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PARAMETRIZATIONS OF G_δ-VALUED MULTIFUNCTIONS

BY

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ABSTRACT. Let T, X be Polish spaces, $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$, the Borel σ-field of T, and $F: T \to X$ a multifunction such that $F(t)$ is a G_δ in X for each $t \in T$. F is $\mathfrak{T}$-measurable and $\mathrm{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$, where $\mathrm{Gr}(F)$ denotes the graph of F. We prove the following three results on F.

(I) There is a map $f: T \times \Sigma \to X$ such that for each $t \in T, f(t, \cdot)$ is a continuous, open map from Σ onto $F(t)$ and for each $\sigma \in \Sigma, f(\cdot, \sigma)$ is $\mathfrak{T}$-measurable, where Σ is the space of irrationals.

(II) The multifunction F is of Souslin type.

(III) If X is uncountable and $F(t)$, $t \in T$, are all dense-in-itself then there is a $\mathfrak{T} \otimes \mathfrak{B}_X$-measurable map $f: T \times X \to X$ such that for each $t \in T, f(t, \cdot)$ is a Borel isomorphism of X onto $F(t)$.

1. Introduction. The object of this paper is to study G_δ-valued multifunctions. We take T, X to be Polish spaces, $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$, the Borel σ-field of T, and $F: T \to X$ a multifunction such that F is $\mathfrak{T}$-measurable, $\mathrm{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$ and $F(t)$ is a G_δ in X for each $t \in T$. Definitions and notation are given in §2. G_δ-valued multifunctions arise in the study of C^*-algebras, group representations, etc. ([**5**], [**12**]).

In [**15**], the existence of a $\mathfrak{T}$-measurable selector for F is established and this article can be viewed as a sequel to [**15**]. Having proved the existence of a measurable selector for F, several questions arise. Can we express $\mathrm{Gr}(F)$ as a union of the graphs of measurable selectors for F? If yes, can we get these graphs to be, moreover, disjoint? Naturally, for the second problem, $F(t)$, $t \in T$, must all be of the same cardinality.

We approach the first problem in more than one way. In §3, we prove a representation theorem for such multifunctions of the kind recently obtained by Ioffe [**4**] and Srivastava [**14**] for closed valued multifunctions. In §4, we prove that these multifunctions are of Souslin type in the sense of Leese [**8**]. This gives us a very important relationship between F and closed valued multifunctions and enables us to answer our question in the affirmative.

We consider the second problem in §5. By a very old and classical result of Luzin ([**9**, p. 252], [**10**]), the answer to this question is "yes" for countable-valued F. In the case, $F(t)$, $t \in T$, are all uncountable, we prove a parametrization theorem, analogous to the one recently obtained by Mauldin [**11**], for F.

2. Preliminaries. The set of positive integers will be denoted by N. S will denote the set of all finite sequences of positive integers, including the empty sequence e.

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For each $k \geq 0$, we denote by S_k the set of elements of S of length k. For $s \in S$, $|s|$ will denote the length of s and if $i \leq |s|$ is a positive integer, s_i will denote the ith co-ordinate of s. If $n \in N$, sn will denote the catenation of s and n. We put $\Sigma = N^N$. Endowed with the product of discrete topologies on N, Σ becomes a homeomorph of the space of irrationals. For $\sigma \in \Sigma$ and $k \in N$, σ_k will denote the kth co-ordinate of σ and $\sigma|k = (\sigma_1, \ldots, \sigma_k)$. If $k = 0$, $\sigma|k = e$. If $s \in S$, Σ_s will denote the set $\{\sigma \in \Sigma: \sigma|k = s\}$.

D will denote the set of all finite sequences of 0's and 1's, including the empty sequence e. C will denote the set $\{0, 1\}^N$. Endowed with the product of discrete topologies on $\{0, 1\}$, it becomes a homeomorph of the Cantor set. For $d \in D$, $k \geq 0$, $i \in \{0, 1\}$ and $\varepsilon \in C$, d_k, ε_k, $\varepsilon|k$, $|d|$ and di are similarly defined.

Let $(X, \mathfrak{A})$ and $(Y, \mathfrak{B})$ be measurable spaces. We denote by $\mathfrak{A} \otimes \mathfrak{B}$ the product of the σ-fields $\mathfrak{A}$ and $\mathfrak{B}$. We say that $\mathfrak{A}$ is *countably generated* if there exist subsets A_n, $n \geq 1$, of X such that $\mathfrak{A}$ is generated by $\{A_n: n \geq 1\}$. A nonempty set $A \in \mathfrak{A}$ is called an $\mathfrak{A}$-*atom* if $A \supseteq B \in \mathfrak{A} \Rightarrow B = A$ or $B = \varnothing$. If X is a metric space, $\mathfrak{B}_X$ will denote the *Borel σ-field* of X. If $E \subseteq X \times Y$ and $x \in X$, E^x will denote the set $\{y \in Y: (x, y) \in E\}$ and will be called the *section* of E at x. The projection maps from $X \times Y$ to X and from $X \times Y$ to Y will be denoted respectively by $\Pi_X^{X \times Y}$ and $\Pi_Y^{X \times Y}$, or simply by Π_X and Π_Y if there is no ambiguity.

A *multifunction* $F: T \to X$ is a function whose domain is T and whose values are nonempty subsets of X. A function $f: T \to X$ is called a *selector* for F if $f(t) \in F(t)$ for each $t \in T$. The set $\{(t, x) \in T \times X: x \in F(t)\}$ is denoted by $\mathrm{Gr}(F)$ and is called the *graph* of F. If X is a metric space and $\mathfrak{T}$ is a σ-field on T, we say that F is $\mathfrak{T}$-*measurable* if the set $\{t \in T: F(t) \cap V \neq \varnothing\} \in \mathfrak{T}$ for every open set V in X. If M is a subset of $T \times X$, we say that $C \subseteq M$ *uniformizes* M if sections of C are at most a singleton and $\Pi_T(C) = \Pi_T(M)$.

Let X, Y be topological spaces. We say that a function $f: X \to Y$ is *open* (resp. *closed*) if for every open (resp. closed) set W in X, $f(W)$ is open (resp. closed) in the range of f.

The rest of our terminology is from **[6]**.

We now state some known results without proof which will be frequently used in the sequel.

LEMMA 2.1 (**[2]**). *Let T be a Polish space and $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$. Let $A \in \mathfrak{B}_T$ be a union of $\mathfrak{T}$-atoms. Then $A \in \mathfrak{T}$.*

LEMMA 2.2. *Let T, X be Polish spaces and $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$. Suppose G is a subset of $T \times X$ such that $G \in \mathfrak{T} \otimes \mathfrak{B}_X$ and G^t is a G_δ in X for every $t \in T$. Then for every closed subset A of X the set $\{t \in T: A \subseteq G^t\} \in \mathfrak{T}$.*

PROOF. Let Y be a metric compactification of X. By a well-known result of Alexandrov and Hausdorff, X is a G_δ in Y. Consequently, $G \in \mathfrak{T} \otimes \mathfrak{B}_Y$ and G^t is a G_δ in Y for each $t \in T$. Let $A \subseteq X$ be closed. Then it is easily verified that

$$\{t \in T: A \subseteq G^t\} = T \setminus \Pi_T((T \times A) \cap ((T \times Y) \setminus G)).$$

By a result of Arsenin and Kunugui **[1]** (see also **[13]**) it follows that the set $\{t \in T$:

$A \subseteq G^t\} \in \mathfrak{B}_T$. Further, this set is a union of $\mathfrak{T}$-atoms. The result now follows from Lemma 2.1.

We now state a very useful result, which is proved in **[15]**, for G_δ-valued multifunctions.

LEMMA 2.3. *Let T, X be Polish spaces and $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$. Let $G \in \mathfrak{T} \otimes \mathfrak{B}_X$ and G^t be a G_δ in X for each $t \in T$. Then there exist sets $G_n \in \mathfrak{T} \otimes \mathfrak{B}_X$ such that G_n^t is open in X for $t \in T$ and $n \geqslant 1$ and $G = \bigcap_{n=1}^{\infty} G_n$.*

In the rest of the paper, T, X will denote arbitrary Polish spaces and $\mathfrak{T}$ a countably generated sub-σ-field of $\mathfrak{B}_T$. X will be given a complete metric such that $\operatorname{diam}(X) < 1$. *$\{V_n: n \geqslant 1\}$ will be a base for the topology of X such that $V_1 = X$. $F: T \to X$ will denote a multifunction such that F is $\mathfrak{T}$-measurable,* $\operatorname{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$ *and $F(t)$ is a G_δ in X for each $t \in T$. G will denote the graph of F and G_n, $n \geqslant 1$, will be a nonincreasing sequence of sets in $\mathfrak{T} \otimes \mathfrak{B}_X$ such that G_n^t is open for $t \in T$ and $n \geqslant 1$ and* $G = \bigcap_{n=1}^{\infty} G_n$. The existence of such a sequence of sets is ensured by Lemma 2.3.

3. A representation theorem.

LEMMA 3.1. *Let X be compact. Then for each $t \in T$, there is a system $\{n_s^t: s \in S\}$ of positive integers such that for $s \in S_k$, $k \geqslant 0$, and $t \in T$,*

(i) *the map $t' \to n_s^{t'}$, defined on T, is $\mathfrak{T}$-measurable,*
(ii) $\operatorname{diam}(V_{n_s^t}) < 2^{-k}$,
(iii) $\overline{V}_{n_{sm}^t} \subseteq G_{k+1}^t \cap V_{n_s^t}$, $m \geqslant 1$,
(iv) $G^t \cap V_{n_s^t} \neq \varnothing$,
(v) $G^t \subseteq V_{n_e^t}$,
(vi) $G^t \cap V_{n_s^t} \subseteq \bigcup_{m=1}^{\infty} V_{n_{sm}^t}$.

PROOF. For each $t \in T$, we define n_s^t, $s \in S$, by induction on $|s|$. We define $n_e^t = 1$, $t \in T$. The above conditions are clearly satisfied for $s = e$. Suppose n_s^t, $t \in T$, are defined satisfying (i)–(vi) for every $s \in \bigcup_{i \leqslant k} S_i$, for some $k \geqslant 0$. Fix an $s \in S_k$. We define n_{sm}^t, $t \in T$, $m \in N$, by induction on m. We first make a simple observation. Let $W \subseteq X$ be closed and $t \in T$. Then

$$W \subseteq G_{k+1}^t \cap V_{n_s^t} \Leftrightarrow (\exists\, l \in N)\left(n_s^t = l \text{ and } W \subseteq G_{k+1}^t \cap V_l\right).$$

By the induction hypothesis and Lemma 2.2, it follows that the set

$$\left\{t \in T: W \subseteq G_{k+1}^t \cap V_{n_s^t}\right\} \in \mathfrak{T}.$$

For $m \geqslant 1$, let

$$\begin{aligned} T_m^0 &= \varnothing \quad \text{if } \operatorname{diam}(V_m) \geqslant 2^{-(k+1)}; \\ &= \Big\{t \in T: G^t \cap V_m \neq \varnothing,\ \overline{V}_m \subseteq G_{k+1}^t \cap V_{n_s^t} \quad \text{and} \\ &\qquad (\forall l < m)\left(\operatorname{diam}(V_l) < 2^{-(k+1)} \Rightarrow \left(G^t \cap V_l = \varnothing \text{ or } \overline{V}_l \nsubseteq G_{k+1}^t \cap V_{n_s^t}\right)\right)\Big\} \\ &\qquad\qquad \text{if } \operatorname{diam}(V_m) < 2^{-(k+1)}. \end{aligned}$$

As F is $\mathfrak{T}$-measurable, by the above observation, it follows that the sets T_m^0, $m \geqslant 1$, belong to $\mathfrak{T}$. Also, these are pairwise disjoint and $T = \bigcup_{m=1}^{\infty} T_m^0$. We define $n_{s1}^t = m$ if $t \in T_m^0$. Clearly, the map $t \to n_{s1}^t$ is $\mathfrak{T}$-measurable. Suppose for some $p \in N$, maps $t \to n_{si}^t$, $i \leqslant p$, have been defined to be $\mathfrak{T}$-measurable. For $m \geqslant 1$, let

$$
\begin{aligned}
T_m^p &= \varnothing \quad \text{if } \operatorname{diam}(V_m) \geqslant 2^{-(k+1)};\\
&= \Big\{ t \in T\colon n_{sp}^t < m,\ G^t \cap V_m \neq \varnothing,\ \overline{V}_m \subseteq G_{k+1}^t \cap V_{n_s^t}, \quad \text{and} \\
&\qquad (\forall l < m)\Big(\operatorname{diam}(V_l) < 2^{-(k+1)} \Rightarrow \Big(n_{sp}^t \geqslant l \text{ or } G^t \cap V_l = \varnothing \\
&\qquad\qquad \text{or } \overline{V}_l \not\subseteq G_{k+1}^t \cap V_{n_s^t}\Big)\Big)\Big\} \\
&\qquad\qquad\qquad \text{if } \operatorname{diam}(V_m) < 2^{-(k+1)}.
\end{aligned}
$$

It is easily checked that the sets T_m^p, $m \geqslant 1$, belong to $\mathfrak{T}$ and are pairwise disjoint. We define

$$
\begin{aligned}
n_{s,p+1}^t &= m \quad \text{if } t \in T_m^p, \\
&= n_{sp}^t \quad \text{if } t \in T \setminus \bigcup_{m=1}^{\infty} T_m^p.
\end{aligned}
$$

The definition of n_s^t, $s \in S$, $t \in T$, is complete. That the conditions (i)–(v) are satisfied follows immediately from the definitions of n_s^t, $s \in S$, $t \in T$. To check (vi) note that $G^t \cap V_{n_s^t} \subseteq G_{k+1}^t \cap V_{n_s^t}$ and $G_{k+1}^t \cap V_{n_s^t}$ is open.

Theorem 3.2. *There is a map $f\colon T \times \Sigma \to X$ such that for each $t \in T$, $f(t, \cdot)$ is a continuous, open map from Σ onto $F(t)$ and for each $\sigma \in \Sigma$, $f(\cdot, \sigma)$ is $\mathfrak{T}$-measurable.*

Proof. Without loss of generality, we assume that X is a compact metric space. For each $t \in T$, we get a system $\{n_s^t\colon s \in S\}$ of positive integers satisfying conditions (i)–(vi) of Lemma 3.1.

Let $f(t, \sigma)$ be the unique point of $\bigcap_k \overline{V}_{n_{\sigma|k}^t}$, $t \in T$, $\sigma \in \Sigma$. By conditions (iii)–(vi) of Lemma 3.1, $f(t, \Sigma) = F(t)$, $t \in T$. By standard arguments we show that for each $t \in T$, $f(t, \cdot)$ is continuous and open. Let $U \subseteq X$ be open, $\sigma \in \Sigma$ and $t \in T$. Then

$$
\begin{aligned}
f(t, \sigma) \in U &\Leftrightarrow \bigcap_k \overline{V}_{n_{\sigma|k}^t} \subseteq U \\
&\Leftrightarrow (\exists\, k \geqslant 1)\,(\exists\, l \geqslant 1)\,\Big(n_{\sigma|k}^t = l \text{ and } \overline{V}_l \subseteq U\Big).
\end{aligned}
$$

Therefore,

$$
f(\cdot, \sigma)^{-1}(U) = \bigcup \bigcup \big\{ t \in T\colon n_{\sigma|k}^t = l \big\} \in \mathfrak{T},
$$

where the inner union is taken over all l such that $\overline{V}_l = U$ and the outer union is over all k. It follows that $f(\cdot, \sigma)$ is $\mathfrak{T}$-measurable for each $\sigma \in \Sigma$.

Corollary 1. *F admits a $\mathfrak{T}$-measurable selector.*

Corollary 2. *There exist $\mathfrak{T}$-measurable selectors $f_1, f_2, \ldots$ for F such that for each $t \in T$, $\{f_n(t)\colon n \geqslant 1\}$ is dense in $F(t)$.*

Proof. