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# A THEOREM ON FREE ENVELOPES

BY

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ABSTRACT. The free envelope of a finite commutative semigroup was defined by Grillet [Trans. Amer. Math. Soc. **149** (1970), 665–682] to be a finitely generated free commutative semigroup $F(S)$ with identity and a homomorphism $\alpha$: $S \to F(S)$ endowed with certain properties. Grillet raised the following question: does $\alpha(S)$ always generate a pure subgroup of the free Abelian group with the same basis as $F(S)$? We prove this is indeed the case. It follows as a result of two lemmas.

Lemma 1: Given a full rank proper subgroup $H$ of a finitely generated free Abelian group $F$ and a basis $X$ of $F$ there exists a surjective homomorphism $f$: $F \to \mathbf{Z}$ such that $f$ is positive on $X$ and $f_{|H}$ is not surjective. Lemma 2: A finitely generated totally cancellative reduced subsemigroup of a finitely generated free Abelian group $F$ is contained in the positive cone of some basis of $F$. The following duality theorem is also proved. Let $S^* \cong \operatorname{Hom}(S, N)$ where $N$ is the nonnegative integers under addition. Then $S \cong S^{**}$ if and only if $S$ is isomorphic to a unitary subsemigroup of a finitely generated free commutative semigroup with identity.

The free envelope of a finitely generated commutative semigroup $S$ was introduced in [**1**]. It is a finitely generated free commutative semigroup $F(S)$ with identity together with a homomorphism $\alpha$: $S \to F(S)$ endowed with certain properties, in particular any homomorphism $f$: $S \to F$, where $F$ is a finitely generated free commutative semigroup with identity, factors, not necessarily uniquely, through $\alpha$.

The following question was raised in [**1**]: Does $\alpha(S)$ always generate a pure subgroup in the free Abelian group with the same basis as $F(S)$? We prove that this is indeed the case. As an application of this result, we prove the following duality theorem (where $S^* = \operatorname{Hom}(S, \mathbf{N})$, and $\mathbf{N}$ is the nonnegative integers under addition): $S \cong S^{**}$ if and only if $S$ is isomorphic to a unitary subsemigroup of a finitely generated free commutative semigroup with identity. The notation is the same as in [**1**], in particular, all semigroups are commutative, denoted additively, and identity elements are denoted by 0.

Recall that a commutative semigroup $S$ is said to be totally cancellative if it is power cancellative (i.e., $na = nb$ implies $a = b$ for all $n \in \mathbf{N} \setminus \{0\}$) and cancellative, and that a totally cancellative semigroup $S$ is said to be reduced if either it has no identity element or its group of units is trivial.

The proof of the main theorem depends on the following lemmas.

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LEMMA 1. *If $F$ is a finitely generated free Abelian group, $H$ is a subgroup of $F$ of the same rank as $F$, and $x_1, x_2, \ldots, x_n$ is a free generating set of $F$, then there exists a surjective homomorphism $\beta$: $F \to Z$ such that*:

(1) $\beta_{|H}$ *is not surjective*;

(2) $\beta(x_i) > 0$ *for* $i = 1,2, \ldots, n$.

PROOF. The proof will proceed by induction on the rank of $F$. If $\dim(F) = 1$, then $F \cong Z$. Thus define $\beta = \theta$ if $\theta(x_1) = 1$ and $\beta = -\theta$ otherwise, where $\theta$ is the isomorphism from $F$ to $Z$ above.

Assume the lemma for all free Abelian groups of rank $n - 1$, and let $F$ have rank $n$. Since $F/H$ is torsion there exists a smallest positive integer $n_i$ such that $n_i x_i \in H$ for $i = 1,2, \ldots, n$.

Let $F'$ be the free Abelian group generated by $x_2, x_3, \ldots, x_n$, and let $H' = H \cap F'$.

*Case* (1). Assume $H' = F'$. Then $n_1 > 1$ since $H$ is a proper subgroup of $F$. Let $\beta(x_1) = 1$ and $\beta(x_i) = n_1$ for $i = 2,3, \ldots, n$. Then $n_1 | \beta(h)$ for all $h \in H$. But $\beta$ is surjective since $\beta(x_1) = 1$.

