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A SPECTRAL THEOREM FOR J-NONNEGATIVE OPERATORS

BY

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ABSTRACT. A J-space is a Hilbert space with the usual inner product denoted $[x, y]$ and an indefinite inner product defined by $(x, y) = [Jx, y]$ where J is a bounded selfadjoint operator whose square is the identity. We define a J-adjoint T^+ of an operator T with respect to the indefinite inner product in the same way as the regular adjoint T^* is defined with respect to $[x, y]$. We say T is J-selfadjoint if $T = T^+$. An operator-valued function is called a J-spectral function with critical point zero if it is defined for all $t \neq 0$, is bounded, J-selfadjoint and has the properties of a resolution of the identity on its domain.

It has been proved by M. G. Krein and Ju. P. Smul'jan that bounded J-selfadjoint operators A with $(Ax, x) > 0$ for all x can be represented as a strongly convergent improper integral of t with respect to a J-spectral function with critical point zero plus a nilpotent of index 2. Further, the product of the nilpotent with the J-spectral function on intervals not containing zero is zero.

The present paper extends this theory to the unbounded case. We show that unbounded J-selfadjoint operators with $(Ax, x) > 0$ are a direct sum of an operator of the above mentioned type and the inverse of a bounded operator of the same type whose nilpotent part is zero.

1. Introduction. A *J-space* is a separable Hilbert space H with, in addition to the usual inner product $[x, y]$, another inner product defined by $(x, y) = [Jx, y]$ for x and y in H. The J is a bounded selfadjoint linear operator in H such that $J^2 = I$ where I is the identity operator in H.

One can define the *J-adjoint* T^+ of an operator T with domain dense in H by the equation $(Tx, y) = (x, T^+y)$ for all appropriate x and y. T^+ is uniquely defined. An operator T is called *J-selfadjoint* if $T = T^+$. The resolvent set of an operator T will be denoted $\rho(T)$. A *J-unitary* operator U is everywhere defined and $(Ux, Ux) = (x, x)$ for all x. Since U has domain H, U^* (the regular adjoint) has domain H. But U^* is closed and so is bounded. Similarly, U^{**} is bounded and equals U. Therefore U is bounded. A *nonnegative subspace* of H is one where $(x, x) > 0$ for each of its members. Two subspaces M and N are called *J-orthogonal* if $(x, y) = 0$ for each x in M and y in N. If M and N are independent and J-orthogonal we write $M \oplus N$ and call this the *J-orthogonal direct sum* of M and N.

The Russians M. G. Krein and Ju. Smul'jan [1] have studied bounded operators T with $(Tx, x) > 0$, called J-nonnegative operators and found that such a $T = \int_{-\infty}^{\infty} t\, dE(t) + S$, the integral being a strongly convergent improper integral with singularity 0. the spectral function $E(t)$ is defined and J-selfadjoint for all nonzero t. It is a resolution of the identity with all the usual properties except that $\|E(t)\|$

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becomes infinite as t approaches 0. Also $(Sx, x) > 0$ for all x and

$$S^2 = [E(s)-E(t)]S = S[E(s)-E(t)] = 0$$

for all intervals $[t, s]$ with 0 in their exterior.

In this paper we consider an unbounded J-selfadjoint operator A such that $(Ax, x) > 0$ for all x in the domain of A. We show that if the resolvent set of A is not empty, then a theorem similar to the above is still valid. If one eigenspace of J has finite dimension, the assumption that $\rho(A) \neq \varnothing$ is unnecessary.

To prove these results we investigate J-unitary operators U such that $\operatorname{Im}(Ux, x) > 0$ for all x. Such an operator U is a J-orthogonal direct sum $U_1 \oplus U_2$ where each U_j is represented as a spectral integral in a fashion analogous to the J-nonnegative operators.

Now, if $\rho(A) \neq \varnothing$, then $U = (A + iI)(A-iI)^{-1}$ is J-unitary and $\operatorname{Im}(Ux, x) > 0$ for all x in H. Since we have a spectral representation for U, we therefore get a representation for A.

