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DECOMPOSITION OF NONNEGATIVE GROUP-MONOTONE MATRICES

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ABSTRACT. A decomposition of nonnegative matrices with nonnegative group inverses has been obtained. This decomposition provides a new approach to the solution of problems relating to nonnegative matrices with nonnegative group inverses. As a consequence, a number of results are derived. Our results, among other things, answer a question of Berman, extend the theorems of Berman and Plemmons, DeMarr and Flor.

1. Introduction. Let A be an $m \times n$ real matrix. Consider the equations: (1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^T = AX$, (4) $(XA)^T = XA$, and (5) $AX = XA$ where X is an $n \times m$ real matrix and T denotes the transpose. For a rectangular matrix A and for a nonempty subset λ of $\{1, 2, 3, 4, 5\}$, X is called a λ-inverse of A if X satisfies equations (i) for each $i \in \lambda$. In particular, the $\{1, 2, 3, 4\}$-inverse of A is the unique Moore-Penrose generalized inverse and is denoted by $A^\dagger$. A $\{1, 2\}$-inverse of A which satisfies (5) is necessarily square and is called a group inverse. The group inverse of a matrix A, if it exists, is unique and is denoted by $A^\#$.

A matrix A is called group-monotone if $A^\#$ exists and is nonnegative. A matrix $A = (a_{ij})$ is called 0-symmetric if $a_{ij} = 0$ implies $a_{ji} = 0$. Thus every symmetric matrix and every positive matrix is 0-symmetric. A is called range-Hermitian (also called EPr) if the range of A is equal to the range of A^T, i.e., $R(A) = R(A^T)$. A is range-Hermitian if and only if $AA^\dagger = A^\dagger A$ and so $A^\dagger = A^\#$. An $m \times n$ matrix $A = (a_{ij})$ is called row (or column) stochastic if $a_{ij} \geqslant 0$ and $\sum_{j=1}^n a_{ij} = 1$, $1 \leqslant i \leqslant n$ (or $\sum_{i=1}^n a_{ij} = 1$, $1 \leqslant j \leqslant n$). If a matrix A is a direct sum of matrices S_i, then S_i will be called summands of A. A nonzero matrix A is called a zero divisor if $AB = 0$ or $BA = 0$ for some nonzero matrix B. For all other terminology the reader is referred to Ben-Israel and Greville [1].

Theorem 1 of this paper characterizes all nonnegative matrices A which have nonnegative group inverses; equivalently, $A^{(1,2)} = p(A) \geqslant 0$, where $p(A)$ is a polynomial in A with scalar coefficients. This theorem generalizes the known results for nonnegative matrices A whose $A^\dagger$ is A [2] or, more generally, $A^\dagger$ is some polynomial in A [7]. The solution to the problem raised by Berman of the characterization of all nonnegative matrices which are equal to a $\{1\}$- or $\{1, 2\}$-inverse of themselves also comes as a special case of Theorem 1. As a consequence of Theorem 1, we show (Corollary 2) that if A is a nonnegative matrix with $A^m = A$, $m > 2$, then $A = A_1 + A_2 + \cdots + A_k$, where $A_i \geqslant 0$; $A_i^m = A_i$; $A_iA_j = 0$, $i \neq j$;

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$d_i = \operatorname{rank} A_i$, $d_i | m - 1$. This generalizes the theorem of DeMarr [5] for nonnegative idempotent matrices. Corollary 3 (Corollary 4) of Theorem 2 shows that for a nonnegative range-Hermitian (row stochastic) matrix A with $A^{\#} > 0$, $A^{\#} = A^{\dagger} = HA^m = A^m H$ ($A^{\#} = A^{\dagger} = A^m$) where H is a diagonal matrix with all entries positive. Theorem 4 characterizes all nonnegative rank factorizations of nonnegative group-monotone matrices. Theorems of Berman and Plemmons [3, Theorem 2 and Theorem 3] are also consequences of the characterizations obtained in Theorem 4. Our results, among others, depend on the following theorems proved in [6] and [7].

