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ON MEROMORPHIC SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

BY

SH. STRELITZ

ABSTRACT. The Malmquist Theorem is generalized for equations of the type $R(z, w, w', \ldots, w^{(n)}) = P(z, w)/Q(z, w)$ where P, Q and R are polynomials of w and $w, w', \ldots, w^{(n)}$ respectively with meromorphic coefficients of finite order.

1. In this paper we consider the differential equation

$$R(z, w, w', \ldots, w^{(n)}) = P(z, w)/Q(z, w) \tag{1.1}$$

where P, Q and R are polynomials of w and $w, w', \ldots, w^{(n)}$ respectively with meromorphic coefficients of z in the plane $|z| < \infty$.

We generalized in [4] the well-known Malmquist Theorem [2] for equations of type (1.1), where P, Q and R were polynomials of all their corresponding variables and obtained there under these conditions that if (1.1) has a transcendental meromorphic solution in $|z| < \infty$ then $Q(z, w)$ does not depend on w. In this paper we generalize the indicated theorem for (1.1) described above.

In order to formulate the main assertion of this paper we rewrite P, Q and R of (1.1) in the following forms:

$$R(z, w, w', \ldots, w^{(n)}) = \sum_{i_0 + i_j + \cdots + i_n = 0}^{m} R_{i_0 i_s \cdots i_n}(z) w^{i_0} (w')^{i_s} \cdots (w^{(n)})^{i_n},$$

$$P(z, w) = \sum_{k=0}^{p} P_k(z) w^{p-k},$$

$$Q(z, w) = \sum_{j=0}^{q} Q_j(z) w^{p-k}. \tag{1.2}$$

We will use the following notations.

(1) We denote by $\rho(f)$ the order of the meromorphic function f and put

$$\lambda = \max\{\rho(R_{i_0 i_1 \cdots i_n}), \rho(P_k), \rho(Q_j)\} \tag{1.3}$$

where the maximum is taken over all the possible values of the indices i_s, k and j.

(2) We denote the set of all the polynomials

$$T(z, w) = \sum_{s=0}^{t} T_s(z) w^{t-s}$$

where all the $T_s(z)$ are meromorphic functions in $|z| < \infty$ by $\mathfrak{M}$.

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DEFINITION 1. The polynomial $T_0(z, w) \in \mathfrak{M}$ which depends explicitly on w is called a nontrivial divisor of $T(z, w)$ in $\mathfrak{M}$ if there is a third nontrivial $T_1 \in \mathfrak{M}$ (that is, $T_1(z, w)$ depends explicitly on w) such that $T \equiv T_1 T_0$.

DEFINITION 2. Two polynomials $T_1, T_2 \in \mathfrak{M}$ are called mutually prime if they have no common nontrivial divisors.

We are now able to formulate the main result of this paper.

THEOREM. *Consider* (1.1). *Let* $P(z, w)$ *and* $Q(z, w)$ *be mutually prime in* $\mathfrak{M}$ *and let* $\lambda < \infty$. *If* (1.1) *has a transcendental meromorphic solution* $w(z)$ *in* $|z| < \infty$ *of order* $\rho > \lambda$ *then* $Q(z, w)$ *does not depend on* w.

REMARK. In this theorem it is impossible in general to put $\rho(w) \geqslant \lambda$. Indeed let $p_k(z)$ and $q_j(z)$ be transcendental meromorphic functions with $\rho(p_k) \leqslant \rho$, $\rho(q_j) \leqslant \rho$ (for all k and j) and $f(z)$ a meromorphic function of order ρ. Then the equation

$$w'' = f'' \cdot \frac{\sum_{k=0}^{n} q_k(z) f^{n-k}}{\sum_{j=0}^{m} p_j(z) f^{m-j}} \cdot \frac{\sum_{j=0}^{m} p_j(z) w^{m-j}}{\sum_{k=0}^{n} q_j(z) w^{n-j}}$$

has a solution $f(z)$ of order ρ.

In the following sections of this paper we prove the theorem formulated above.

