ξ_1 in Step 0 is provided with an equivariant G-structure under the assumption that $H^2(G, V^* \otimes \mathrm{Aut}(1_\varepsilon)) = 0$. Thus assume that ξ_1 is extended to a G-equivariant ξ_n, which is versal for $\mathfrak{F}|\mathfrak{B}_n$. We want to extend ξ_n to ξ_{n+1}, which is versal for $\mathfrak{F}|\mathfrak{B}_{n+1}$ and also G-equivariant.

Step I. Set $s = \gamma(\xi)$ where $\xi = \xi_n$. If $s = \mathrm{Spec}(R)$ with the exact sequence $0 \to I \to \mathrm{Symm}(V^*) \to R \to 0$, then set $s'' = \mathrm{Spec}(\mathrm{Symm}(V^*)/\mathfrak{m}I)$, and consider all the subschemes of s'' containing s. If ξ can be extended to $s_i \subset s''(i = 1, 2)$, then it can be extended to $s_1 \cup s_2$ since $s_1 \cup s_2 = s_1 \amalg s_2'$ for some subscheme s_2' of s'' containing s. Therefore if we set s' to be the largest subscheme of s'' on which ξ can be extended, then we must have $s' \cup \sigma(s') \subset s'$ for all $\sigma \in G$ i.e., s' has to be G-invariant. We now pick some η over s', which extends ξ, and set $s' = s_{n+1}$.

Step II. Set $s = \mathrm{Spec}(R)$, $s' = \mathrm{Spec}(R')$, and let $0 \to W \to R' \to R \to 0$ be exact. Then $\mathfrak{m}W = 0$, and W is a G-invariant quotient of I and hence is a G-invariant subquotient of $\mathrm{Symm}(V^*)$. The canonical isomorphism

$$s' \amalg_s s' \simeq s' \amalg A_1(W^*)$$

entails a canonical equivalence of categories:

$$\mathfrak{F}(s') \overset{2}{\underset{\mathfrak{F}(s)}{\times}} \mathfrak{F}(s') \simeq \mathfrak{F}(s') \times \mathfrak{F}(A_1(W^*)). \tag{$*$}$$

Now η is an extension of ξ which is G-equivariant, and hence, for each $\sigma \in G$, the pair $(\eta, \sigma(\eta))$ defines an object in $\mathfrak{F}(s') \times^2_{\mathfrak{F}(s)} \mathfrak{F}(s')$ and hence an object λ_σ in $\mathfrak{F}(A_1(W^*))$ such that $\eta \amalg_\xi \sigma(\eta) \simeq \eta \amalg \lambda_\sigma$, where λ_σ is uniquely determined up to canonical isomorphism. By abuse of notation, we denote by the same symbol λ_σ its isomorphism class in $|\mathfrak{F}(A_1(W^*))| = W \otimes |\mathfrak{F}(\varepsilon)| = W \otimes V$. Now the commutative diagram

$$\begin{tikzcd}
s' \coprod_s s' \coprod_s s' \arrow[r, "\sim"] & s' \coprod_s (s' \amalg A_1(W^*)) \arrow[r, "\sim"] & s' \amalg A_1(W^*) \amalg A_1(W^*) \\
s' \coprod_s s' \arrow[u, "i_{1,3}"'] \arrow[rr, "\sim"] & & s' \amalg A_1(W^*) \arrow[u, "(1, \nabla)"']
\end{tikzcd}$$

applied to $\eta \amalg_\xi \sigma(\eta) \amalg_\xi \sigma\tau(\eta)$ over $s' \amalg_s s' \amalg_s s'$ yields a canonical isomorphism $\eta \amalg \lambda_{\sigma\tau} \simeq \eta \amalg \nabla^*(\lambda_\sigma, \sigma(\lambda_\tau))$ over $s' \amalg A_1(W^*)$, identity on the 1st factor, and hence we have $\lambda_{\sigma\tau} = \lambda_\sigma + \sigma(\lambda_\tau)$ in $|\mathfrak{F}(A_1(W^*))| = W \otimes V$. In other words, $\lambda\colon G \to W \otimes V$ determines a 1-cocycle. Consequently, if we have $H^1(G, W \otimes V) = 0$ then there exists $\zeta \in \operatorname{ob}(\mathfrak{F}(A_1(W^*)))$ such that $\lambda_\sigma = \zeta - \sigma(\zeta)$ for all $\sigma \in G$.

By virtue of $(*)$ above, we may and shall pick an extension η' of ξ over s' determined by the isomorphism $\eta \amalg_\xi \eta' \simeq \eta \amalg \zeta$. Then the above commutative diagram, applied to $\eta \amalg_\xi \sigma(\eta) \amalg_\xi \sigma(\eta')$ on $s' \amalg_s s' \amalg_s s'$, yields an isomorphism $\phi_\sigma\colon \sigma(\eta') \to \eta'$ extending the given isomorphism $\sigma\colon \sigma(\xi) \to \xi$. Replacing η by η', this proves an existence of data of G-equivariance on η, extending the given data on ξ.

Step III. Let $\xi \to \eta$ be an extension to s', admitting a data of G-equivariance $\{\phi_\sigma\colon \sigma(\eta) \to \eta \mid \sigma \in G\}$ extending that on ξ. By virtue of the above lemma, the given equivariant G-structure on ξ can be extended to η under the assumption that we have $H^2(G, W \otimes \operatorname{Aut}(1_\varepsilon)) = 0$.

Step IV. Assume that $H^1(G, W \otimes V) = 0 = H^2(G, W \otimes \operatorname{Aut}(1_\varepsilon))$ for all G-invariant subquotients W of $\operatorname{Symm}(V^*)$ where $V = |\mathfrak{F}(\varepsilon)|$. Pick ξ_1 over $A_1(V)$ as in Step 0. Then ξ_1 is provided with an equivariant G-structure, and $\tilde{\xi}_1\colon \operatorname{Mor}(\varepsilon, \gamma(\xi_1)) \to |\mathfrak{F}(\varepsilon)|$ is an isomorphism. Our hypothesis on the cohomology of G insures that we can proceed with the Steps I, II, III, getting a G-equivariant extension $\xi_1 \to \xi_2$. Since ξ_2 is G-equivariant, our hypothesis on the cohomology of G enables us to repeat the Steps I, II, III, getting a G-equivariant extension $\xi_2 \to \xi_3$, etc. Then $\xi = \varinjlim_n \xi_n$ is a versal object for $\mathfrak{F}$, and carries an equivariant G-structure.

We thus have established an existence under the hypothesis that $H^1(G, W \otimes V) = 0 = H^2(G, W \otimes \operatorname{Aut}(1_\varepsilon))$ for every G-invariant subquotient W of $\operatorname{Symm}(V^*)$ where $V = |\mathfrak{F}(\varepsilon)|$. We now deal with its uniqueness, which is simpler as usual.

Uniqueness. Given $\xi \in \operatorname{ob}(\hat{\mathfrak{F}})$ with $s = \gamma(\xi)$, we set $s_n =$ the maximal closed subscheme of s of order $< n$. We thus have $\cdot = s_0 \subset s_1 \subset s_2 \subset s_3 \subset \ldots$ and $s = \varinjlim s_n$. We set $\xi_n = \xi | s_n$. Now let ξ, η be any two G-equivariant versal object for $\mathfrak{F}$. We then have an isomorphism

$$\xi_1 \underset{\approx}{\xrightarrow{\phi_1}} \eta_1$$

by the nature of versality. We now assume that ϕ_1 had been extended to an equivariant isomorphism