THEOREM A ([6, THEOREM 2]). *If E is a nonnegative idempotent matrix of rank r with no row or column completely zero. Then there exists a permutation matrix P such that*

$$PEP^T = \begin{bmatrix} x_1 y_1^T & & 0 \\ & \ddots & \\ 0 & & x_r y_r^T \end{bmatrix}$$

where x_i, y_i are positive vectors with $y_i^T x_i = 1$. In particular, E is 0-symmetric.

THEOREM B ([7, REMARK 3]). *Let A be a nonnegative matrix and $p(A) = \alpha_1 A^{m_1} + \cdots + \alpha_k A^{m_k}$, $\alpha_i \neq 0$, $m_i > 0$, such that $p(A) > 0$, $Ap(A)$ is 0-symmetric, $Ap(A)A = A$, and* $\operatorname{rank} A = \operatorname{rank} p(A)$. *Then there exists a permutation matrix P such that PAP^T is a direct sum of matrices of the following three types (not necessarily all)*

(I) *$\beta x y^T$, where x and y are positive vectors with $y^T x = 1$, and β is some positive number satisfying $\sum_{m_i} \alpha_i \beta^{m_i+1} = 1$:*

(II)

$$\begin{bmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta_{d-1d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 \end{bmatrix}$$

where x_i and y_i are positive vectors of the same order with $y_i^T x_i = 1$, x_i and x_j, $i \neq j$, are not necessarily of the same order, and $\beta_{12}, \beta_{23}, \ldots, \beta_{d1}$ are arbitrary positive numbers with $d > 1$ and $d | m_i + 1$ for some m_i such that the product $\beta_{12}\beta_{23} \cdots \beta_{d1}$ is a common root of the following system of at most d equations in t:

$$\sum_{d|(m_i+1)} \alpha_i t^{(m_i+1)/d} = 1,$$

$$\sum_{d|(m_i+1-k)} \alpha_i t^{(m_i+1-k)/d} = 0, \qquad k \in \{1, \ldots, d-1\},$$

where the summation in each of the above equations runs over all those m_i for which $d|(m_i + 1 - k)$, $k = 0, 1, \ldots, d-1$, with the convention that if there is no m_i for which $d|(m_i + 1 - k)$, $k \in \{1, \ldots, d-1\}$, then the corresponding equation is absent.

(III) *A zero matrix.*

In particular, if all $\alpha_i > 0$ then β in type (I) *and the product $\beta_{12}\beta_{23} \cdots \beta_{d1}$ in type* (II) *are unique. Further, in this case the positive integer d, i.e. the rank of a matrix of type* (II), *must divide each $m_i + 1$.*

The concept of 0-symmetry has played a crucial role in the development of this paper.

2. Main results. Let A be any $n \times n$ matrix. Let us group the indices $i = 1, 2, 3, \ldots, n$ into four disjoint sets according to whether the ith row and the ith column of A are both zero, or the ith row is zero but the ith column is not, and so on. Then by simultaneously rearranging rows and columns, we can find a permutation matrix P such that

$$PAP^T = \begin{bmatrix} K & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M & N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the diagonal blocks are square matrices such that K and L have no zero rows in common, and K and M have no zero columns in common. It may be noted that in certain situations some of the blocks may be absent. For example, if A is row stochastic then

$$PAP^T = \begin{pmatrix} K & 0 \\ M & 0 \end{pmatrix}.$$

In view of the frequent use of the above representation of a matrix throughout this paper, we record it in the following lemma.

LEMMA 1. *Let A be a square matrix. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{bmatrix} K & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M & N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where K and L have no zero rows in common, and K and M have no zero columns in common.

THEOREM 1. *Let A be a nonnegative matrix and $A^{(1,2)} = p(A) > 0$, where $p(A) = \alpha_1 A^{m_1} + \cdots + \alpha_k A^{m_k}$, $\alpha_i \neq 0$, $m_i > 0$. Then there exists a permutation matrix P such that*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are some nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (*not necessarily both*):

(I) *βxy^T, where x and y are positive vectors with $y^Tx = 1$ and β is a positive root of*

$$\sum_{m_i} \alpha_i t^{m_i+1} = 1. \tag{6}$$

(II)