2. We can always assume that the degree, $d(P)$, of $P(z, w)$ in respect to w is less than the degree, $d(Q)$, of $Q(z, w)$ in respect to w; $d(P) < d(Q)$. Indeed if $d(P) \geqslant d(Q)$ then dividing $P(z, w)$ by $Q(z, w)$ as polynomials of w, we will separate the entire polynomial part (in respect to w), $s(z, w)$, and will obtain a remainder, $P_1(z, w)/Q(z, w)$, with a degree, $d(P_1)$, of $P_1(z, w)$ lower than the degree, $d(Q)$, of $Q(z, w)$:

$$P(z, w)/Q(z, w) = S(z, w) + P_1(z, w)/Q(z, w)$$

with $d(P_1) < d(Q)$. From (1.1) now follows $R(z, w, w', \ldots, w^{(n)}) - S(z, w) = P_1(z, w)/Q(z, w)$. Denoting

$$R(z, w, w', \ldots, w^{(n)}) - S(z, w) = R_1(z, w, w', \ldots, w^{(n)})$$

we get a new equation

$$R_1(z, w, w', \ldots, w^{(n)}) = P_1(z, w)/Q(z, w)$$

where $P_1(z, w)/Q(z, w)$ has the required property. We assume that already in (1.1) the degree of $P(z, w)$ is lower than the degree of $Q(z, w)$ in respect of w.

In order to prove our theorem we suppose that $Q(z, w)$ depends explicitly on w. Let $w(z)$ be a meromorphic solution of order $\rho > \lambda$ of (1.1). We substitute in (1.1) w for $w(z)$. By a suitable transformation

$$w = (\beta u + 1)/(u + 1) \tag{2.1}$$

with an appropriately chosen β, (1.1) will be reduced to

$$R^*(z, u, u', \ldots, u^{(n)}) = P^*(z, u)/Q^*(z, u) \tag{2.2}$$

with the following properties:

(i) $P^*(z, u)$, $Q^*(z, u)$ and $R^*(z, u, u', \ldots, u^{(n)})$ are polynomials of u and $u, u', \ldots, u^{(n)}$ respectively with meromorphic coefficients of order no more than λ;

(ii) $P^*(z, w)$ and $Q^*(z, w)$ are mutually prime in $\mathfrak{M}$;

(iii) the function $Q^*(z, u(z))$, where $u(z)$ is the meromorphic solution of (2.2) corresponding to $w(z)$ (of order ρ: $\rho > \lambda$), does not vanish at the poles of the function $u(z)$ and

(iv) $a = \infty$ is not a deficiency value of $u(z)$.

In order to prove these properties consider now the zeros of the function $Q^*(z, u(z))$. In the poles of $u(z)$ according to (2.1)

$$w(z)|_{u=\infty} = \beta. \tag{2.3}$$

Further,

$$Q\left(z, \frac{\beta u + 1}{u + 1}\right) = \sum_{j=0}^{q} Q_j(z)\left(\frac{\beta u + 1}{u + 1}\right)^{q-j}$$
$$= \frac{1}{(u + 1)^q} \sum_{j=0}^{q} Q_j(z)(\beta u + 1)^{q-j}(u + 1)^j = \frac{Q^*(z, u)}{(1 + u)^q} \tag{2.4}$$

and

$$\frac{P(z, w)}{Q(z, w)} = \frac{P\left(z, \frac{\beta u + 1}{u + 1}\right)(1 + u)^q}{Q^*(z, u)} = \frac{P^*(z, u)}{Q^*(z, u)}$$

with the polynomial

$$P^*(z, u) = P\left(z, \frac{\beta u + 1}{u + 1}\right)(1 + u)^q.$$

We have

$$Q\left(z, \frac{\beta u + 1}{u + 1}\right)\bigg|_{u=\infty} = Q(z, \beta). \tag{2.5}$$