*Case* (2). Assume $H' \neq F'$.

Since $H' \subseteq F'$ contains $n_2 x_2, \ldots, n_n x_n$, it has rank $n - 1$ and by the induction hypothesis there exists a surjective homomorphism $\beta'$: $F' \to Z$ such that $\beta'_{|H'}$ is not surjective and $c_i = \beta'(x_i) > 0$ for $i = 2,3, \ldots, n$. Let $c$ be the smallest positive integer in $\beta'(H')$. Then $\gcd_{i>1}(c_i) = 1$ and $c > 1$. Clearly there exists a smallest $k$ such that $0 < k \leqslant n_1$ and $(kx_1 + F') \cap H \neq \varnothing$. Thus there exists $\Sigma_{i>1} p_i x_i$ such that $0 \leqslant p_i < n_i$ and $kx_1 + \Sigma_{i>1} p_i x_i \in H$.

Let $j = \operatorname{lcm}_{i>1}(c_i)$ and $b = \gcd(k, \Sigma_{i>1}(cjn_i - c_i p_i))$. Define $c_1' = (\Sigma_{i>1} cjn_i - c_i p_i)/b$ and $c_i' = (k/b)c_i$ for $i = 2,3, \ldots, n$. It is clear that $\gcd_{i\geqslant 1}(c_i') = 1$ and $c_i' > 0$. Define $\beta(\Sigma_{i\geqslant 1} b_i x_i) = \Sigma_{i\geqslant 1} b_i c_i'$. Then $\beta$ is surjective since $\gcd_{i\geqslant 1}(c_i') = 1$ and $\beta(x_i) = c_i'$.

Let $x = \Sigma\, b_i x_i \in H$. If $b_1 = 0$ then $x \in H'$. Thus assume $b_1 \neq 0$, then $k | b_1$ by the choice of $k$, say $b_1 = sk$. Thus $x - s(kx_1 + \Sigma p_i x_i) \in H'$. Therefore, $H$ is generated by $H'$ and $kx_1 + \Sigma\, p_i x_i$. Hence

$$\left\langle H' \cup \left\{ kx_1 + \sum p_i x_i \right\} \right\rangle = H.$$

By the definition of $\beta$ it is clear that for all $r \in H'$, $c | \beta(r)$ and $\beta(kx_1 + \Sigma\, p_i x_i) = c(k/b)j\Sigma\, n_i$. Thus $c | \beta(kx_1 + \Sigma\, p_i x_i)$. Therefore, for all $r \in H$, $c | \beta(r)$, and thus the lemma is proved.

In what follows, $\langle x_1, \ldots, x_n \rangle$ denotes the subsemigroup (not the subgroup) generated by $x_1, \ldots, x_n$.

Let $S$ and $T$ be cancellative commutative semigroups, then $G(S)$ denotes the universal group of $S$, and if $f$: $S \to T$ is a homomorphism, then $G(f)$: $G(S) \to G(T)$ is the induced homomorphism.

LEMMA 2. *Let $S$ be a finitely generated totally cancellative reduced semigroup which is embedded into a finitely generated free Abelian group $F$. There exists a basis of $F$, $x_1, \ldots, x_n$, such that $S \subseteq \langle x_1, \ldots, x_n \rangle$.*

PROOF. From [1] there is a homomorphism $f\colon S \to \mathbf{N}$ which is positive on $S \setminus \{0\}$. Since $S$ is both commutative and cancellative $f$ can be extended to $G(f)\colon G(S) \to \mathbf{Z}$. Let $G(S)^d$ be the divisible hull of $G(S)$. Then $G(f)$ can be extended to $f'\colon G(S)^d \to \mathbf{Q}$. Since divisible groups are injective there is a homomorphism $g\colon G(F) \to \mathbf{Q}$ such that $g$ is positive on $S \setminus \{0\}$. Let $T = \operatorname{Ker}(g)$, the kernel of $g$. Then $T$ has rank $n - 1$ since $g$ has rank $n$, and hence $G(F)$ has a basis $x_1, \ldots, x_n$ such that $m_2x_2, \ldots, m_nx_n$ is a basis of $T$. Since $g(mx) = 0$ implies $g(x) = 0$, we see that $x_2, \ldots, x_n$ is a basis of $T$. Replacing $x_1$ by $-x_1$ if necessary we can also arrange that $g(x_1) > 0$. When $x = \sum p_ix_i \in S \setminus \{0\}$ we have $f(x) = p_1 f(x_1) > 0$. Thus $p_1 > 0$.