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where x_i and y_i are positive vectors of the same order with $y_i^Tx_i = 1$; x_i and x_j, $i \neq j$, are not necessarily of the same order; and $\beta_{12}, \ldots, \beta_{d1}$ are arbitrary positive numbers with $d > 1$ and $d|m_i + 1$ for some m_i such that the product $\beta_{12}\beta_{23}\cdots\beta_{d1}$ is a common root of the following system of at most d equations in t

$$\sum_{d|(m_i+1)} \alpha_i t^{(m_i+1)/d} = 1, \tag{7}$$

$$\sum_{d|(m_i+1-k)} \alpha_i t^{(m_i+1-k)/d} = 0, \qquad k \in \{1, 2, \ldots, d-1\}, \tag{8}$$

where the summation in each of the above equations runs over all those m_i for which $d|(m_i + 1 - k)$, $k = 0, 1, 2, \ldots, d-1$, with the convention that if there is no m_i for which $d|(m_i + 1 - k)$, $k \in \{1, \ldots, d-1\}$, then the corresponding equation is absent.

In particular, if all $\alpha_i > 0$ then β in type (I) *and the product $\beta_{12}\beta_{23}\cdots\beta_{d1}$ in type* (II) *are unique. Further, in this case the positive integer d, i.e. the rank of a matrix of type* (II), *must divide each $m_i + 1$.*

Conversely, if for some permutation matrix P

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are arbitrary nonnegative matrices of appropriate sizes and J is a direct sum of matrices of the following types (not necessarily both):

(I′) βxy^T, $\beta > 0$, *x, y are positive vectors with* $y^Tx = 1$.

(II′)

$$\begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $\beta_{ij} > 0$, x_i and y_i are positive vectors with $y_i^Tx_i = 1$, then $A^{(1,2)} \geqslant 0$ and is equal to some polynomial in A with scalar coefficients.

PROOF. By Lemma 1, there exists a permutation matrix P_1 such that

$$P_1AP_1^T = \begin{bmatrix} K & L & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M & N & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where K, L, M, and N are nonnegative matrices such that K and L have no zero rows in common, and K and M have no zero columns in common. Since $A^{(1,2)} = p(A)$, we have $A^2p(A) = A$ and $A(p(A))^2 = p(A)$. From $A^2p(A) = A$ we obtain $K^2p(K) = K$, $Kp(K)L = L$, $MKp(K) = M$, and $Mp(K)L = N$. Hence $Kp(K)$ is a nonnegative idempotent matrix. Since $Kp(K)K = K$ and $Kp(K)L = L$ have no zero rows in common, $Kp(K)$ cannot have a zero row. Similarly, no column of $Kp(K)$ is zero. Thus by Theorem A, $Kp(K)$ is 0-symmetric. Similarly, from $A(p(A))^2 = p(A)$, we obtain $K(p(K))^2 = p(K)$, which, together with $K^2p(K) = K$, gives rank $K = \operatorname{rank} p(K)$. But then by Theorem B, there exists a permutation matrix P_2 such that $P_2KP_2^T$ is a direct sum of matrices of types (I) or (II) (not necessarily both). Set

$$P = \begin{pmatrix} P_2 & 0 \\ 0 & I \end{pmatrix} P_1,$$

where the block matrices are of suitable orders. Then PAP^T is of the desired form.

The converse is trivial.

COROLLARY 1. *Let A be a nonnegative matrix of rank r and let $A^{(1,2)} = p(A) \geqslant 0$, $p(A) = \alpha_1A^{m_1} + \cdots + \alpha_kA^{m_k}$, $\alpha_i \neq 0$, $m_i \geqslant 0$. Then $A = A_1 + A_2 + \cdots + A_k$, where $A_i \geqslant 0$; $A_iA_j = 0$, $i \neq j$; $A_i^{(1,2)} = p(A_i)$; $\operatorname{rank} A_i = d_i$, $\sum_{i=1}^k d_i = r$, and d_i divides some $m_j + 1$.*

PROOF. By Theorem 1, there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J is a direct sum of the matrices of the types (I) or (II) (not necessarily both) and C, D are nonnegative matrices of appropriate orders. Thus

$$J = \begin{bmatrix} S_1 & & & & 0 \\ & \ddots & & & \\ & & S_i & & \\ & & & \ddots & \\ 0 & & & & S_k \end{bmatrix},$$

where S_i's are of the types (I) or (II) and rank $S_i = d_i$. Set

$$J_i = \begin{bmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & S_i & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{bmatrix}.$$

Then $A = \sum_{i=1}^k A_i$, where

$$A_i = P^T \begin{bmatrix} J_i & J_iD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ_i & CJ_iD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P.$$

It can be easily verified that $A_i > 0$; $A_iA_j = 0$, $i \neq j$; $A_i^{(1,2)} = p(A_i)$; rank $A_i = d_i$, $\sum_{i=1}^k d_i = r$, and d_i divides some $m_j + 1$.