Let $\{z_n\}$ be the sequence of all the zeros of the function $Q(z, w(z))$: $Q(z_n, w(z_n)) = 0$, $n = 1, 2, 3, \ldots$. We now point out a number β such that

$$w(z_n) \neq \beta, \qquad n = 1, 2, 3, \ldots. \tag{2.6}$$

Suppose $\{z_p'\}$ is the sequence of all the solutions of the equation $w(z) = \beta$:

$$w(z_j') = \beta, \qquad j = 1, 2, 3, \ldots. \tag{2.7}$$

Obviously $z_j' \neq z_k$; $j, k = 1, 2, 3, \ldots$. At the point z_j', in view of their construction,

$$Q(z_j', w(z_j')) \neq 0, \qquad j = 1, 2, 3, \ldots. \tag{2.8}$$

So that $Q(z, (\beta u + 1)/(u + 1))$ does not vanish in the poles of u. From (2.4) we now infer that all the solutions of $Q^*(z, u) = (u + 1)^q Q(z, (\beta u + 1)/(u + 1)) = 0$ are different from the poles of $u(z)$. Besides β can be chosen in such a manner that $Q^*(z, u)$ will depend explicitly on u and additionally so that $a = \infty$ will be a nondeficiency value of $u(z)$ since the deficiency values form only a countable set. It is obvious also that if in (1.1) P and Q are mutually prime in $\mathfrak{M}$, then $P^*(z, u)$ and $Q^*(z, u)$ will be mutually prime in $\mathfrak{M}$ too.

Thus without restricting the generality we can assume that already in (1.1)

$Q(z, w(z)) \neq 0$ at the poles of $w(z)$ and that $a = \infty$ is not a deficiency value of $w(z)$.

3. It follows now from (1.1) under our assumptions of the §2 that $P(z, w(z))$ vanishes at every point where $Q(z, w(z))$ vanishes with the exception maybe of a sequence of poles of the function $R_{i_0 i_1 \cdots i_n}(z)$ (see (1.2)) whose convergence exponent is not greater than λ. Indeed, at a point z_0 where $Q(z_0, w(z_0)) = 0$, according to our construction $w(z_0) \neq \infty$, so that if $P(z_0, w(z_0)) \neq 0$ then it is indispensable that

$$R_0(z_0) = R(z_0, w(z_0), \ldots, w^{(n)}(z_0)) = \infty, \qquad R_0(z) \equiv R(z, w(z), \ldots, w^{(n)}(z))$$

i.e., the function $R_0(z)$ will have a pole in z_0. But this is possible only if z_0 is a pole of at least one of the coefficients $R_{i_0 i_1 \cdots i_n}(z)$ in (1.2). Since $w(z_0) \neq \infty$ the multiplicity of the pole of $R_0(z)$ cannot be greater than the multiplicity of the pole of the corresponding functions $R_{i_0 i_1 \cdots i_n}(z)$. But the last functions are of order not more than λ, so that the convergence exponent of the poles of $R_0(z)$ is not exceeding λ.

Consider now the common zeros of $P(z, w)$ and $Q(z, w)$, that is the set of all the solutions of the system of equations

$$P(z, w) = 0, \qquad Q(z, w) = 0. \tag{3.1}$$

The resultant $D(z)$ of this system is a meromorphic function of order no greater than λ and cannot vanish identically because $P(z, w)$ and $Q(z, w)$ are mutually prime in $\mathfrak{M}$ according to our conditions of the theorem. Consequently the sequence of the zeros of the function $Q(z, w(z))$ has a convergence exponent not greater than λ. Thus there is a representation

$$Q(z, w(z)) \equiv f(z)/F(z), \tag{3.2}$$

where $f(z)$ is an entire function of order not greater than λ, $F(z)$ is another entire function.

Lemma 1. *$F(z)$ in the representation* (3.2) *is an entire function of order ρ.*

Remark. We restrict ourself with $\rho < \infty$. The case $\rho = \infty$ can be dealt with literally in the same way.