2. J-unitary-dissipative operators.

DEFINITION 1. A linear operator U in a J-space H is called *J-unitary-dissipative* if U is J-unitary and $\operatorname{Im}(Ux, x) > 0$ for all x in H.

DEFINITION 2. Let a and b be real numbers with $a < 0 < b$. A *J-spectral function with critical point* 0 defined on the interval (a, b) is an operator-valued function $F(t)$ defined on (a, b) except at 0 taking values in a J-space H. Also

(I) $F(t)$ is a J-orthogonal projector,

(II) $F(s)F(t) = F(\operatorname{Min}(s, t))$,

(III) $F(t-0) = F(t)$,

(IV) $\operatorname{Lim}_{t\uparrow b} F(t) = I$ and $\operatorname{Lim}_{t\downarrow a} F(t) = 0$,

where the limits are in the strong operator topology.

LEMMA 1. *Let A be a bounded everywhere defined linear operator in a J-space H with $(Ax, x) > 0$ for all x in H. Let $f(z)$ be any function analytic near the spectrum of A. Then*

$$f(A) = f(0)I + f'(0)S + \int_{-\infty}^{\infty} [f(t)-f(0)]\,dE(t)$$

where $(Sx, x) > 0$ for all x and

$$[E(s)-E(t)]S = S[E(s)-E(t)] = S^2 = 0$$

where $t < s$ are nonzero real numbers such that $0 \notin [t, s]$. The function E is a J-spectral function with critical point 0 defined on $(-\infty, \infty)$ and I is the identity operator in H. The integral is improper at 0 and converges in the strong operator topology.

PROOF. It has been shown by M. G. Krein and J. L. Smul'jan [1] that operators A as above have a resolvent given by

$$(zI-A)^{-1} = z^{-1}I + z^{-2}S + \int_{-\infty}^{\infty} t\, dF(t)/z(z-t).$$

These integrals and all those in the sequel converge in the strong operator topology.

Let f be any function analtyic near the spectrum of A and let C be a curve consisting of a finite number of rectifiable Jordan arcs lying in the resolvent set of A and in the domain of f. Then

$$f(A) = \frac{1}{2\pi i}\int_C f(z)(zI-A)^{-1}\,dz$$

and therefore the lemma follows. Q.E.D.

THEOREM 1. *Let U be J-unitary-dissipative in a J-space H with $-1 \in \rho(U)$. Then the spectrum of U lies on the unit circle, and*

$$U = I + 2iS + \int_{-\pi}^{\pi}(e^{it}-1)dF(t)$$

where $S^2 = S[F(s)-F(t)] = [F(s)-F(t)]S = 0$ for all real $t < s$ in $(-\pi, \pi)$ wth $0 \notin [t, s]$. The function F is a J-spectral function with critical point 0 defined on $(-\pi, \pi)$. For each x in H the function $(F(t)x, x) \leqslant 0$ and is nonincreasing for $-\pi < t < 0$ while for $0 < t < \pi$; $(F(t)x, x) \leqslant (x, x)$ and is nondecreasing. $(Sx, x) \geqslant 0$. Lastly, if g is any function analytic near the spectrum of U, then

$$g(U) = g(1)I + 2ig'(1)S + \int_{-\pi}^{\pi}(g(e^{it})-g(1))dF(t).$$

The integral is improper at 0 and converges in the strong operator topology.