The corollary which follows generalizes DeMarr's theorem for nonnegative idempotent matrices [5].

COROLLARY 2. *Let A be a nonnegative matrix of rank r and let $A = A^m$, where $m > 2$ is a positive integer. Then $A = A_1 + A_2 + \cdots + A_k$, where $A_i > 0$; $A_iA_j = 0$, $i \neq j$; $A_i^m = A_i$;* rank *$A_i = d_i$, $d_i | m - 1$, $\sum_{i=1}^k d_i = r$.*

PROOF. Follows from Corollary 1.

REMARK 1. Theorem 1 provides a complete solution, in a more general case, to the problem raised by Berman of characterization of the class of nonnegative matrices A with $\{1\}$-inverse or $\{1, 2\}$-inverse equal to A itself [2, Remark 5].

Henceforth by matrices of types (I) or (II), we will mean the matrices of types (I) or (II) described in Theorem 1.

THEOREM 2. *Let A be a nonnegative matrix having a nonnegative group inverse $A^{\#}$. Then $A^{\#} = K_1A^m = A^mK_2$, where*

$$K_1 = P^T \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P, \quad K_2 = P^T \begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P,$$

P is a permutation matrix, K is a diagonal matrix with positive diagonal entries, C, D are some nonnegative matrices of appropriate sizes, and m is a positive integer. Indeed, we may also choose

$$K_1 = K_2 = P^T \begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & CKD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P.$$

PROOF. By Theorem 1 there exists a permutation matrix P such that

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where C, D are some nonnegative matrices of appropriate sizes and J is a direct sum of matrices of types (I) or (II) (not necessarily both). We note that if S is a summand of type (I) then $S^{(1,2)} = \beta^{-1}xy^T = \beta^{-2}S = \cdots = \beta^{-k-1}S^k$, for any positive integer k. We show that if S is a summand of type (II) then $S^{(1,2)} = (\beta_{12}\beta_{23}\cdots\beta_{d1})^{-k}S^{kd-1}$ for any positive integer k. A straightforward verification shows that

$$S^{(1,2)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\beta_{d1}}x_1y_d^T \\ \frac{1}{\beta_{12}}x_2y_1^T & & \vdots & & 0 \\ 0 & \frac{1}{\beta_{23}}x_3y_2^T & \vdots & \vdots & \vdots \\ 0 & 0 & & & \\ \vdots & \vdots & \vdots & 0 & 0 \\ 0 & 0 & & \frac{1}{\beta_{d-1d}}x_dy_{d-1}^T & 0 \end{bmatrix}$$

$$= (\beta_{12}\beta_{23}\cdots\beta_{d1})^{-1}S^{d-1}.$$

Also

$$S^{kd} = (\beta_{12}\beta_{23}\cdots\beta_{d1})^k \begin{bmatrix} x_1y_1^T & & 0 \\ & \ddots & \\ 0 & & x_dy_d^T \end{bmatrix}$$

for any positive integer k. Thus $S^{(1,2)} = (\beta_{12}\beta_{23}\cdots\beta_{d1})^{-k}S^{kd-1}$ as asserted. Now if $S_{11}, S_{12}, \ldots, S_{1r}$ are summands of type (I) and $S_{21}, S_{22}, \ldots, S_{2s}$ are summands of type (II) of ranks $d_{21}, \ldots, d_{2s}$ respectively, then

$$J^{(1,2)} = \begin{bmatrix} S_{11}^{(1,2)} & & & & & 0 \\ & \ddots & & & & \\ & & S_{1r}^{(1,2)} & & & \\ & & & S_{21}^{(1,2)} & & \\ & & & & \ddots & \\ 0 & & & & & S_{2s}^{(1,2)} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11}I & & & & & 0 \\ & \ddots & & & & \\ & & \alpha_{1r}I & & & \\ & & & \alpha_{21}I & & \\ & & & & \ddots & \\ 0 & & & & & \alpha_{2s}I \end{bmatrix}$$

