TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 257, Number 2, February 1980

THE LATTICE OF l-GROUP VARIETIES

BY

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ABSTRACT. For any type of abstract algebra, a variety is an equationally defined class of such algebras. Recently, attempts have been made to study varieties of lattice-ordered groups (l-groups). Martinez has shown that the set **L** of all l-group varieties forms a lattice under set inclusion with a compatible associative multiplication. Certain varieties $\mathcal{S}_p$ (p prime) have been proved by Scrimger to be minimal nonabelian varieties in **L**. In the present paper, it is shown that these varieties can be used to produce varieties minimal with respect to properly containing various other varieties in **L**. Also discussed are the relations among the $\mathcal{S}_n$ ($n \in N$), and it is established that all infinite collections of the $\mathcal{S}_n$ have the same least upper bound in **L**. Martinez has also classified l-groups using torsion classes, a generalization of the idea of varieties. It is proved here that **L** is not a sublattice of **T**, the lattice of torsion classes.

1. Introduction. As the study of varieties of groups has proved to be such a useful tool in group theory, it is natural that varieties of lattice-ordered groups (l-groups) are of interest to those concerned with l-group theory. The first works to appear in this area dealt mainly with specific varieties. Weinberg's work on abelian l-groups **[14]** led to the conclusion that the abelian variety $\mathcal{A}$ is the smallest nontrivial l-group variety. Martinez **[9]** considered a family of varieties $\mathcal{L}_n$ ($n \in N$) defined by the equation $x^n y^n = y^n x^n$. Scrimger **[13]**, working within the $\mathcal{L}_n$ showed that certain varieties $\mathcal{S}_p \subseteq \mathcal{L}_p$ cover $\mathcal{A}$ for all primes p. Holland **[6]** proved that the variety $\mathcal{N}$ of normal-valued l-groups contains every proper l-group variety.

A more comprehensive study of l-group varieties was made by Martinez **[10]**, in which the set **L** of all l-group varieties was shown to form a complete lattice, using containment as the partial order, with a compatible multiplication. The present paper is an attempt to extend the work on the structure of **L** and to further investigate the role of the $\mathcal{S}_n$'s within this lattice. The notation to be used will follow the lines of that used by H. Neumann **[12]**.

Starting near the top of the lattice **L**, we show in (2.2) that $\mathcal{N}$ is finitely join-irreducible. At the other extreme of the lattice, it is first shown in (3.11) and (3.12) that the sublattice of **L** generated by $\{\mathcal{S}_p \mid p \text{ is prime}\}$ is lattice-isomorphic to the lattice of finite sets of primes. However, this isomorphism does not extend to the complete sublattice generated by $\{\mathcal{S}_p \mid p \text{ is prime}\}$, as it is shown in (3.14) and (3.17) that the join of any infinite collection of $\mathcal{S}_n$'s is the variety generated by $Z \text{ Wr } Z$.

Received by the editors August 24, 1977.

AMS (MOS) subject classifications (1970). Primary 06A55, 08A15; Secondary 06A23, 06A35, 06A50.

Key words and phrases. Lattice-ordered group, variety, lattice, Scrimger variety, torsion class.

[1]This paper is a portion of the author's doctoral dissertation, written under the direction of W. Charles Holland.

0002-9947/80/0000-0052/$03.75

The other main result of this paper deals with the relationship between **L** and **T**, the set of all torsion classes of l-groups. It has been shown by Martinez [**11**] that **T** also forms a lattice and a semigroup. Holland [7] has shown that $\mathbf{L} \subseteq \mathbf{T}$. As both the meet and multiplication operations are identical in **L** and **T**, it is natural to ask whether **L** forms a sublattice of **T**. In (4.3) and (4.4) it is shown that this is not the case and that in general the join operations of **L** and **T** do not yield the same results.

2. The lattice-ordered semigroup L. A *variety* of l-groups is the class of all l-groups for which a given set of words (involving the group and lattice operations) are laws. Equivalently a variety of l-groups is a class of l-groups which is closed with respect to l-subgroups, l-homomorphic images, and cardinal products (direct products with componentwise ordering). Some of the l-group varieties of particular interest are $\mathfrak{E}$ the trivial variety, $\mathcal{A}$ the abelian variety, $\mathcal{R}$ the representable variety, $\mathcal{N}$ the normal-valued variety, and $\mathfrak{L}$ the variety of all l-groups.

The set **L** of all l-group varieties is known to be a dually-Brouwer and hence distributive lattice, with intersection functioning as the meet operation, and the join of a collection of varieties defined to be the intersection of all varieties containing those in the collection. Similarly, it is possible to describe the variety generated by an l-group G, l-var(G), as the intersection of all l-group varieties containing G. In addition, we may define an associative operation on **L** such that for $\mathcal{U}, \mathcal{V} \in \mathbf{L}$, $\mathcal{U}\mathcal{V}$ contains all l-groups G having an l-ideal in $\mathcal{U}$ whose quotient lies in $\mathcal{V}$. This operation is compatible with the set containment ordering on **L** and distributes over all meets and finite joins from the left.

Martinez [**10**] has shown that $G \in \mathcal{V}_1 \vee \mathcal{V}_2$ iff G is a subdirect product of l-groups from $\mathcal{V}_1$ and $\mathcal{V}_2$. He has also shown that $\mathcal{N}$ is idempotent in **L**, i.e. $\mathcal{N}\mathcal{N} = \mathcal{N}$. It is now to be shown that $\mathcal{N}$ is finitely join-irreducible as well; however, the proof of this fact will require the use of wreath products.