$$\xi_n \xrightarrow{\phi_n} \eta_n.$$

By versality, ϕ_n can be extended to some isomorphism

$$\psi\colon \xi_{n+1} \to \eta_{n+1}.$$

We want to adjust ψ so that it becomes equivariant, i.e., $\psi \cdot \sigma = \sigma \cdot \sigma(\psi)$ i.e., $\psi^\sigma = \psi$ for all $\sigma \in G$. Set $u_\sigma = \psi^\sigma \cdot \psi^{-1}$. It is an automorphism of η_{n+1} inducing the identity on η_n, i.e., $u_\sigma \in \mathrm{Aut}_{\eta_n}(\eta_{n+1})$. Since $u_\tau^\sigma \cdot u_\sigma = (\psi^\tau \cdot \psi^{-1})^\sigma \cdot \psi^\sigma \cdot \psi^{-1} = \psi^{\sigma\tau} \cdot \psi^{-1} = u_{\sigma\tau}$, it is 1-cocycle. Consequently if we have $H^1(G, \mathrm{Aut}_{\eta_n}(\eta_{n+1})) = 0$, then $u_\sigma = u^\sigma \cdot u^{-1}$ for some element $u \in \mathrm{Aut}_{\eta_n}(\eta_{n+1})$ i.e., $\psi^\sigma \cdot \psi^{-1} = u^\sigma \cdot u^{-1}$, i.e., $(\psi \cdot u^{-1})^\sigma = \psi \cdot u^{-1}$ for all $\sigma \in G$. Thus if we set $\phi_{n+1} = \psi \cdot u^{-1}$, then it is an equivariant extension of ϕ_n. Since $\mathrm{Aut}_{\eta_n}(\eta_{n+1}) \simeq W \otimes \mathrm{Aut}(1_\varepsilon)$ as G-modules for some G-invariant subquotient W of $\mathrm{Symm}(V^*)$, we shall have uniqueness provided $H^1(G, W \otimes \mathrm{Aut}(1_\varepsilon)) = 0$ for every G-invariant subquotient W of $\mathrm{Symm}(V^*)$.

We thus have established

THEOREM. *Let $\gamma\colon \mathfrak{F} \to \mathfrak{B}$ be a homogeneous fibred category in groupoid together with equivariant G-action. We assume that $\mathfrak{F}(\cdot) = \{1\}$ and $V = |\mathfrak{F}(\varepsilon)|$ is finite-dimensional. If we have $H^1(G, W \otimes V) = 0 = H^2(G, W \otimes \mathrm{Aut}(1_\varepsilon))$ for every G-invariant subquotient W of* Symm(V^*), *then there exists a G-equivariant versal object for $\mathfrak{F}$. If $H^1(G, W \otimes \mathrm{Aut}(1_\varepsilon)) = 0$ for every G-invariant subquotient W of* Symm(V^*), *then a G-equivariant versal object is unique up to G-equivariant isomorphism.*

Now let a variety X_0 be either complete or affine with only a finite number of nonsmooth points. Then $|\mathfrak{D}_{X_0}(\varepsilon)|$ is well known to be finite-dimensional. Now let an algebraic group G act on X_0 algebraically. Then $\mathrm{Aut}(1_\varepsilon) = \mathrm{Aut}_{X_0}(X_0 \times \varepsilon)$ may be infinite-dimensional in case when X_0 is affine, but in any case it is a rational G-module. Consequently, if G is linearly reductive, then $H^i(G, W \otimes |\mathfrak{D}_{X_0}(\varepsilon)|) = 0 = H^i(G, W \otimes \mathrm{Aut}(1_\varepsilon))$ for all $i > 0$ and for every finite-dimensional G-module W. Therefore we have

COROLLARY. *Let G be an algebraic group and X_0 an algebraic variety together with an algebraic G-action where X_0 is assumed to be either complete or affine with only a finite number of nonsmooth points. If G is linearly reductive, then X_0 admits an equivariant G-structure on versal deformation, unique up to G-equivariant isomorphism.*

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