Let $\{s_1, s_2, \ldots, s_k\}$ be a generating set of $S$, and for each $i$, $s_i = \sum_{j>1} b_j^i x_j$. Then by the choice of $x_1$, $b_1^i > 0$ for each $i$. Therefore, pick $p \in \mathbf{N}$ such that $b_j^i + pb_1^i > 0$ for all $i$ and $j$. Define a new basis for $F$, $x_1', x_2', \ldots, x_n'$, from the old basis $x_1, x_2, \ldots, x_n$ by $x_1' = x_1 - \sum_{j>2} px_j$ and $x_i' = x_i$ for $i = 2, 3, \ldots, n$. Thus

$$s_i = b_1^i x_1' + \sum_{j>2} \left(b_j^i + pb_1^i\right) x_j',$$

and hence

$$S \subseteq \langle x_1', x_2', \ldots, x_n' \rangle.$$

THEOREM 3. *If $S$ is a finitely generated totally cancellative reduced semigroup, then $G(\alpha(S))$ is pure in $G(F(S))$, the universal group of the free envelope $(f(S), \alpha)$ of $S$.*

PROOF. Let $U$ denote the subgroup of $G(F(S))$ generated by $\alpha(S)$. Let $P$ denote the pure subgroup of $G(F(S))$ generated by $\alpha(S)$. Denote $P \cap F(S)$ by $F$. Now $U \subseteq P$, and let us assume that $P \neq U$. Then

(1) There exists a basis of $P$, $x_1, x_2, \ldots, x_n$ such that $\alpha(S) \subseteq \langle x_1, x_2, \ldots, x_n \rangle$, by Lemma 2.

(2) There exists a surjective homomorphism $\beta\colon P \to \mathbf{Z}$ such that $\beta_{|U}$ is not surjective and $\beta(x_i) > 0$ for all $i$, by Lemma 1.

Let $c$ be the smallest positive integer in the set $\beta(U)$. Then by (2), $c > 1$. Thus define $\beta'(a) = \beta(a)/c$ for all $a \in U$. Let $\beta'' = \beta' | S$. Then by (1) and (2), $\beta$, $\beta'$ and $\beta''$ are all positive on $S \setminus \{0\}$. Since $\beta'$ is the unique extension of $\beta''$ to $U$, and $\beta'$ has a unique extension to $\gamma\colon P \to \mathbf{Q}$ (as $x \in P$ implies $nx \in U$ for some $n \in \mathbf{N}$). Since $\beta' = \beta/c$ on $U$, then $\gamma = \beta/c$, and because $c > 1$ and $\gcd(\beta(P)) = 1$, $\gamma$ must have nonintegral values. Therefore, $\beta'$ cannot be extended to $P$ (as an integral valued homomorphism). On the other hand, the quasi-universal property of free envelopes says that $\beta''$ and hence $\beta'$ can be extended to $F(S)$ and thus to $G(F(S)) \supseteq P$, a contradiction. Therefore, $U = P$, and hence $U$ is pure in $G(F(S))$ which completes the proof of the theorem.

A subsemigroup $S$ of a semigroup $K$ is said to be unitary if $x + a \in S$ and $a \in S$ implies $x \in S$ for all $x, a \in S$.

Definition. If $S$ is a subsemigroup of a semigroup $K$, then the *unitary closure* of $S$ with respect to $K$, $U_k(S)$, is the smallest unitary subsemigroup of $K$ containing $S$. Note: If $S$ is a finitely generated totally cancellative reduced semigroup which is contained in a finitely generated free semigroup $F$, then $U_k(S) = G(S) \cap F$.