Let G and H be l-groups. Due to a theorem of Holland [**5**], there exist chains Ω_1 and Ω_2 such that G and H can be considered as l-groups of order-preserving permutations (l-permutation groups) on Ω_1 and Ω_2 respectively. It should be noted that an order-preserving permutation g on a chain Γ is considered positive if $\gamma g > \gamma$ for all $\gamma \in \Gamma$. Using this representation for G and H, let $(W, \Omega) = (G, \Omega_1) \operatorname{Wr} (H, \Omega_2)$ be the order-preserving permutation group on $\Omega = \Omega_1 \overleftarrow{\times} \Omega_2$, where all elements of W are of the form (θ, h) for $h \in H$ and $\theta \colon \Omega_2 \to G$, and the action of W on Ω defined by $(\alpha_1, \alpha_2)(\theta, h) = (\alpha_1\theta(\alpha_2), \alpha_2 h)$ for any $(\alpha_1, \alpha_2) \in \Omega$. It can be verified that the group operation on W is such that $(\theta, h)(\phi, k) = (\psi, hk)$, where for $\alpha \in \Omega_2$, $\psi(\alpha) = \theta(\alpha)\phi(\alpha h)$. Using this multiplication on W and the usual ordering of order-preserving permutations discussed above, W is an l-group [**8**], and is called the Wreath product of G with H. Within W,

$$W' = \{(\theta, h) \mid \theta(\alpha) = e_G \text{ for all but a finite subset of } \Omega_2\}$$

forms an l-subgroup which also acts as an l-permutation group on $\Omega = \Omega_1 \overleftarrow{\times} \Omega_2$. This l-subgroup is called the wreath product of G with H and is denoted by $(W', \Omega) = (G, \Omega_1) \operatorname{wr} (H, \Omega_2)$. When an l-group G is itself totally ordered, it is possible to consider G as an l-permutation group on itself, using right translation as the action. In such a case, G is usually preferred over the more cumbersome notation (G, G); so the notation (G, Ω_1) Wr H, G Wr (H, Ω_2), G Wr H, etc. make sense in this context and will be used where appropriate.

2.1. LEMMA. *$\mathcal{V} \in \mathbf{L}$ is finitely join-irreducible if $\mathcal{V}$ is closed under wreath (Wreath) products.*

PROOF. By way of contradiction, suppose $\mathcal{V}$ is finitely join-reducible. Then there exist $\mathcal{U}_1, \mathcal{U}_2 \in \mathbf{L}$ such that $\mathcal{U}_1 \subset \mathcal{V}$, $\mathcal{U}_2 \subset \mathcal{V}$ and $\mathcal{V} = \mathcal{U}_1 \vee \mathcal{U}_2$. Since a variety is generated by its subdirectly irreducible l-groups [4], there must exist subdirectly irreducible l-groups $H \in \mathcal{V} \setminus \mathcal{U}_1$ and $K \in \mathcal{V} \setminus \mathcal{U}_2$. Holland [5] has shown that there exist chains Ω_1 and Ω_2 such that (H, Ω_1) and (K, Ω_2) are transitive l-permutation groups, i.e. a group G acts transitively on a set Γ if for all $\gamma, \delta \in \Gamma$ there exists $g \in G$ such that $\gamma g = \delta$. Then $(W, \Omega) = (H, \Omega_1)$ wr (K, Ω_2) is also a transitive l-permutation group (similarly for the Wreath product), and $W \in \mathcal{V}$ by assumption. Thus, $W \in \mathcal{U}_1 \vee \mathcal{U}_2$, so W must be a subdirect product of l-groups from $\mathcal{U}_1$ and $\mathcal{U}_2$, i.e. there must exist l-ideals A_1 and A_2 in W such that $A_1 \cap A_2 = \{e\}$, $W/A_1 \in \mathcal{U}_1$, and $W/A_2 \in \mathcal{U}_2$. If A_1 were trivial, then $W \in \mathcal{U}_1$; and since H is l-embeddable in W, then $H \in \mathcal{U}_1$, contradicting our selection of H. If A_2 were trivial, then $W \in \mathcal{U}_2$; and since K is an l-homomorphic image of W, $K \in \mathcal{U}_2$, contradicting our selection of K. Thus, neither A_1 nor A_2 can be trivial. Hence, there must exist $h_1 \in A_1^+$, $h_2 \in A_2^+$, and $\alpha, \beta \in \Omega$ such that $\alpha h_1 > \alpha$ and $\beta h_2 > \beta$. Also, by transitivity, there exists $g \in W$ such that $\alpha g = \beta$. Now since $\beta g^{-1} h_1 g = \alpha h_1 g > \alpha g = \beta$, both $g^{-1} h_1 g \in A_1$ and $h_2 \in A_2$ move β. But this is impossible for disjoint l-ideals. Therefore, $\mathcal{V}$ must be finitely join-irreducible.

2.2. THEOREM. *$\mathcal{N}$ is finitely join-irreducible in* $\mathbf{L}$.

PROOF. Martinez [10] has shown that $\mathcal{N}$ is idempotent, and hence closed with respect to all extensions including Wreath and wreath products. Thus, by (2.1) $\mathcal{N}$ is finitely join-irreducible.

3. Properties of the Scrimger varieties. It is known that $\mathcal{A}$ is the smallest nontrivial variety in $\mathbf{L}$. In an attempt to generalize $\mathcal{A}$, the family of varieties $\mathcal{S}_n$ for $n \in N$ were introduced, where $\mathcal{S}_n$ is the variety of all l-groups in which all nth powers commute. Obviously, $\mathcal{S}_1 = \mathcal{A}$ and $\mathcal{S}_m \subseteq \mathcal{S}_n$ if $m|n$. Also, within each $\mathcal{S}_n$ is an l-group [9] of the form

$$G_n = \{(\phi, k) \in Z \text{ Wr } Z | \phi(i) = \phi(j) \text{ if } i \equiv j \ (\text{mod } n)\}.$$

Such an l-group may also be viewed as a lexicographically-ordered splitting extension by Z of a cardinal product of n copies of Z, with the multiplication defined by

$$((r_0, \ldots, r_{n-1}), k)((s_0, \ldots, s_{n-1}), l) \\ = ((r_0 + s_{0+k}, \ldots, r_{n-1} + s_{n-1+k}), k + l),$$

where r_i, s_j, k, $l \in Z$ and all subscripts are considered mod n. These l-groups and the varieties they generate, where $\mathcal{S}_n = l\text{-var}(G_n)$ is known as the nth Scrimger variety, have been shown to be of great interest in the study of **L**. Some of the more useful facts concerning the G_n and $\mathcal{S}_n$ are:

3.1. For any $n \in N$, G_n is an l-subgroup of $Z \text{ Wr } Z$.

3.2. For any $n \in N$, G_n is a subdirectly irreducible l-group.

3.3. For any $n \in N$, $G_n \in \mathcal{A}^2 = \mathcal{A}\mathcal{A}$.

3.4. For $m, n \in N$, $G_m \in \mathcal{L}_n$ iff $m|n$.

3.5. For $m, n \in N$, $\mathcal{L}_m \subseteq \mathcal{L}_n$ iff $m|n$.

3.6. For $m, n > 1$, $\mathcal{S}_m \vee \mathcal{S}_n \subset \mathcal{S}_{mn}$ and $\mathcal{L}_m \vee \mathcal{L}_n \subset \mathcal{L}_{mn}$.

3.7. For any $n \in N$, $\mathcal{R} \wedge \mathcal{L}_n = \mathcal{A}$ and $\mathcal{R} \wedge \mathcal{S}_n = \mathcal{A}$ **[9]**.

3.8. For $m, n \in N$ such that $(m, n) = 1$, $\mathcal{L}_m \wedge \mathcal{L}_n = \mathcal{A}$, $\mathcal{S}_m \wedge \mathcal{S}_n = \mathcal{A}$, and $\mathcal{L}_m \wedge \mathcal{S}_n = \mathcal{A}$ **[13]**.

3.9. For any prime p, $\mathcal{S}_p$ covers $\mathcal{A}$ in **L**, i.e. no variety lies between $\mathcal{A}$ and $\mathcal{S}_p$ **[13]**.

3.10. Proposition. *For any variety $\mathcal{V} \in \mathbf{L}$ and any prime p such that $\mathcal{V} \wedge \mathcal{S}_p = \mathcal{A}$, $\mathcal{V} \vee \mathcal{S}_p$ covers $\mathcal{V}$ in* **L**.

Proof. Since **L** is a distributive lattice, the interval $\{\mathcal{W} \in \mathbf{L} | \mathcal{V} \subset \mathcal{W} \subset \mathcal{V} \vee \mathcal{S}_p\}$ is lattice-isomorphic to the interval $\{\mathcal{U} \in \mathbf{L} | \mathcal{A} = \mathcal{V} \wedge \mathcal{S}_p \subset \mathcal{U} \subset \mathcal{S}_p\}$ **[3]**, and hence is empty.

Thus, it is known that for any prime p, $\mathcal{R} \vee \mathcal{S}_p$ covers $\mathcal{R}$. Also, for any integer n and prime p such that $p \nmid n$, $\mathcal{S}_n \vee \mathcal{S}_p$ covers $\mathcal{S}_n$ and $\mathcal{L}_n \vee \mathcal{S}_p$ covers $\mathcal{L}_n$. Extending this notion by means of induction leads us to the following result.

3.11. Corollary. *For any finite set of distinct primes $\{p_1, p_2, \ldots, p_m\}$, $\bigvee\{\mathcal{S}_{p_i} | i = 1, \ldots, m\}$ covers $\bigvee\{\mathcal{S}_{p_i} | i = 1, \ldots, m-1\}$.*

3.12. Proposition. *For any primes $p_1, \ldots, p_m, q_1, \ldots, q_n$,*

$$\bigvee\{\mathcal{S}_{p_i} | i = 1, \ldots, m\} \subseteq \bigvee\{\mathcal{S}_{q_j} | j = 1, \ldots, n\}$$

iff $\{p_1, \ldots, p_m\} \subseteq \{q_1, \ldots, q_n\}$.

Proof. Sufficiency is obvious. To prove necessity, suppose $\bigvee\{\mathcal{S}_{p_i} | i = 1, \ldots, m\} \subseteq \bigvee\{\mathcal{S}_{q_j} | j = 1, \ldots, n\}$. Then for any p_i,

$$\mathcal{S}_{p_i} \subseteq \mathcal{S}_{p_i} \wedge \left(\bigvee\{\mathcal{S}_{q_j} | j = 1, \ldots, n\}\right) = \bigvee\{(\mathcal{S}_{p_i} \wedge \mathcal{S}_{q_j}) | j = 1, \ldots, n\}.$$

Hence, for some j, $\mathcal{S}_{p_i} = \mathcal{S}_{p_i} \wedge \mathcal{S}_{q_j}$, implying that $p_i = q_j$.