$$\begin{bmatrix} S_{11} & & & & & 0 \\ & \ddots & & & & \\ & & S_{1r} & & & \\ & & & S_{21} & & \\ & & & & \ddots & \\ 0 & & & & & S_{2s} \end{bmatrix}^{(d_{21}d_{22}\cdots d_{2s})-1}$$

$$= \begin{bmatrix} S_{11} & & & & & 0 \\ & \ddots & & & & \\ & & S_{1r} & & & \\ & & & S_{21} & & \\ & & & & \ddots & \\ 0 & & & & & S_{2s} \end{bmatrix}^{(d_{21}d_{22}\cdots d_{2s})-1}$$

$$\begin{bmatrix} \alpha_{11}I & & & & & 0 \\ & \ddots & & & & \\ & & \alpha_{1r}I & & & \\ & & & \alpha_{21}I & & \\ & & & & \ddots & \\ 0 & & & & & \alpha_{2s}I \end{bmatrix}$$

where $\alpha_{ij}I$ are scalar matrices of appropriate sizes, $\alpha_{ij} > 0$. Thus $J^{(1,2)} = KJ^m = J^mK$, where

$$K=\begin{bmatrix} \alpha_{11}I & & & & & 0 \\ & \ddots & & & & \\ & & \alpha_{1r}I & & & \\ & & & \alpha_{21}I & & \\ & & & & \ddots & \\ 0 & & & & & \alpha_{2s}I \end{bmatrix}, \qquad m=(d_{21}\cdots d_{2s})-1.$$

Thus

$$\begin{aligned} PA^{(1,2)}P^T &= \begin{bmatrix} KJ^m & KJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CKJ^m & CKJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} J^m & J^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^m & CJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} J^m & J^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ^m & CJ^mD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Hence

$$A^{(1,2)}=P^T\begin{bmatrix} K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CK & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} PA^m = A^mP^T\begin{bmatrix} K & KD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} P.$$

COROLLARY 3. *Under the hypothesis of Theorem 2, if A is also range-Hermitian, that is, $A^{\#}=A^{\dagger}$, then $A^{\dagger}=A^{\#}=HA^m=A^mH$ where H is a diagonal matrix with positive diagonal entries and m is a positive integer.*

PROOF. Let A be range-Hermitian. Then $A^{\dagger}=A^{\#}$. By Theorem 1,

$$PAP^T=\begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where J is a direct sum of matrices of types (I) or (II). But then by using $A^{\dagger}=A^{\#}$, we obtain that C and D must be zero. Thus

$$PAP^T=\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned}
A^{\#} &= P^T\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} PA^m = A^m P^T\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P \\
&= P^T\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix} P = P^T\begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} P \\
&= P^T\begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix} P = P^T\begin{pmatrix} J^m & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P \\
&= P^T\begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} PA^m = A^m P^T\begin{pmatrix} K & 0 \\ 0 & I \end{pmatrix} P \\
&= HA^m = A^m H,
\end{aligned}$$

where H is a diagonal matrix with positive diagonal entries.

Before we prove Corollary 4 we record below a simple fact which we state without proof.

SUBLEMMA. *If $\beta x y^T$ is a row (or column) stochastic matrix where $\beta > 0$ and x, y are positive vectors such that $y^T x = 1$ then $\beta = 1$.*

COROLLARY 4. *If A is an $n \times n$ nonnegative row (or column) stochastic matrix such that $A^{(1,2)} = p(A) \gg 0$, $p(A)$ is a polynomial in A with scalar coefficients, then $A^{(1,2)} = A^m$ for some positive integer m and $A^{(1,2)}$ is row (or column) stochastic.*