Definition. Let $S$ be a subsemigroup of a cancellative semigroup $K$, and let $P$ be the pure subgroup of $G(K)$ generated by $S$. Then the *pure unitary closure* of $S$ in $K$, $P_k(S)$, is $P \cap K$.

Corollary 4. *If $S$ is a finitely generated totally cancellative reduced semigroup, then the unitary closure of $\alpha(S)$ in the free envelope $(F(S), \alpha)$ of $S$ coincides with the pure unitary closure of $\alpha(S)$ in $F(S)$.*

Proof. This follows directly from the theorem.

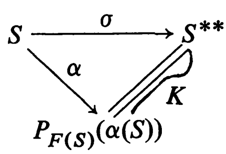
Definition. Let $S$ be a semigroup, and N be the positive integers under addition. Then the *dual* of $S$, $S^*$, is equal to $\mathrm{Hom}(S, \mathbf{N}^0)$. From [1] it is known that:

(1) $S^* \cong S^{***}$;

(2) The evaluation map $\sigma$: $S \to S^{**}$ is injective if $S$ is finitely generated totally cancellative reduced;

(3) If $f$: $S^{**} \to T$ is injective when restricted to $\sigma(S)$, then $f$ is injective;

(4) If $(F(S), \alpha)$ is the free envelope of $S$, then there exists an isomorphism, $K$, such that the following diagram commutes:




(5) If $f$: $S^{**} \to T$ is a homomorphism and $f|\sigma(S)$ is an epimorphism, then $f$ is an epimorphism.

Corollary 5. *If $S$ is a finitely generated totally cancellative reduced semigroup, then $S^{**} \cong U_{F(S)}(\alpha(S))$.*

Proof. The proof follows directly from Corollary 4 and Proposition 3.7 of [1].

Corollary 6. *Let $S$ be a finitely generated totally cancellative reduced semigroup. Then $S \cong S^{**}$. If and only if $S$ can be embedded as a unitary subsemigroup of a finitely generated free semigroup.*

Proof. Let $S \cong S^{**}$. Then by Corollary 4, $S \cong S^{**} U_{F(S)}(\alpha(S))$, and $U_{F(S)}(\alpha(S))$ is unitary in $F(S)$.

Conversely, assume that $S$ is a unitary subsemigroup of a finitely generated free semigroup $F$. Then by hypothesis $S = \bar{S}_F$. Therefore, there exists a homomorphism $f$: $F(S) \to F$ such that the following diagram commutes:

$$\begin{array}{ccc} U_{F(S)}(\alpha(S)) & \underset{i}{\subseteq} & F(S) \\ \uparrow \alpha & & \downarrow f \\ S & \underset{j}{\subseteq} & F \end{array}$$

Hence $f\alpha = 1_S$.

If $x \in F(S)$, $\alpha(s) \in \alpha(S)$ and $x + \alpha(s) \in \alpha(S)$, then $x \in U_{F(S)}(\alpha(S))$. Hence $a + x \in \alpha(S)$ for some $a \in \alpha(S)$. Thus $f(a) + f(x) \in S$. Therefore, $f(s) \in S$. Thus $U_{F(S)}(\alpha(S)) \subseteq S$. Thus there is an induced homomorphism $g$: $U_{F(S)}(\alpha(S)) \to S$ induced by the following commutative diagram:

$$\begin{array}{ccccc} & & U_{F(S)}(\alpha(S)) & \underset{i}{\subseteq} & F(S) \\ & \overset{\alpha}{\nearrow} & \downarrow g & & \downarrow f \\ S & = & S & \underset{j}{\subseteq} & F \end{array}$$

Therefore, $g\alpha = 1_S$ and also $\alpha g \alpha = \alpha = 1_{U_{F(S)}}(\alpha(S))\alpha$. Thus $\alpha g = 1_{U_{F(S)}}(\alpha(S))$, by property (5) above. Therefore, by Corollary 4, $S \cong U_{F(S)}(\alpha(S)) \cong S^{**}$.

## References

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