Accordingly, the sublattice of **L** generated by $\{\mathcal{S}_p | p \text{ is prime}\}$ is lattice-isomorphic to the lattice of finite sets of primes. To investigate whether or not the complete sublattice of **L** generated by $\{\mathcal{S}_p | p \text{ is prime}\}$ is lattice-isomorphic to the full Boolean lattice of sets of primes, it is helpful to consider conditions under which an l-group is l-isomorphic to G_n for some $n \in N$.

3.13. Theorem. *An l-group G is l-isomorphic to G_n iff it is generated (as a group) by two positive elements a and b such that*

(i) *$a < b$, and*

(ii) *for $1 \leq i \leq n-1$, $b^{-i}ab^i \wedge a = e$ and $b^{-n}ab^n = a$.*

PROOF. To prove necessity, it need only be noted that the elements $((1, 0, \ldots, 0), 0)$ and $((0, \ldots, 0), 1)$ of S_n satisfy the conditions on a and b respectively.

To prove sufficiency, let us assume that G is an l-group generated by positive elements a and b satisfying conditions (i) and (ii). Define a function ν: $G_n \to G$ by:

$$((r_0, \ldots, r_{n-1}), k)\nu = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right)b^k,$$

where $r_i, k \in Z$. To verify that ν preserves the group operations, it should be noted that condition (ii) guarantees that all conjugates of a by powers of b commute in G. So, for arbitrary elements of G_n,

$$\begin{aligned}
&\left[((r_0, \ldots, r_{n-1}), k)((s_0, \ldots, s_{n-1}), l)\right]\nu \\
&\qquad = \left[((r_0 + s_{0+k}, \ldots, r_{n-1} + s_{n-1+k}), k + l)\right]\nu \\
&\qquad = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i + s_{i+k}}\right)b^{k+l},
\end{aligned}$$

while

$$\begin{aligned}
&((r_0, \ldots, r_{n-1}), k)\nu((s_0, \ldots, s_{n-1}), l)\nu \\
&\qquad = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right)b^k\left(\prod_{j=0}^{n-1} (b^{-j}ab^j)^{s_j}\right)b^l \\
&\qquad = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right)\left(\prod_{j=0}^{n-1} (b^{-j+k}ab^{j-k})^{s_j}\right)b^{k+l} \\
&\qquad = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right)\left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{s_{i+k}}\right)b^{k+l} \\
&\qquad = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i + s_{i+k}}\right)b^{k+l},
\end{aligned}$$

with all subscripts considered mod n. Hence, ν is a group homomorphism.

In order to prove that ν preserves the lattice operations as well, it is helpful to first show that b exceeds the l-ideal of G generated by a. Suppose there exists $m \in N$ such that $a^m \nless b$. If $a^m = b$, then a and b would commute. So, in fact, $a^m \nleq b$, and then $(b^{-1}ab)^m \nleq b$. By Holland's theorem [5], there exists a chain Γ such that G acts as an l-group of order-preserving permutations on Γ. Thus, there exists some $\gamma \in \Gamma$ for which

$$\gamma(b^{-1}ab)^m > \gamma b \geqslant \gamma. \tag{$*$}$$

Since γ is moved by $(b^{-1}ab)^m$, it is moved by $b^{-1}ab$. Then condition (ii) implies that $\gamma a = \gamma$, so $(\gamma b)(b^{-1}ab) = \gamma b$ and $(\gamma b)(b^{-1}ab)^m = \gamma b$. But by $(*)$, $(\gamma b)(b^{-1}ab)^m \geqslant \gamma(b^{-1}ab)^m > \gamma b$. Thus $a^m < b$ for all $m \in Z$, and similarly $(b^{-i}ab^i)^m < b$ for all $i, m \in Z$. Now this and the disjointness of the conjugates leads us to:

$$\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} = \bigvee_{i=0}^{n-1} (b^{-i}ab^i)^{r_i} < b$$

for any $r_0, \dots, r_{n-1} \in Z$. Again this inequality must in fact be proper, as a and b do not commute in G. Hence, b exceeds the l-ideal of G generated by a and so dominates the order on G.

Now for any $((r_0, \dots, r_{n-1}), k) \in G_n$, $((r_0, \dots, r_{n-1}), k) \vee ((0, \dots, 0), 0)$ is computed to be $((r_0, \dots, r_{n-1}), k)$ if $k > 0$, $((0, \dots, 0), 0)$ if $k < 0$, and $((r_0 \vee 0, \dots, r_{n-1} \vee 0), 0)$ if $k = 0$. In the case where $k > 0$,

$$\left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right) b^k \vee e = \left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right) b^k$$

since b^k dominates. Similarly, for $k < 0$,

$$\left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right) b^k \vee e = e.$$

For $k = 0$, let us divide $\{0, \dots, n-1\}$ into two disjoint sets $I = \{i | r_i > 0\}$ and $J = \{j | r_j < 0\}$. Then

$$\begin{aligned}\left(\prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i}\right) \vee e &= \left[\prod_{i \in I} (b^{-i}ab^i)^{r_i} \vee \prod_{j \in J} (b^{-j}ab^j)^{-r_j}\right] \prod_{j \in J} (b^{-j}ab^j)^{r_j} \\ &= \left[\prod_{i \in I} (b^{-i}ab^i)^{r_i} \prod_{j \in J} (b^{-j}ab^j)^{-r_j}\right] \prod_{j \in J} (b^{-j}ab^j)^{r_j} \\ &= \prod_{i \in I} (b^{-i}ab^i)^{r_i} = \prod_{i=0}^{n-1} (b^{-i}ab^i)^{r_i \vee 0}.\end{aligned}$$

Thus ν is an l-homomorphism.