PROOF. For definiteness, let us assume that A is row stochastic. By appropriate application of Lemma 1 and Theorem 1, we can find a permutation matrix P such that

$$PAP^T = \begin{pmatrix} J & 0 \\ CJ & 0 \end{pmatrix},$$

where J is a direct sum of matrices of types (I) or (II) (not necessarily both). We note that if S is a summand of type (I) then by sublemma $\beta = 1$ and so $S^{(1,2)} = xy^T = S$. Next, let S be a summand of type (II). Then S is a stochastic matrix and

$$S = \begin{bmatrix} 0 & \beta_{12}x_1y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23}x_2y_3^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1d}x_{d-1}y_d^T \\ \beta_{d1}x_dy_1^T & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\beta_{ij} > 0$ and x_i, y_i are positive vectors with $y_i^T x_i = 1$. A straightforward verification shows that $S^{(1,2)} = (\beta_{12}\beta_{23} \cdots \beta_{d1})^{-1} S^{d-1}$ and S^{d-1} is a row stochastic matrix. Thus again by the sublemma we get $(\beta_{12}\beta_{23} \cdots \beta_{d1}) = 1$. Then as in the proof of Theorem 2, we get $J^{(1,2)} = J^m$, and hence

$$A^{(1,2)} = P^T\begin{pmatrix} J^m & 0 \\ CJ^m & 0 \end{pmatrix} P = A^m.$$

THEOREM 3. *Every nonnegative rank factorization of nonnegative matrices J which are direct sum of matrices of types* (I′) *or* (II′) (*not necessarily both*) *is of the form* $J = (FQ)(Q^TG)$, *where Q is a permutation matrix, F and G are respectively the direct sum of matrices of the form* (1) *or* (2) *and* (1′) *or* (2′):

(1) $\beta' x$,

(1′) $\beta'' y^T$,

(2)

$$\begin{bmatrix} 0 & \gamma_1 x_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_2 x_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \gamma_{d-1} x_{d-1} \\ \gamma_d x_d & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

(2′)

$$\begin{bmatrix} \gamma_1' y_1^T & & & 0 \\ & \gamma_2' y_2^T & & \\ & & \ddots & \\ 0 & & & \gamma_d' y_d^T \end{bmatrix}$$

such that $\beta' > 0$, $\beta'' > 0$, $\gamma_i > 0$, $\gamma_i' > 0$, x, y, x_i, y_i *are positive vectors with* $y^Tx = 1$ *and* $y_i^Tx_i = 1$. *Moreover,* $J^{\#} = p(J)$ *where* $p(t) = \sum_{i=1}^{k} \alpha_i t^{m_i}$, $\alpha_i \neq 0$, $m_i > 0$, *is some polynomial in t, $\beta'\beta''$ is a root of equation* (6), *and the product* $\gamma_1\gamma_2 \cdots \gamma_d \gamma_1' \gamma_2' \cdots \gamma_d'$ *is a common root of the system of at most d equations* (7) *and* (8). *It is understood that in forming the product* $(FQ)(Q^TG)$ *if F has a summand of type* (1) *or* (2) *in the ith place of its direct sum then G has a corresponding summand of type* (1′) *or* (2′) *at the same ith place.*

Also, for each nonnegative rank factorization $J = FG$ of J, $(GF)^{-1} = p(GF)$.

PROOF. Let $S = \beta x y^T$ with $y^Tx = 1$ be a summand of J of type (I′). Clearly, the only possible nonnegative rank factorization of S is $S = FG$, $F = \beta' x$, $G = \beta'' y^T$ with $\beta'\beta'' = \beta$. This gives $GF = \beta y^T x = \beta$. By equation (6), it then follows that $(GF)^{-1} = p(GF)$.

Next let

$$S = \begin{bmatrix} 0 & \beta_{12} x_1 y_2^T & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_{23} x_2 y_3^T & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \beta_{d-1d} x_{d-1} y_d^T \\ \beta_{d1} x_d y_1^T & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