Proof of Lemma 1. We consider the function

$$Q(z, w(z)) \equiv \sum_{j=0}^{q} Q_j(z) w^{q-j}(z). \tag{3.3}$$

Since $a = \infty$ is not a deficiency value of $w(z)$ (see property (iv) in §2) then the convergence exponent of the sequence of the poles of (3.3) equals ρ and $Q(z, w(z))$ is of order ρ. Indeed

$$\sum_{j=0}^{q} Q_j(z) w^{q-j}(z) = w^q(z)\left[Q_0(z) - \sum_{j=1}^{q} Q_j(z)/w^j(z) \right]. \tag{3.4}$$

As we saw in §2

$$Q_0(z) - \sum_{j=1}^{q} Q_j(z)/w^j(z) \neq 0 \tag{3.5}$$

at the poles of $w(z)$ so that $Q(z, w(z))$ is of order not less than ρ, because the convergence exponent of its poles as it follows from (3.4) and (3.5) is not less than ρ. On the other hand, $Q(z, w(z))$ is at most of order ρ. Thus the order of $Q(z, w(z))$ equals ρ.

Equality (3.2) now shows that $F(z)$ is an entire function of order ρ since $Q(z, w(z))$ is of order ρ and $f(z)$ of order $\lambda < \rho$. Lemma 1 is proven.

4. We continue the proof of the theorem.

Let ζ be a maximal point of the function $|F(z)|$ on the circle $|z| = r$, that is,

$$|F(\zeta)| = \max_{|r|=r} |F(z)| = M(r, F), \qquad |\zeta| = r. \tag{4.1}$$

Denote

$$K(r_0, F) = \lim_{r \to r_0+0} \frac{rM'(r, F)}{M(r, F)}. \tag{4.2}$$

It is known [3, p. 148] that for each ζ, $|\zeta| = r$ with the exception of a sequence of intervals $\bar{I}$ on the r-axis with bounded logarithmic measure (that is $\int_{\bar{I}}(dr/r) < \infty$).

$$F(\zeta e^{\eta}) = (1 + o(1))F(\zeta), \qquad |F(\zeta e^{\eta})| = (1 + o(1))M(r, F), \qquad r \notin \bar{I}, \tag{4.3}$$

for

$$|\eta| \leqslant 1/K \ln^{1+\alpha} K, \qquad K = K(r, F), \tag{4.4}$$

with an arbitrary fixed $\alpha > 0$, $\bar{I} = \bar{I}(\alpha)$. For such η

$$|\zeta|(1 - 2|\eta|) < |\zeta e^{\eta}| < |\zeta|(1 + 2|\eta|) \tag{4.5}$$

because $K(r, F) \to \infty$ (see [3, p. 66]). Hence according to equation (4.3), $|F(z)| = (1 + o(1))M(r, F)$ in the circle C_ζ:

$$|z - \zeta| \leqslant |\zeta|\,|\eta|/2 \tag{4.6}$$

(η is given by (4.4)).

Denote $H = \cup_\zeta C_\zeta$, $|\zeta| \notin \bar{I}$. We will now construct a sequence $I_1 = \cup_j (R_j, R_j')$ on the r-axis such that $(R_i, R_i') \cap (R_j, R_j') = \varnothing$, $i \neq j$, $I_1 \cap \bar{I} = \varnothing$, mes $I_1 = \infty$ and

$$\lim_{\substack{r \to \infty \\ r \in I_1}} \frac{\ln \ln M(r, F)}{\ln r} = \rho. \tag{4.7}$$

In order to prove it we note that there is a sequence I_0 of intervals $I_0 = \cup_{j=1}^{\infty} (R_j, R_j')$, $(R_j, R_j') \cap (R_i, R_i') = \varnothing$, $i \neq j$, mes $I_0 = \infty$ such that

$$\lim_{\substack{r \to \infty \\ r \in I_0}} \frac{\ln \ln M(r, F)}{\ln r} = \rho. \tag{4.8}$$

This is obvious if the lower and upper orders of $F(z)$ coincide. To prove (4.8) suppose now that

$$\varliminf_{r \to \infty} \frac{\ln \ln M(r, F)}{\ln r} < \varlimsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r} = \rho.$$

In this case there is an infinite sequence of local maximal points $R_j \uparrow \infty$ such that

$$\lim_{j\to\infty} \frac{\ln \ln M(R_j, F)}{\ln R_j} = \rho.$$

Let $N > 1$ be an arbitrary fixed number and R_j^* an arbitrary point on the segment $[R_j, NR_j], j = 1, 2, 3, \ldots$. Evidently

$$\frac{\ln \ln M(R_j^*, F)}{\ln R_j^*} \geqslant \frac{\ln \ln M(R_j, F)}{\ln R_j} \cdot \frac{\ln R_j}{\ln NR_j} \underset{j\to\infty}{\to} \rho.$$

We assume $I_0 = \bigcup_{j=1}^{\infty}(R_j, NR_j)$. Obviously the sequence $\{R_j\}$ can be chosen so rare that $(R_j, NR_j) \cap (R_i, NR_i) = \varnothing$, $i \neq j$. Define $I_1 = I_0 \setminus \bar{I}$. We will show now that mes $I_1 = \infty$. Indeed

$$\int_{I_1} \frac{dr}{r} = \int_{I_0 \setminus \bar{I}} \frac{dr}{r} = \sum_{j=1}^{\infty} \int_{R_j}^{NR_j} \frac{dr}{r} - \int_{\bar{I}} \frac{dr}{r} = \sum_{j=1}^{\infty} \ln N - \int_{\bar{I}} \frac{dr}{r} = \infty$$

since $\int_{\bar{I}} (dr/r) < \infty$ (see above in this section). Thus

$$\infty = \int_{I_1 \cap \{r > 1\}} \frac{dr}{r} < \int_{I_1} dr = \text{mes } I_1.$$

We return now to the representation (3.2). The function $f(z)$ there is of order λ, so that for a given $\varepsilon > 0$

$$|f(z)| < \exp r^{\lambda+\varepsilon}. \tag{4.9}$$

On the other hand for $\zeta \in H$, $r = |\zeta| \in I_1$ and a given $\varepsilon > 0$ according to (4.3)

$$|F(\zeta)| = (1 + o(1))M(r, P) > \exp r^{\rho-\varepsilon}. \tag{4.10}$$

From (4.9) and (4.10) now follows: for $\zeta \in H$, $|\zeta| = r \in I_1$

$$|Q(\zeta, w(\zeta))| = \left|\frac{f(\zeta)}{F(\zeta)}\right| < \frac{\exp r^{\lambda+\varepsilon}}{\exp r^{\rho-\varepsilon}} = e^{-(1+o(1))r^{\rho-\varepsilon}}. \tag{4.11}$$

5. LEMMA 2. *For each $\omega > 0$ with $\rho > \lambda + \omega$ there is a sequence of intervals I on the r-axis with* mes $I = \infty$ *such that for all $\zeta \in H$, $|\zeta| \in I$*

$$|w^{(\nu)}(\zeta)| < r^{\nu(\rho+1)}e^{(2r)^{\rho-\omega}}, \qquad \nu = 0, 1, 2, 3, \ldots, N_0, \tag{5.1}$$

where N_0 is an arbitrary positive integer and $I = I(N_0)$.

PROOF. Let $h(z)$ be an entire function of order μ. It is known (see for example [1, p. 22]) that outside a set H_0 of circles E_k with union I^* of their radii I_k^*: $\bigcup_{k=1}^{\infty} I_k^* = I^*$ of finite measure: mes $I^* < \infty$

$$e^{-r^{\mu+\varepsilon}} < |h(z)| < e^{r^{\mu+\varepsilon}} \tag{5.2$'$}$$

for an arbitrary $\varepsilon > 0$ and $r > r_0(\varepsilon)$ $(r \notin I^*)$. Each function $Q_k(z)$ (see (1.2)) is a meromorphic function of order not more than λ and can be represented as a fraction of two entire functions of the same order as $Q_k(z)$: $Q_k(z) = h_1(z)/h_2(z)$. As above, there is a set H_0 of circles with union of radii I^* of finite measure, mes $I^* < \infty$, such that for $z \notin H_0$ the functors $h_1(z)$ and $h_2(z)$ satisfy inequality (5.2′). Then for $z \notin H_0$

$$e^{-(2r)^{\lambda+\varepsilon}} < |Q_k(z)| < e^{(2r)^{\lambda+\varepsilon}}, \tag{5.2}$$

for $r > r_0(\varepsilon)$.