Suppose $(\prod_{i=0}^{n-1}(b^{-i}ab^i)^{r_i})b^k = e$. Then $k = 0$, since b strictly exceeds all products of conjugates of a. Now let us divide $\{0, \dots, n-1\}$ into I and J as above. Then $\prod_{i \in I}(b^{-i}ab^i)^{r_i} = \prod_{j \in J}(b^{-j}ab^j)^{-r_j}$. So if some $r_i > 0$, then

$$\begin{aligned}(b^{-i}ab^i)^{r_i} &= (b^{-i}ab^i)^{r_i} \wedge \prod_{j \in J} (b^{-j}ab^j)^{-r_j} \\ &= \prod_{j \in J} \left((b^{-i}ab^i)^{r_i} \wedge (b^{-j}ab^j)^{-r_j}\right) = e,\end{aligned}$$

contradicting the fact that $b^{-i}ab^i > e$. Thus $r_0 = r_1 = \dots = r_{n-1} = 0$ and ν is one-to-one.

To complete the proof it is necessary only to note that since every element of G is of the form $a^{m_0}b^{m_1}a^{m_2} \cdots b^{m_q}$, such a term can be re-expressed as the image of some element of G_n by merely adding and subtracting appropriate powers of b throughout.

3.14. THEOREM. *If $\{m_\alpha | \alpha \in N\}$ is any infinite subset of N, then $\bigvee\{\mathcal{S}_{m_\alpha} | \alpha \in N\} = \bigvee \{\mathcal{S}_n | n \in N\}$, i.e. there is only one variety in* $\mathbf{L}$ *that is an infinite join of $\mathcal{S}_n$'s.*

PROOF. It is sufficient to show that for $n \in N$, $G_n \in \bigvee \{\mathcal{S}_{m_\alpha} | \alpha \in N\}$. If $n | m_\beta$ for some β, then $\mathcal{S}_n \subseteq \mathcal{S}_{m_\beta} \subseteq \bigvee \{\mathcal{S}_{m_\alpha} | \alpha \in N\}$; therefore, we may assume that for all m_α, $n \nmid m_\alpha$. We may also assume that $n < m_\alpha$ for all $\alpha \in N$, since there is only a finite set for which this is not the case, and that the set $\{\mathcal{S}_{m_\alpha}\}$ is ordered so that $m_\alpha < m_{\alpha+1}$ for all $\alpha \in N$.

Now let G be the cardinal product of the G_{m_α}'s. So elements of G are of the form $((\phi_1, k_1), (\phi_2, k_2), (\phi_3, k_3), \ldots)$, where for all $\alpha \in N$, $k_\alpha \in Z$ and $\phi_\alpha: Z \to Z$ such that $\phi_\alpha(j + m_\alpha) = \phi_\alpha(j)$ for every $j \in Z$, and the k_α dominate the order.

Let us pick $a, b \in G$ of the form $a = ((\psi_1, 0), (\psi_2, 0), (\psi_3, 0), \ldots)$ and $b = ((\mathbf{0}, 1), (\mathbf{0}, 1), (\mathbf{0}, 1), \ldots)$ where for all $\alpha \in N$, $\psi_\alpha(j)$ is 1 if $j \equiv 0 \pmod{n}$ and is 0 otherwise and $\mathbf{0}(j) = 0$ for $0 \leq j < m_\alpha$ and extended mod m_α to all $j \in Z$. Then certainly $b > a > e$ in G. For $1 \leq i \leq n - 1$,

$$b^{-i}ab^i = ((\psi_{1,i}, 0), (\psi_{2,i}, 0), (\psi_{3,i}, 0), \ldots),$$

where $\psi_{\alpha,i}(j) = \psi_\alpha(j - i)$ for all $\alpha \in N$ and $j \in Z$. Therefore,

$$b^{-i}ab^i \wedge a = ((\psi_{1,i} \wedge \psi_1, 0), (\psi_{2,i} \wedge \psi_2, 0)\ (\psi_{3,i} \wedge \psi_3, 0), \ldots),$$

where $(\psi_{\alpha,i} \wedge \psi_\alpha)(j) = 0$ for all $\alpha \in N$ and $0 < j \leq m_\alpha - 1$ and $(\psi_{\alpha,i} \wedge \psi_\alpha)(0)$ is 1 if $i \equiv m_\alpha \pmod{n}$ and is 0 otherwise. Similarly,

$$b^{-n}ab^n = ((\psi_{1,n}, 0), (\psi_{2,n}, 0), (\psi_{3,n}, 0), \ldots)$$

with $\psi_{\alpha,n}(j) = \psi_\alpha(j - n)$ for all $\alpha \in N$ and $j \in Z$. Hence,

$$b^{-n}ab^na^{-1} = ((\psi_{1,n} - \psi_1, 0), (\psi_{2,n} - \psi_2, 0), (\psi_{3,n} - \psi_3, 0), \ldots),$$

where for all $\alpha \in N$, $(\psi_{\alpha,n} - \psi_\alpha)(0) = \psi_\alpha(0 - n) - \psi_\alpha(0) = \psi_\alpha(m_\alpha - n) - \psi_\alpha(0) = 0 - 1 = -1$ since $n < m_\alpha$ and $n \nmid m_\alpha$; $(\psi_{\alpha,n} - \psi_\alpha)(j)$ is 1 if $m_\alpha \equiv (n - j) \pmod{n}$ and is 0 otherwise for $0 < j \leq n - 1$; and $(\psi_{\alpha,n} - \psi_\alpha)(j) = 0$ for $n \leq j \leq m_\alpha - 1$. So for any $\alpha \in N$, $(\psi_{\alpha,n} - \psi_\alpha)$ has exactly two nonzero entries, both of which occur within the first n terms.