PROOF. Since $-1 \in \rho(U)$, then $(U + I)^{-1}$ exists and is a bounded everywhere defined linear operator in H. Therefore the operator $A = -i(U-I)(U + I)^{-1}$ is a bounded operator with domain H. A calculation shows that A is J-selfadjoint $(A = A^+)$ and $-i \in \rho(A)$. Also if $y = (A + iI)^{-1}x$,

$$\begin{aligned}(Ux, x) &= -\left((A-iI)(A + iI)^{-1}x, x\right) = -((A-iI)y, (A + iI)y)\\ &= -(y,y)-(Ay,Ay) + 2i(Ay,y).\end{aligned}$$

Therefore $\operatorname{Im}(Ux, x) = 2(Ay, y)$ for all x in H. Since U is J-unitary-dissipative, and $-i \in \rho(A)$, A will have domain H and $(Ay,y) \geqslant 0$. By Lemma 1, $A = S + \int_{-\infty}^{\infty} t\,dE(t)$ where E is a J-spectral function with critical point 0 defined on $(-\infty, \infty)$. Also $[E(s)-E(t)]S = S[E(s)-E(t)] = S^2 = 0$ for all real numbers $s < t$ with $0 \notin [s, t]$. $(Sx, x) \geqslant 0$ for all x. For $t < 0$ the function $(E(t)x, x) \leqslant 0$ and nonincreasing, while for $t > 0$ we have $(E(t)x, x) \leqslant (x, x)$ and is nondecreasing. The spectrum of A is real.

If $f(z) = -(z-i)(z + i)^{-1}$, then f is analytic near the spectrum of A and by Lemma 1,

$$U = f(A) = f(0)I + f'(0)S + \int_{-\infty}^{\infty}[f(t)-f(0)]\,dE(t).$$

But $f(0) = 1$ and $f'(0) = 2i$ and so

$$U = I + 2iS + \int_{-\infty}^{\infty}\left[-(t-i)(t + i)^{-1}-1\right]dE(t).$$

If we make the change of variables t to $\tan(\frac{1}{2}t)$ our integral becomes

$$U = I + 2iS + \int_{-\pi}^{\pi} (e^{it}-1)\, dF(t)$$

where $F(t) = E(\tan(\frac{1}{2}t))$. The change of variables makes F a J-spectral function with critical point 0 on $(-\pi, \pi)$. Clearly F has all the properties stated in the theorem and the spectrum of U lies on the unit circle.

Now let g be any function analytic near the spectrum of U and

$$f(z) = -(z-i)(z+i)^{-1}.$$

Then the composite function $g \circ f = h$ is analytic near the spectrum of A and so

$$g(U) = h(A) = h(0)I + h'(0)S + \int_{-\infty}^{\infty} [h(t)-h(0)]\, dE(t)$$

$$= g(1)I + g'(1)S + \int_{-\pi}^{\pi} [g(e^{it})-g(1)]\, dF(t)$$

and this proves the theorem. Q.E.D.

THEOREM 2. *Let U be a J-unitary-dissipative operator in a J-space H. Then U is a J-orthogonal direct sum $U_1 \oplus U_2$ where*

$$U_1 = I_1 + 2iS_1 + \int_{-\pi}^{\pi} (e^{it}-1)\, dF_1(t),$$

$$U_2 = -I_2 - 2iS_2 - \int_{-\pi}^{\pi} (e^{it}-1)\, dF_2(t)$$

and F_j $(j = 1, 2)$ is a J-spectral function with critical point 0 on $(-\pi, \pi)$. $(S_j x, x) > 0$ for appropriate x and F_j has the properties stated in Theorem 1 for F.

PROOF. We need only show $U = U_1 \oplus U_2$ where $(-1)^j \in \rho(U_j)$ for each j. Therefore let $c_n = ([\operatorname{Im} U]U^n x, x)$. Then

$$\sum c_{p-q} w_p \bar{w}_q = \operatorname{Im}\left(U \sum U^p w_p x, \sum U^q w_q x\right) > 0$$

where the summations are for all integral values of p and q from 0 to $n-1$. Since the sequence is thus positive definite we have

$$\left([\operatorname{Im} U] U^n x, x\right) = \int_0^{2\pi} e^{int}\, d\mu(t; x)$$

for every integer n and where for each x the function $\mu(t; x)$ is a bounded nondecreasing function of t. It is uniquely determined if it is normalized by the requirement that $\mu(0; x) = 0$, and $\mu(t-0; x) = \mu(t; x)$. By polarization we can write

$$\left([\operatorname{Im} U] U^n x, y\right) = \int_0^{2\pi} e^{int}\, d_\mu(t; x, y),$$

$$\mu(t; x, y) = (1/4)\left[\mu(t; x+y) - \mu(t; x-y) + i\mu(t; x+iy) - i\mu(t; x-iy)\right].$$