Denote $I = I_1 \setminus I^* = I_1 \setminus \bigcup_{k=1}^{\infty} I_k^*$ (the definition of I_1 is given in the previous section). Suppose now that (5.1) is already wrong for $\nu = 0$. Then there is a certain $\omega_0 > 0$ with $\rho > \lambda + \omega_0$ and a sequence $\{\zeta_j\}$, $\{\zeta_j\} \in I$ such that

$$|w(\zeta_j)| \geqslant \exp r_j^{\rho-\omega_0}, \qquad r_j = |\zeta_j| \in I, \quad r_j \uparrow \infty. \tag{5.3}$$

We have

$$|Q(\zeta, w(\zeta))| = \left| \sum_{k=0}^{q} Q_k(\zeta) w^{q-k}(\zeta) \right| \geqslant |Q_0(\zeta) w^q(\zeta)| - \sum_{k=1}^{q} |Q_k(\zeta)|\, |w^{q-k}(\zeta)|$$

$$= |w^q(\zeta)| \left\{ |Q_0(\zeta)| - \sum_{k=1}^{q} |Q_k(\zeta)|\, |w(\zeta)|^{-k} \right\},$$

whence in view of (4.3) and (4.2),

$$|Q(\zeta_j, w(\zeta_j))| \geqslant \exp q r_j^{\rho-\omega_0} \left\{ \exp -r_j^{\lambda+\varepsilon} - \exp r_j^{\lambda+\varepsilon} \sum_{k=1}^{q} \exp -k r_j^{\rho-\omega_0} \right\}$$

$$\geqslant \exp q r_j^{\rho-\omega_0} \left\{ \exp -r_j^{\lambda+\varepsilon} - q \exp\left(-r_j^{\rho+\omega_0} + r_j^{\lambda+\varepsilon}\right) \right\}$$

$$= \exp((1 + o(1)) q r_j^{\rho+\omega_0})$$

for large enough j and sufficiently small ε (since $\rho > \lambda + \omega_0$). The last inequality contradicts (4.11). Thus for each $\omega > 0$ with $\rho - \omega > \lambda$

$$|w(\zeta)| < e^{r^{\rho-\omega}}, \qquad r = |\zeta| \in I, \quad r > r_0(\omega). \tag{5.4}$$

Now let $\zeta \in H$ with $|\zeta| \in I$. Then

$$w^{(\nu)}(\zeta) = \frac{\nu!}{2\pi i} \int_C \frac{w(u)}{(u - \zeta)^{\nu+1}}\, du, \qquad C\colon |u - \zeta| = \frac{|\zeta|\,|\eta|}{2},$$

where according to (3.4) we can choose

$$|\eta| = (K \ln^{l+\alpha} K)^{-1}, \qquad K = K(r, F). \tag{5.5}$$

We obtain for $|\zeta| = r \in I$

$$|w^{(\nu)}(\zeta)| \leqslant \frac{2^\nu \cdot \nu!}{|\zeta|^\nu |\eta|^\nu} \max_{|u| \leqslant |\zeta|(1+|\eta|/2)} |w(u)|$$

$$= \frac{2^\nu}{r^\nu} (K \ln^{1+\alpha} K)^\nu M\left(r + \frac{r|\eta|}{2}, w\right), \qquad K = K(r, F),$$

and in respect to (5.4) and (5.5) for $\zeta \in H$, $|\zeta| \in I$

$$|w^{(\nu)}(\zeta)| \leqslant \frac{2^\nu \cdot \nu!}{|\zeta|^\nu} (K \ln^{1+\alpha} K)^\nu \exp\left[r + \frac{r}{2} (K \ln^{1+\alpha} K)^{-1} \right]^{\rho-\omega}. \tag{5.6}$$

It is known ([3, p. 35]), that

$$\overline{\lim_{r\to\infty}} \frac{\ln K(r, F)}{\ln r} = \overline{\lim_{r\to\infty}} \frac{\ln \ln M(r, F)}{\ln r} = \rho.$$

Thus for a given ε_0: $1 > \varepsilon_0 > 0$ and $r > r_0(\varepsilon_0)$

$$K(r, F) < r^{\rho+\varepsilon_0}. \tag{5.7}$$

From (5.6) and (5.7) we obtain

$$|w^{(\nu)}(\zeta)| \leqslant \nu!\left(\frac{2}{r}\right)^{\nu}(r^{\rho+\varepsilon_0}\ln^{1+\alpha} r^{\rho+\varepsilon_0})^{\nu}\exp(2r)^{\rho-\omega} < r^{\nu(\rho+1)}\exp(2r)^{\rho-\omega}, \qquad r \in I.$$

Lemma 2 is proven.

6. We fix now the number ω so that $\rho > \lambda + \omega$.

LEMMA 3. *The inequality*

$$|R(\zeta, w(\zeta), w'(\zeta), \ldots, w^{(n)}(\zeta))| < r^s e^{(3r)^{\rho-\omega}} \tag{6.1}$$

holds for all $\zeta \in H$ *with large enough* $|\zeta| = r \in I$ *(I is defined in Lemma 2) and a certain constant s.*

PROOF. From (1.2) follows

$$|R(\zeta, w(\zeta), w'(\zeta), \ldots, w^{(n)}(\zeta))| \leqslant \sum_{i_0+i_1+\cdots+i_n=0}^{m} |R_{i_0 i_1\cdots i_n}(\zeta)|\,|w^{i_0}(\zeta)|\,|w'(\zeta)|^{i_1}\cdots|w^{(n)}(\zeta)|^{i_n}.$$

According to (5.1) we obtain

$$|R(\zeta, w(\zeta), w'(\zeta), \ldots, w^{(n)}(\zeta))| \leqslant \sum_{i_0+i_1+\cdots+i_n=0}^{m} |R_{i_0 i_1\cdots i_n}(\zeta)|\,r^{(\rho+1)(i_0+i_1+2i_2+\cdots+ni_n)(2r)}. \tag{6.2}$$

The functions $R_{i_0 i_1\cdots i_n(z)}$ are meromorphic functions of order not higher than λ and therefore

$$|R_{i_0 i_1\cdots i_n}(\zeta)| < e^{(2r)^{\lambda+\varepsilon}}, \qquad r = |\zeta| > r'(\varepsilon), \tag{6.3}$$

for $\varepsilon \in H$, $|\zeta| \in I$ (see (5.2)).

From (6.2) and (6.3) now follows

$$|R(\zeta, w(\zeta), w'(\zeta), \ldots, w^{(n)}(\zeta))| < Ar^{nm(\rho+1)}\exp((2r)^{\rho-\omega} + (2r)^{\lambda+\varepsilon}), \qquad \rho-\omega > \lambda+\varepsilon,$$

where

$$A = \sum_{i_0+i_1+\cdots+i_n=0}^{m} 1.$$

We see that for $r = |\zeta| > A$, $r \in I$, $\zeta \in H$

$$|R(\zeta, w(\zeta), w'(\zeta), \ldots, w^{(n)}(\zeta))| < r^s e^{(3r)^{\rho-\omega}}$$

with $s = mn(\rho + 1) + 1$. Thus Lemma 3 is proven.

7. Completion of the proof of the theorem. We know the representation $Q(z, w(z)) = f(z)/F(z)$ where $f(z)$ and $F(z)$ are entire functions of order $<\lambda$ and ρ: $\rho > \lambda$ respectively. For $\zeta \in H$, $|\zeta| \in I$ (mes $I = \infty$) according to (4.3)

$$|Q(\zeta, w(\zeta))| < \exp r^{\lambda+\varepsilon}/\exp r^{\rho+\beta(r)} = \exp -r^{\rho+\beta(r)}(1 + o(1)) \tag{7.1}$$

where $\beta(r) \to_{r\to\infty} 0$. In view of (6.1) and (7.1) we get for $\zeta \in H$, $|\zeta| \in I$

$$|R(\zeta, w(\zeta), w'(\zeta), \ldots, w^{(n)}(\zeta))| \, |Q(\zeta, w(\zeta))| < r^s \exp(3r)^{\rho-\omega} \exp -r^{\rho+\beta(r)\cdot(1+o(1))} = \exp -(1 + o(1))r^{\rho+\beta(r)}. \tag{7.2}$$

(1.1) now gives for the same ζ ($\zeta \in H$, $|\zeta| \in I$)

$$P(\zeta, w(\zeta)) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}). \tag{7.3}$$

The relations

$$P(\zeta, w) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}),$$
$$Q(\zeta, w) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}), \qquad \beta(r) \underset{r\to\infty}{\to} 0,$$

on the set $\tilde{H} = \{\zeta: \zeta \in H, |\zeta| \in I\}$ define a system of equations, from which $w(\zeta)$ can be found. After eliminating w we obtain

$$\varphi(\zeta) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}),$$

where $\varphi(z)$ is a meromorphic function of order not greater than λ. Indeed replacing w by $w(\zeta)$, we get according to (1.2)

$$P_0(\zeta)w^p + P_1(\zeta)w^{p-1} + \cdots + P_p(\zeta) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}),$$
$$Q_0(\zeta)w^q + Q_1(\zeta)w^{q-1} + \cdots + Q_p(\zeta) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}),$$
$$\left(|\zeta| = r, w = w(\zeta), \beta(r) \underset{r\to\infty}{\to} 0\right). \tag{7.4}$$

Suppose $p \leqslant q$. We can now eliminate w^q by multiplying the first equation of the last system with $Q_0(\zeta)w^{q-p}(\zeta)$ and the second with $-P_0(\zeta)$ and then adding them. Remembering that $w(\zeta)$ satisfies on $\tilde{H}$ inequality (1.5) (with $\nu = 0$ there), we obtain

$$Q_0(\zeta)w^{q-p}(\zeta)O(\exp -(1 + o(1))r^{\rho+\beta(r)}) = O(\exp -(1 + o(1))r^{\rho+\beta(r)})$$

and

$$-P_0(\zeta)O(\exp -(1 + o(1))r^{\rho+\beta(r)}) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}).$$

Therefore after the above indicated operations we get

$$-Q_1(\zeta)P_0(\zeta)w^{q-1} - \cdots - Q_q(\zeta)P_0(\zeta) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}).$$

Thus we reduced the degree of one of the equations of (7.4) in respect to w by a unit and the new system from which w has to be eliminated contains the first equation of (7.4) and the last one. In such a way we lower the degrees of w in the equation of (7.4) step by step until w will be completely eliminated and we finally come to

$$\varphi(\zeta) = O(\exp -(1 + o(1))r^{\rho+\beta(r)}), \tag{7.5}$$

where $\varphi(z)$ is a meromorphic function of order no more than λ but which does not vanish identically because $\varphi(z)$ is the resultant of the system (3.1), where $P(z, w)$ and $Q(z, w)$ are mutually prime polynomials in $\mathfrak{M}$. We saw earlier that outside a set E' of circles with a finite sum of radii

$$|\varphi(z)| > e^{-r^{\lambda+\epsilon}}. \tag{7.6}$$

Equality (7.5) now shows that all the solutions of this equation are within E'. But

(7.5) is correct for all $\zeta \in \tilde{H}$ ($|\zeta| \in I$, mes $I = \infty$) so that the set I on the r-axis has to be of finite measure: mes $I < \infty$. But mes $I = \infty$. We obtained a contradiction which shows that our assumption about $Q(z, w)$ depending on w is not possible. The theorem is proven.

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Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Current address: Department of Mathematics, University of Haifa, Haifa, Israel