Now, let M be the l-ideal of G generated by $b^{-n}ab^na^{-1}$ and $\{b^{-i}ab^i \wedge a | 1 \leq i \leq n - 1\}$. Obviously, $b \notin M$. Also, for any $g \in M$, where $g = ((\phi_1, 0), (\phi_2, 0), (\phi_3, 0), \ldots)$, there is a uniform bound on the number of nonzero entries for the ϕ_α. However, this is not true for the ψ_α, so $a \notin M$. Obviously, aM and bM in G/M satisfy the conditions of (3.13), so the l-subgroup of G/M generated by aM and bM is l-isomorphic to G_n. Thus, $G_n \in \bigvee \{\mathcal{S}_{m_\alpha} | \alpha \in N\}$.

Using this theorem, it is now possible to answer a question posed by Martinez **[10]**.

3.15. COROLLARY. **L** *is not a Brouwer lattice.*

PROOF. $\mathcal{S}_3 = \mathcal{S}_3 \wedge (\bigvee\{\mathcal{S}_{2^n} | n \in N\})$ according to (3.14), but $\bigvee\{(\mathcal{S}_3 \wedge \mathcal{S}_{2^n}) | n \in N\} = \mathcal{Q}$ by (3.8).

Note that the proof above is actually of a stronger result, i.e. meets do not distribute over infinite joins even of towers in **L**. Also (3.14) eliminates the possibility of dropping the idea of taking quotients when generating an infinite join of varieties.

Since for all $n \in N$, G_n is an l-subgroup of $Z \text{ Wr } Z$, $\bigvee\{\mathcal{S}_n | n \in N\} \subseteq l$-var$(Z \text{ Wr } Z)$. In order to determine whether this containment is proper, it is necessary to look at a generalization of (3.13).

3.16. THEOREM. *An l-group G is l-isomorphic to Z wr Z iff it is generated (as a group) by two positive elements a and b such that*
(i) $a < b$, *and*
(ii) $b^{-i}ab^i \wedge a = e$, *for all* $i \in N$.

PROOF. To prove necessity, the elements $a = (\theta, 0)$ and $b = (\mathbf{0}, 1)$ of Z wr Z, where $\theta(j)$ is 1 if $j = 0$ and is 0 otherwise and $\mathbf{0}(j) = 0$ for all $j \in Z$, can readily be shown to generate Z wr Z and satisfy conditions (i) and (ii).

For the proof of sufficiency, define ν: Z wr $Z \to G$ by $(\phi, k)\nu = (\Pi_{i \in Z}(b^{-i}ab^i)^{\phi(i)})b^k$. Since for only a finite number of integers i is $\phi(i) \neq 0$, this function is well defined. By arguments analogous to those of (3.13), it can be shown that ν is an l-isomorphism.

3.17. THEOREM. $\bigvee\{\mathfrak{S}_n | n \in N\} = l\text{-var}(Z \text{ Wr } Z) = l\text{-var}(Z \text{ wr } Z)$.

PROOF. First it should be noted that Z Wr Z is an $\mathfrak{L}$-extension of Z wr Z, i.e. every element of Z Wr Z can be expressed as a join of a pairwise disjoint subset of elements of Z wr Z. As Bernau [1] has shown that l-group varieties are closed under $\mathfrak{L}$-extensions, it suffices simply to show that Z wr $Z \in \bigvee\{\mathfrak{S}_n | n \in N\}$.

Let G be the cardinal product of the G_n. Then elements of G can be expressed in the form $((\phi_1, k_1), (\phi_2, k_2), (\phi_3, k_3), \ldots)$, where for every $n \in N$, $k_n \in Z$ and ϕ_n: $Z \to Z$ such that $\phi_n(j + n) = \phi_n(j)$ for all $j \in Z$, and where the k_n dominate the order in G. Let us pick $a, b \in G$ of the form $a = ((\psi_1, 0), (\psi_2, 0), (\psi_3, 0), \ldots)$ and $b = ((\mathbf{0}, 1), (\mathbf{0}, 1), \ldots)$ where for all $n \in N$, $\psi_n(j)$ is 1 if $j \equiv 0 \pmod n$ and is 0 otherwise and $\mathbf{0}(j) = 0$ for all $j \in Z$. Thus, $b > a > e$.

For any $i \in N$, $b^{-i}ab^i = ((\psi_{1,i}, 0), (\psi_{2,i}, 0), (\psi_{3,i}, 0), \ldots)$ where $\psi_{n,i}(j) = \psi_n(j - i)$ for all $n \in N$ and $j \in Z$. Therefore,

$$b^{-i}ab^i \wedge a = ((\psi_{1,i} \wedge \psi_1, 0), (\psi_{2,i} \wedge \psi_2, 0), (\psi_{3,i} \wedge \psi_3, 0), \ldots)$$

where for any $n \in N$, $(\psi_{n,i} \wedge \psi_n)(j) = 1$ if both $i, j \equiv 0 \pmod n$ and is 0 otherwise. So for any given i, only a finite number of n's will divide i; thus, only a finite number of the $(\psi_{n,i} \wedge \psi_n)$ will be nonzero functions for that i.

Let M be the l-ideal of G generated by $\{b^{-i}ab^i \wedge a | i \in N\}$. Obviously, $b \notin M$; and since all of the ψ_n are nonzero functions, $a \notin M$. Hence the l-subgroup of G/M generated by aM and bM is l-isomorphic to Z wr Z by (3.16), and Z wr Z $\in \bigvee \{\mathfrak{S}_n | n \in N\}$.

4. Varieties and torsion classes. An alternate system of classification of l-groups is that of torsion classes [11]. A *torsion class* is a collection of l-groups that is closed under taking convex l-subgroups, l-homomorphic images, and joins of convex l-subgroups in the collection. Let **T** denote the class of all torsion classes of l-groups. Then using definitions analogous to those for varieties, it is possible to define the operations of meet, join, and multiplication on **T**. It is also worthy of note that for any l-group G, the join of all convex l-subgroups of G belonging to a torsion class $\mathfrak{T}$ is itself in $\mathfrak{T}$. This join is called the $\mathfrak{T}$-torsion radical of G, denoted $\mathfrak{T}(G)$, and is a characteristic l-ideal of G. It follows that $G \in \mathfrak{T}$ iff $G = \mathfrak{T}(G)$, and $G \in \mathfrak{T} \vee \mathfrak{U}$ iff $G = \mathfrak{T}(G) + \mathfrak{U}(G)$ for any $\mathfrak{T}, \mathfrak{U} \in \mathbf{T}$.

Holland has shown that all l-group varieties are torsion classes [7], and Martinez has shown that **T** is a complete Brouwer lattice [11]. Hence, by (3.15) we have that **L** is not a complete sublattice of **T**, as otherwise it would inherit the Brouwerian structure. It then remains to ask whether **L** is even a sublattice of **T**.

4.1. DEFINITION. A variety $\mathcal{V}$ is said to have the *lex property* if for every $G \in \mathcal{V}$, $G_\# \in \mathcal{V}$, where $G_\# = G \overleftarrow{\times} Z$.

4.2. PROPOSITION. *If* $\{\mathcal{V}_\alpha | \alpha \in \mathfrak{A}\}$ *is a collection of varieties having the lex property and* $\mathcal{U} \in \mathbf{L}$, *then*:

(i) $\bigwedge \mathcal{V}_\alpha$ *and* $\bigvee \mathcal{V}_\alpha$ *have the lex property*,

(ii) $\mathcal{U}\mathcal{V}_\alpha$ *has the lex property*.

PROOF. (i) That $\bigwedge \mathcal{V}_\alpha$ has the lex property is obvious. If $G \in \bigvee \mathcal{V}_\alpha$, then $G \approx H/K$ where H has l-ideals N_α with $\bigcap N_\alpha = \{e\}$ and $H/N_\alpha \in \mathcal{V}_\alpha$ for all $\alpha \in \mathfrak{A}$. Then $G_\# \approx (H/K)_\# \approx H_\#/(K \times \{0\})$, where $N_\alpha \times \{0\}$ is an l-ideal of $H_\#$ with $\bigcap (N_\alpha \times \{0\}) = \{(e, 0)\}$ and $H_\#/(N_\alpha \times \{0\}) \approx (H/N_\alpha)_\# \in \mathcal{V}_\alpha$, since $\mathcal{V}_\alpha$ has the lex property. Thus, $G_\# \in \bigvee \mathcal{V}_\alpha$.

(ii) If $G \in \mathcal{U}\mathcal{V}_\alpha$, then G has an l-ideal $N \in \mathcal{U}$ with $G/N \in \mathcal{V}_\alpha$. Therefore, $N \times \{0\}$ is an l-ideal of $G_\#$ with $G_\#/(N \times \{0\}) \approx (G/N)_\# \in \mathcal{V}_\alpha$ since $\mathcal{V}_\alpha$ has the lex property. Hence, $G_\# \in \mathcal{U}\mathcal{V}_\alpha$.

4.3. THEOREM. *Let* $\{\mathcal{V}_\alpha | \alpha \in \mathfrak{A}\}$ *be a collection of varieties, all having the lex property, such that* $\mathcal{V}_\beta \subset {}_{\mathbf{L}}\bigvee \{\mathcal{V}_\alpha | \alpha \in \mathfrak{A}\}$ *for all* $\beta \in \mathfrak{A}$. *Then* ${}_{\mathbf{L}}\bigvee \mathcal{V}_\alpha \supset {}_{\mathbf{T}}\bigvee \mathcal{V}_\alpha$ (*where* ${}_{\mathbf{L}}\bigvee$ *and* ${}_{\mathbf{T}}\bigvee$ *denote joins in* **L** *and* **T** *respectively*).

PROOF. Since $\mathbf{L} \subseteq \mathbf{T}$, ${}_{\mathbf{L}}\bigvee \mathcal{V}_\alpha \supseteq {}_{\mathbf{T}}\bigvee \mathcal{V}_\alpha$. Thus, we need only show the containment to be proper. Without loss of generality, we may assume that $\mathfrak{A}$ is well-ordered and that for any $\alpha \in \mathfrak{A}$, $\mathcal{V}_\alpha \not\subseteq \bigcup \{\mathcal{V}_\beta | \beta < \alpha\}$; since otherwise, such "unneeded" varieties could be thrown out without affecting the final join in either **L** or **T**.

If $\mathfrak{A}$ has no largest element, then choose $G_\alpha \in \mathcal{V}_\alpha \setminus \bigcup \{\mathcal{V}_\beta | \beta < \alpha\}$, for each $\alpha \in \mathfrak{A}$. Then by (4.2), $G = (\prod G_\alpha)_\# \in {}_{\mathbf{L}}\bigvee \mathcal{V}_\alpha$. If $G \in {}_{\mathbf{T}}\bigvee \mathcal{V}_\alpha$, then $G = \sum \mathcal{V}_\alpha(G)$, where $\mathcal{V}_\alpha(G)$ is the $\mathcal{V}_\alpha$-torsion radical of G and hence an l-ideal of G. All proper l-ideals of G are contained within $(\prod G_\alpha) \times \{0\}$, and such l-ideals cannot generate G. Thus, for some $\beta \in \mathfrak{A}$, $G = \mathcal{V}_\beta(G)$, implying that $G \in \mathcal{V}_\beta$. Since $G_{\beta+1}$ is an l-subgroup of G chosen outside of $\mathcal{V}_\beta$, the desired contradiction has been reached.

If $\mathfrak{A}$ has a largest element α_0, then for some $\gamma \in \mathfrak{A}$, $\mathcal{V}_\gamma \not\subseteq \mathcal{V}_{\alpha_0}$. So, choose $G_\alpha \in \mathcal{V}_\alpha \setminus \bigcup \{\mathcal{V}_\beta | \beta < \alpha\}$ for each $\alpha \in \mathfrak{A}$ as before, and choose $H_\gamma \in \mathcal{V}_\gamma \setminus \mathcal{V}_{\alpha_0}$. Then $G = (H_\gamma \times \prod G_\alpha)_\# \in {}_{\mathbf{L}}\bigvee \mathcal{V}_\alpha$, and by an argument similar to that above, $G \notin {}_{\mathbf{T}}\bigvee \mathcal{V}_\alpha$.

In order to state finally that **L** is not a sublattice of **T**, it remains to be determined whether or not there exist varieties having the lex property, and if some exist, how prevalent they are in **L**. First, as a lex product is simply a direct product of the groups involved, any l-group variety definable solely in terms of its group operations will have the lex property. Thus, $\mathcal{A}$ and $\mathcal{L}_n$ $(n \in N)$ have the lex property. In fact, all of the varieties to which reference has been made within this paper have the lex property, making **L** very far indeed from being a sublattice of **T**.

4.4. **Proposition.** $\mathcal{N}$, $\mathcal{R}$, $\mathcal{W}$ *(the weakly abelian variety), and* $\mathcal{S}_n$ $(n \in N)$ *have the lex property.*

Proof. (i) For $G \in \mathcal{N}$, we wish to show that $x_1^2x_2^2 > x_2x_1$ for all positive elements $x_1, x_2 \in G_{\#}$. Let $x_1 = (g, k)$ and $x_2 = (h, l)$ where $g, h \in G$ and $k, l \in Z$. These elements are positive iff $k > 0$ or $k = 0$ and $g \in G^+$ and $l > 0$ or $l = 0$ and $h \in G^+$. If either $k > 0$ or $l > 0$, then $2k + 2l > l + k$, making

$$(g, k)^2(h, l)^2 = (g^2h^2, 2k + 2l) > (hg, l + k) = (h, l)(g, k).$$

If both $k, l = 0$, then $(g, 0)^2(h, 0)^2 = (g^2h^2, 0) > (hg, 0) = (h, 0)(g, 0)$ since $G \in \mathcal{N}$. Thus, $G_{\#} \in \mathcal{N}$.

(ii) For $G \in \mathcal{R}$, we wish to show that $x_1 \wedge (x_2^{-1}x_1^{-1}x_2) < e$ for all $x_1, x_2 \in G_{\#}$. Again letting $x_1 = (g, k)$ and $x_2 = (h, l)$, we have that

$$(g, k) \wedge (h, l)^{-1}(g, k)^{-1}(h, l) = (g, k) \wedge (h^{-1}g^{-1}h, -k),$$

which equals (g, k) if $k < 0$ and $(g \wedge h^{-1}g^{-1}h, 0)$ if $k = 0$ and $(h^{-1}g^{-1}h, -k)$ if $k > 0$. In all cases, the outcome is less than or equal to $(e, 0)$; so $G_{\#} \in \mathcal{R}$.

(iii) For $G \in \mathcal{W}$, it is sufficient to show that $x_1^2 > x_2^{-1}x_1x_2$ for any x_2 and positive x_1 in $G_{\#}$. Letting $x_1 = (g, k)$ and $x_2 = (h, l)$, with either $k > 0$ or $k = 0$ and $g \in G^+$, then clearly $(g, k)^2 = (g^2, 2k) > (h^{-1}gh, k) = (h, l)^{-1}(g, k)(h, l)$. So $G_{\#} \in \mathcal{W}$.

(iv) To show that $\mathcal{S}_n$ has the lex property, it suffices to show that $(G_n)_{\#} \in \mathcal{S}_n$. Let $G = (\prod_{i=1}^{\infty} G_n)/(\sum_{i=1}^{\infty} G_n)$; then $G \in \mathcal{S}_n$. Let H_n be the diagonal l-subgroup of $\prod_{i=1}^{\infty} G_n$, with K_n its canonical image in G. Clearly K_n and G_n are l-isomorphic; let us denote this l-isomorphism by ν. Let $g \in G$ correspond to $((\mathbf{0}, n), (\mathbf{0}, 2n), (\mathbf{0}, 3n), \dots) \in \prod_{i=1}^{\infty} G_n$. Then for any $h \in K_n$, $g > h$ and $gh = hg$. Hence, the l-subgroup of G generated by K_n and g is l-isomorphic to $(G_n)_{\#}$ via the map which sends hg^m to $(h\nu, m)$ for $h \in K_n$ and $m \in Z$. So $(G_n)_{\#} \in \mathcal{S}_n$.

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