We also have $0 < \mu(t;\ x, x) = \mu(t;\ x)$ and, taking $n = 0$ in the integral, $([\operatorname{Im} U]x, x) = \mu(2\pi;\ x)$. Therefore $0 < \mu(t;\ x) < |([\operatorname{Im} U]x, x)|$ and thus $0 < \mu(t;\ x) < \|\operatorname{Im} U\|\ \|x\|^2$. Therefore the bilinear form $\mu(t;\ x, y)$ is bounded and there is an operator-valued function $M(t)$ defined on $[0, 2\pi]$, $M(t)$ is a bounded everywhere-defined linear operator in H, $(M(t)x, x) \geq 0$, and $\mu(t; x, y) = (M(t)x, y)$ for all x and y in H. Further, $M(0) = 0$ and $M(t-0) = M(t)$ for $0 < t \leq 2\pi$. Lastly, the function $(M(t)x, x)$ is a nondecreasing function of t for each x.

We now show that the function $M(t)$ has bounded variation on $[0, 2\pi]$. To this end let $0 = t_0 < t_1 < t_2 < \cdots < t_n = 2\pi$ be a partition of $[0, 2\pi]$. Since $([M(t_i)-M(t_{i-1})]x, y)$ is a hermitian bilinear form for each i and $([M(t_i)-M(t_{i-1})]x, x) \geq 0$ then

$$\|M(t_i)-M(t_{i-1})\| = \operatorname{Sup}\left[\left(\left[M(t_i)-M(t_{i-1})\right]x, x\right): \|x\| = 1\right].$$

Therefore there is an x_0 with norm 1 such that

$$\|M(t_i)-M(t_{i-1})\|-n^{-1} < \left(\left[M(t_i)-M(t_{i-1})\right]x_0, x_0\right),$$

$$\sum_{i=0}^{n} \|M(t_i)-M(t_{i-1})\|-1 < \left(\sum_{i=0}^{n}\left[M(t_i)-M(t_{i-1})\right]x_0, x_0\right) \leq \|M(2\pi)\|,$$

$$\sum_{i=0}^{n} \|M(t_i)-M(t_{i-1})\| \leq \sum_{i=0}^{n} \|M(t_i)-M(t_{i-1})\| \leq \|M(2\pi)\| + 1$$

for all partitions of $[0, 2\pi]$. Therefore $M(t)$ has bounded variation. The strong limits $M(t \pm 0)$ exist for all t. Also from above

$$\left(\left[\operatorname{Im} U\right] U^n x, y\right) = \int_0^{2\pi} e^{int}\, d(M(t)x, y).$$

This integral converges in the strong operator topology in H. From the above equation we get

$$U^p\left[U^2-I\right] = 2i\int_0^{2\pi} e^{i(p+1)t}\, dM(t), \qquad p = 0, \pm 1, \ldots.$$

$$U^{2k} = I + 2i\int_0^{2\pi}\left[\sum_{j=0}^{k-1} e^{i(2j+1)t}\right] dM(t), \qquad k = 1, 2, \ldots.$$

Therefore, for $|z| < \|U^{-2}\|^{-1}$, the resolvent transformation is

$$\begin{aligned}(U-zI)^{-1} &= \sum_0^\infty z^k U^{-2k-2} \\ &= \sum_0^\infty \left[I-2i\int_0^{2\pi} \sum_0^k e^{-i(2j+1)t}\, dM(t)\right] z^k \\ &= \sum_0^\infty z^k I-2i\int_0^{2\pi}\left[\sum_{k=0}^\infty \sum_{j=0}^k e^{-i(2j+1)t} z^k\right] dM(t).\end{aligned}$$