be a summand of J of type (II′) of rank d. Let $S = FG$ be a nonnegative rank factorization of S. Partition $F = (F_{ij})$ and $G = (G_{ij})$ into matrix blocks such that for all i,j,k, $F_{ij}G_{jk}$ is defined and is of the same order as that of the (i, k)th block entry in S. Since F is of full column rank, no column of F is zero. Also no row of F can be zero. For otherwise, $S = FG$ shall have a zero row which is not true. Similarly, no row or column of G is zero. Since no column of F is zero, for each j there exists an i (depending on j) such that $F_{ij} \neq 0$. But then $G_{jk} = 0$ for all $k \neq i + 1$. Thus each row of the partitioned matrix G has at most one (and hence exactly one) nonzero entry. Clearly, then each column of G has also exactly one nonzero entry. The same reason yields that the partitioned matrix F has exactly one nonzero entry in each row and in each column. This implies there exists a permutation $\sigma \in S_d$ such that $F_{ij} = 0$, for all $j \neq \sigma(i)$, $G_{\sigma(i)k} = 0$, for all $k \neq i + 1$ and $F_{i\sigma(i)}G_{\sigma(i),i+1} = \beta_{i,i+1}x_iy_{i+1}^T$. But then the only solutions for $F_{i\sigma(i)}$ and $G_{\sigma(i),i+1}$ are given by $F_{i\sigma(i)} = \gamma_i x_i$ and $G_{\sigma(i),i+1} = \gamma_i' y_{i+1}^T$ where $\gamma_i\gamma_i' = \beta_{i,i+1}$. It is now clear that $J = (FQ)(Q^TG)$ where F and G are, respectively, the direct sum of matrices of the form (1) or (2) and (1′) or (2′) and Q is some permutation matrix. For summand of J of type (I′) we have already shown that $(GF)^{-1} = p(GF)$. We now show the same for summand of type (II′). It is sufficient to prove the result for $S = FG$ where

$$F = \begin{bmatrix} 0 & \gamma_1x_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_2x_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \gamma_{d-1}x_{d-1} \\ \gamma_dx_d & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} \gamma_1'y_1^T & & & 0 \\ & \gamma_2'y_2^T & & \\ & & \ddots & \\ 0 & & & \gamma_d'y_d^T \end{bmatrix}.$$

Then

$$GF = \begin{bmatrix} 0 & \gamma_1\gamma_1' & 0 & 0 & \cdots & 0 \\ 0 & 0 & \gamma_2\gamma_2' & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \gamma_{d-1}\gamma_{d-1}' \\ \gamma_d\gamma_d' & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and $(GF)^d = (\gamma_1\gamma_2 \cdots \gamma_d\gamma_1'\gamma_2' \cdots \gamma_d')I$.

From equations (7) and (8), it follows that $(\gamma_1\gamma_2 \cdots \gamma_d\gamma_1'\gamma_2' \cdots \gamma_d')^{1/d}$ is a root of $\sum_{i=1}^k \alpha_i t^{m_i+1} = 1$. Thus $\sum_{i=1}^k \alpha_i(GF)^{m_i+1} = I$, since $(GF)^d = (\gamma_1\gamma_2 \cdots \gamma_d\gamma_1'\gamma_2' \cdots \gamma_d')I$. Hence $(GF)p(GF) = I$, completing the proof.

The next theorem describes all nonnegative rank factorizations of a nonnegative matrix with a nonnegative group inverse.

THEOREM 4. (a) *Let* $A \geqslant 0$ *and* P *be a permutation matrix such that*

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

J is a direct sum of matrices of types (I′) *or* (II′), *C, D are some nonnegative matrices of suitable sizes* (*equivalently*, $A \geqslant 0$ *and* $A^{\#} = p(A) \geqslant 0$). *Then we have the following*:

(a_1) *If* $J = FG$ *is a nonnegative rank factorization of* J *then it "lifts" to a nonnegative rank factorization*

$$A = P^T \begin{bmatrix} F \\ 0 \\ CF \\ 0 \end{bmatrix} (G \quad GD \quad 0 \quad 0) P$$

of A.

(a_2) *If* $A = F'G'$ *is a nonnegative rank factorization of* A *then it "contracts" to a nonnegative rank factorization* $J = F'_{11}G'_{11}$ *of* J *where* F'_{11} (G'_{11}) *consists of first n rows* (*columns*) *of* PF' ($G'P^T$), *n being the order of the matrix J.*

(a_3) *If* σ *denotes the operation of "lifting" defined in* (a_1), *and* η *denotes the operation of "contracting" defined in* (a_2) *then* $\sigma\eta = \text{identity} = \eta\sigma$. *Thus there is a* 1-1 *correspondence between the class of nonnegative rank factorizations of J and the class of nonnegative rank factorizations of A.*

(a_4) *If* $A = F'G'$ *and* $J = FG$ *are corresponding nonnegative rank factorizations of A and J, respectively, then* $GF = G'F'$ *and* $(GF)^{-1} = p(GF)$, *where* $A^{\#} = p(A)$.

(b) *If* $A = FG$ *is a nonnegative rank factorization of A such that* $(GF)^{-1} = p(GF)$ *where $p(t)$ is some polynomial in t with scalar coefficients then $A^{\#}$ exists,* $A^{\#} \geqslant 0$, *and* $A^{\#} = p(A)$.

PROOF. (a_1) Straightforward verification.

(a_2) Let n be the order of J. Partition

$$PAP^T = \begin{bmatrix} J & JD & 0 & 0 \\ 0 & 0 & 0 & 0 \\ CJ & CJD & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad PF' = \begin{pmatrix} F'_{11} \\ F'_{21} \end{pmatrix}, \quad \text{and} \quad G'P^T = (G'_{11}G'_{12})$$

where F'_{11} (G'_{11}) consists of first n rows (columns) of PF' ($G'P^T$) respectively. By comparing we get $J = F'_{11}G'_{11}$ which is clearly nonnegative rank factorization of J.

(a_3) It is obvious that if we perform the operation of lifting followed by the operation of contracting then the composition is identity operation, i.e. $\eta\sigma =$ identity. On the other hand, it is not clear to us that $\sigma\eta$ is also identity, in general. However, we can show $\sigma\eta =$ identity under our hypothesis. Let $A = F'G'$ be a nonnegative rank factorization of A. By performing the operation η we get

$J = F'_{11}G'_{11}$ where F'_{11} and G'_{11} are respectively the first n rows and the first n columns of PF' and $G'P^T$. Then by performing σ, we get

$$A = P^T \begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix} (G'_{11} \quad G'_{11}D \quad 0 \quad 0)P.$$

To prove $\sigma\eta$ = identity, we need to show

$$PF' = \begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix}, \qquad G'P^T = (G'_{11} \quad G'_{11}D \quad 0 \quad 0). \tag{9}$$

Since by Theorem 3, F'_{11} (G'_{11}) is the direct sum of matrices of the form (1) or (2) ((1′) or (2′)) it is sufficient to prove (9) when F'_{11} is of the form (1) or (2) and G'_{11} is of the corresponding form (1′) or (2′). Partition

$$PF' = \begin{pmatrix} F'_{11} \\ F'_{21} \\ F'_{31} \\ F'_{41} \end{pmatrix}, \qquad G'P^T = (G'_{11} \quad G'_{12} \quad G'_{13} \quad G'_{14})$$

such that the size of F'_{j1}, $j = 1, 2, 3, 4$, is the same as that of the corresponding block in

$$\begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix}.$$

Similarly, for the size of G'_{1j}, $j = 1, 2, 3, 4$. Then comparing the corresponding blocks in

$$\begin{pmatrix} F'_{11} \\ 0 \\ CF'_{11} \\ 0 \end{pmatrix} (G'_{11} \quad G'_{11}D \quad 0 \quad 0) = \begin{pmatrix} F'_{11} \\ F'_{21} \\ F'_{31} \\ F'_{41} \end{pmatrix} (G'_{11} \quad G'_{12} \quad G'_{13} \quad G'_{14})$$

we get the result by observing that F'_{11} and G'_{11} are not zero divisors.

(a_4) Follows from (a_1)–(a_3) and Theorem 3.

(b) Recall that if $A = FG$ is a rank factorization of A then $A^{\#}$ exists if and only if $(GF)^{-1}$ exists [4]. In this case $A^{\#} = F(GF)^{-2}G$. A straightforward computation then yields $A^{\#} = p(A)$ where $(GF)^{-1} = p(GF)$.

Remark 2. Theorem 3 along with Theorem 4(a_3) characterizes all the nonnegative rank factorizations of a nonnegative matrix with nonnegative group inverse.

REMARK 3. Another proof of Theorem 4(a_4) can be given on the same lines as the proof for the special case when $p(A) = A$ given by Berman and Plemmons [3]. However, the purpose of Theorem 4 is to characterize all nonnegative rank factorizations of nonnegative matrices A with $A^{\#} > 0$, and (a_4) comes out as an offshoot.

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