Interchanging the order of summation we have

$$\begin{aligned}(U^2-zI)^{-1} &= (1-z)^{-1}I-2i\int_0^{2\pi}\left[\sum_{j=0}^{\infty} e^{-i(2j+1)t}\sum_{k=j}^{\infty} z^k\right] dM(t)\\ &= (1-z)^{-1}I-2i\int_0^{2\pi}\left[\sum_{j=0}^{\infty} e^{-i(2j+1)t}z^k(1-z)^{-1}\right] dM(t)\\ &= (1-z)^{-1}I-2i(1-z)^{-1}\int_0^{2\pi} e^{-it}\sum_{j=0}^{\infty} e^{-2ijt}z^j\, dM(t)\\ &= (1-z)^{-1}I-2i(1-z)^{-1}\int_0^{2\pi} e^{-it}(1-ze^{-2it})^{-1}\, dM(t).\end{aligned}$$

Since the right-hand side is analytic inside the unit circle, so is the resolvent $(U^2-zI)^{-1}$. In a similar fashion we can show that for $|z| > \|U^2\|$, we have the same representation for $(U^2-zI)^{-1}$. That is,

$$(U^2-zI)^{-1} = (1-z)^{-1}I-2i(1-z)^{-1}\int_0^{2\pi} e^{-it}(1-ze^{-2it})^{-1}\, dM(t).$$

Since the right-hand side is analytic outside the unit circle, so is the resolvent. In summary, then, the resolvent of U^2 is analytic off the unit circle and therefore the spectrum of U lies on the unit circle.

Let us recall that

$$U^p[U^2-I] = 2i\int_0^{2\pi} e^{i(p+1)t}\, dM(t)$$

for all integers p. From this we deduce in the same way we did for the resolvent of U^2 that for z in the resolvent set of U

$$(zI-U)^{-1}U^n(U^2-I) = 2i\int_0^{2\pi}(z-e^{it})^{-1}e^{i(n+1)t}\, dM(t)$$

for all nonnegative integers n. The function defined by $dW(t) = 2ie^{int}\, dM(t)$ has bounded variation in the interval $[0, 2\pi]$ and therefore the function

$$g(z) = \int_0^{2\pi}(z-e^{it})^{-1}e^{it}\, dW(t)$$

has radial limits $\operatorname{Lim}_{r\uparrow 1} g(re^{is}) = g(e^{is})$ except on a set Z of Lebesgue measure 0. Also the limit function $g(e^{is})$ is in $L^2(0, 2\pi)$.

Consider for $0 < \alpha < \beta < 2\pi$ the path $C_{\alpha,\beta}$ constructed as follows. We proceed along the ray $re^{i\alpha}$ from $r = \frac{1}{2}$ to $r = 2$, then along the arc $2e^{it}$ from $t = \alpha$ to $t = \beta$, then along the ray $re^{i\beta}$ from $r = 2$ to $r = \frac{1}{2}$ and lastly along the arc $\frac{1}{2}e^{it}$ from $t = \beta$ to $t = \alpha$. The contour integral of $g(z)$ around $C_{\alpha,\beta}$ exists as a Cauchy principal value if neither α nor β are in Z. For such an α and β we have

$$\frac{1}{2\pi i}\int'_{C_{\alpha,\beta}}(zI-U)^{-1}\, dz U^n(U^2-I)$$

and this equals

$$2i\int_0^{2\pi}\left[\frac{1}{2\pi i}\int'_{C_{\alpha,\beta}}(z-e^{it})^{-1}\,dz\right]e^{i(n+1)t}\,dM(t).$$

That is

$$P_{\alpha,\beta}U^n(U^2-I) = 2i\int_\alpha^\beta e^{i(n+1)t}\,dM(t)$$

where

$$P_{\alpha,\beta} = \frac{1}{2\pi i}\int'_{C_{\alpha,\beta}}(zI-U)^{-1}\,dz.$$

By our choice of $C_{\alpha,\beta}$ it is clear that $P_{\alpha,\beta}$ is J-selfadjoint and a calculation shows that it is a projector in H.

Now choose α and β not in Z and such that $0 < \alpha < \pi < \beta < 2\pi$ and let $P_2 = P_{\alpha,\beta}$ and $P_1 = I-P_2$. Then P_1 and P_2 are J-orthogonal, i.e., they are bounded and J-selfadjoint and $0 = P_1P_2 = P_2P_1$ and their sum is I. Let $H_j = P_jH$. Then the pair (H_1, H_2) reduces U and U^{-1}. Let $U_j = U|_{H_j}$. We need only show that $(-1)^j \in \rho(U_j)$ to complete the proof. A calculation yields

$$(zI-U)^{-1}(U^2-I)P_2 = 2i\int_\alpha^\beta(z-e^{it})^{-1}e^{it}\,dM(t),$$

$$(zI-U)^{-2}(U^2-I)P_2 = 2i\int_\alpha^\beta(z-e^{it})^{-2}e^{it}\,dM(t).$$

If we restrict ouselves to H_2, this equation becomes

$$(zI-U_2)^{-2}(U_2^2-I_2) = 2i\int_\alpha^\beta(z-e^{it})^{-2}e^{it}\,dM(t)$$

where I_2 is the identity operator in H_2.

Since $0 < \alpha < \pi < \beta < 2\pi$, the above integral is a bounded everywhere defined linear operator for $z = 1$ in H_2. Hence the domain of $(I_2-U_2)^{-2}$ is H_2. But this implies that the domain of $(I_2-U_2)^{-1}$ is H_2. A calculation yields

$$(I_2-U_2)^{-2}(U_2^2-I_2) = I_2 + 2(U_2-I_2)^{-1}$$

in H_2. Therefore $(U_2-I_2)^{-1}$ is bounded and $1 \in \rho(U_2)$. Similarly, $-1 \in \rho(U_1)$. Apply Theorem 1 to U_1 and $-U_2$. Q.E.D.

3. Unbounded J-selfadjoint operators A with $(Ax, x) \geqslant 0$.

THEOREM 3. *Let A be a densely defined J-selfadjoint operator such that $(Ax, x) \geqslant 0$ for x in the domain D of A and such that $\rho(A) \neq \varnothing$. Then A is reduced by the direct sum $H_1 \oplus H_2 = H$. If A_j is A restricted to H_j $(j = 1, 2)$ then A_1 is bounded and*

$$A_1 = \int_{-\infty}^{\infty} t\,dE_1(t) + S_1 \quad \textit{where } (S_1x, x) \geqslant 0$$

and S_1 is bounded on H. Further, for each $t < s$ with $0 \notin [t, s]$,

$$S_1^2 = S_1(E_1(s)-E_1(t)) = (E_1(s)-E_1(t))S_1 = 0.$$

Also, $A_2x = \int_{-\infty}^{\infty} t^{-1}\,dE_2(t)x$ for each x in D.

The function E_j is a J-spectral function with critical point 0 defined on $(-\infty, \infty)$. For each x in H_j ($j = 1, 2$) the function $(E_j(t)x, x)$ is nonpositive and nonincreasing for $t < 0$, while it is less than or equal to (x, x) and nondecreasing for $t > 0$. The above integrals are improper at 0 and converge in the strong operator topology.

PROOF. As $(Ax, x) \gg 0$ and $\rho(A) \neq \varnothing$, $\sigma(A)$ is real [1]. Thus $\pm i \in \rho(A)$. Therefore the operator $U = -(A-iI)(A + iI)^{-1}$ is a bounded everywhere-defined linear operator mapping H onto itself. A calculation shows that U is J-unitary and that $(Ax, x) = 2\operatorname{Im}(Uy, y)$ where $(U + I)y = x$. We can recover A from U as follows. U is defined only for elements of form $y = (A + iI)x$. For such a y we have $Uy = -(A-iI)x$. From this we deduce that $(U-I)y = -2Ax$ and $(U + I)y = 2ix$. Therefore, if $(U + I)y = 0$, then $x = 0$ and so $y = 0$. The set of all vectors of form $(U + I)y$ is the domain D of A and is dense in H. Thus $(U + I)^{-1}$ exists and we can write $A = -i(U-I)(U + I)^{-1}$. Clearly U is J-unitary-dissipative. By Theorem 2, therefore, U is a J-orthogonal direct sum $U_1 \oplus U_2$. Hence A splits into a J-orthogonal direct sum $A_1 \oplus A_2$. Also $(-1)^j \in \rho(U_j)$ and

$$U_j = -(A_j - iI_j)(A_j + iI_j)^{-1}$$

for each $j = 1, 2$. Therefore we need only deal with A_1 and A_2 separately.

Consider A_1 first. Since $-1 \in \rho(U_1)$ and $A_1 = -i(U_1-I_1)(U_1 + I_1)^{-1}$ we see that A_1 is bounded. By the functional calculus given for U_1 in Theorem 1 with the function $g(z) = -i(z-1)(z + 1)^{-1}$ we find that

$$A_1 = S_1 + \int_{-\infty}^{\infty} t\, dE_1(t),$$

$$S_1[E_1(s)-E_1(t)] = [E_1(s)-E_1(t)]S_1 = S_1^2 = 0$$

for all $t < s$ such that $0 \notin [t, s]$. The operator S_1 is bounded and $(S_1x, x) \gg 0$. E_1 is a J-spectral function with critical point 0 defined on $(-\infty, \infty)$. The integral is improper at 0 and converges in the strong operator topology. The function E_2 has all the properties stated in the conclusion of the theorem.

Now consider A_2. As above we can show that $(U_2 + I_2)^{-1}$ exists and is densely defined and this set is the domain of A_2. Therefore -1 is in the continuous spectrum of U_2. But $1 \in \rho(U_2)$. Thus

$$U_2 = -I_2-2iS_2-\int_{-\pi}^{\pi} (e^{it}-1)\, dF_2(t)$$

in H_2. Now S_2 and the integral above have product zero and $S_2^2 = 0$. Therefore if $S_2 \neq 0$ then -1 will be an eigenvalue of U_2, a contradiction. Hence

$$U_2 = -I_2-\int_{-\pi}^{\pi} (e^{it}-1)\, dF_2(t).$$

To simplify notation we drop the subscript 2 in what follows. So we consider a densely defined operator A in a J-space H with $(Ax, x) \gg 0$ and

$$A = -i(U-I)(U + I)^{-1}, \qquad U = -I-\int_{-\pi}^{\pi} (e^{it}-1)\, dF(t).$$

Since $Ax = -i(U-I)(U+I)^{-1}x$, for x in the domain of A we have

$$x = -(1/2i)(U+I)y, \qquad Ax = (1/2)(U-I)y$$

where $y = -(A+iI)x$. If $s \neq 0$ we have

$$\begin{aligned}(F(s)x, x) &= (F(s)[1/2i](U+I)y, [1/2i](U+I)y) \\ &= \tfrac{1}{4}(F(s)[2I + U + U^{-1}]y, y) \\ &= \int_{-\pi}^{s} \sin^2\left(\tfrac{1}{2}t\right) d(F(t)y, y), \end{aligned} \tag{A}$$

by Theorem 1 with $g(z) = \frac{1}{4}(2-z-z^{-1})$. By the above

$$\begin{aligned}(Ax, x) &= -([1/2](U-I)y, [1/2i](U+I)y) \\ &= [1/4i]([U-U^{-1}]y, y) \\ &= \int_{-\pi}^{\pi} \{[e^{it}-e^{-it}]/4i\}\, d(F(t)y, y) \\ &= \int_{-\pi}^{\pi} \sin\left(\tfrac{1}{2}t\right)\cos\left(\tfrac{1}{2}t\right) d(F(t)y, y).\end{aligned}$$

Recall that the integrals

$$\int_{-\pi}^{s} \sin^2\left(\tfrac{1}{2}t\right) d(F(t)y, y) \qquad [s > 0]$$

and

$$\int_{-\pi}^{\pi} \sin\left(\tfrac{1}{2}t\right)\cos\left(\tfrac{1}{2}t\right) d(F(t)y, y)$$

are convergent improper integrals with singularity 0.

Let $\gamma > 0$ and consider

$$I_\gamma = \int_{-\pi}^{-\gamma} \sin\left(\tfrac{1}{2}t\right)\cos\left(\tfrac{1}{2}t\right) d(F(t)y, y).$$

Since $[-\pi, -\gamma]$ does not contain 0, $(F(t)y, y)$ has bounded variation there. Therefore

$$\begin{aligned}I_\gamma &= \int_{-\pi}^{-\gamma} \cot\left(\tfrac{1}{2}t\right)\sin^2\left(\tfrac{1}{2}t\right) d(F(t)y, y) \\ &= \int_{-\pi}^{-\gamma} \cot\left(\tfrac{1}{2}t\right) d(F(t)x, x)\end{aligned}$$

from (A) above. Therefore

$$\lim_{\gamma\downarrow 0} I_\gamma = \lim_{\gamma\downarrow 0} \int_{-\pi}^{-\gamma} \cot\left(\tfrac{1}{2}t\right) d(F(t)x, x).$$

Let $\delta > 0$ and consider

$$\begin{aligned}I_\delta &= \int_{\delta}^{\pi} \cos\left(\tfrac{1}{2}t\right)\sin\left(\tfrac{1}{2}t\right) d(F(t)y, y) \\ &= \int_{\delta}^{\pi} \cot\left(\tfrac{1}{2}t\right)\sin^2\left(\tfrac{1}{2}t\right) d(F(t)y, y).\end{aligned}$$

For $s > 0$

$$([I-F(s)]x, x) = \int_s^\pi \sin^2\left(\tfrac{1}{2}t\right) d(F(t)y, y).$$

Since $(F(t)x, x)$ is nondecreasing for t negative, the function $w(t) = -([I-F(t)]x, x)$ is negative and increasing on $[\delta, \pi]$. Thus

$$\int_\delta^\pi \cot\left(\tfrac{1}{2}t\right) dw(t) = \int_\delta^\pi \cot\left(\tfrac{1}{2}t\right) d(F(t)x, x)$$

and this equals

$$\int_\delta^\pi \cot\left(\tfrac{1}{2}t\right)\sin^2\left(\tfrac{1}{2}t\right) d(F(t)y, y) = I_\delta.$$

Since $(Ax, x) = \lim_{\gamma\downarrow 0} I_\gamma + \lim_{\delta\downarrow 0} I_\delta$, then

$$(Ax, x) = \int_{-\pi}^\pi \cot\left(\tfrac{1}{2}t\right) d(F(t)x, x)$$

for x in the domain of A. This is an integral convergent for each x in the domain of A and improper at 0. If we make the transformation $\tan(\frac{1}{2}t) \to t$ we obtain

$$(Ax, x) = \int_{-\infty}^\infty t^{-1} d(E(t)x, x)$$

where $E(\tan(\frac{1}{2}t)) = F(t)$. Translating the properties of the above F by the same transformation we see that E is a J-spectral function with critical point 0 on $(-\infty, \infty)$. That is, $E = E_2$ and $A = A_2$ have the properties listed in the statement of this theorem. Q.E.D.

Corollary. *Let A satisfy the conditions of Theorem* 3. *Then A is the direct sum of a bounded operator and the inverse of a bounded operator.*

Proof. This is just a restatement of Theorem 3. Q.E.D.

Bibliography

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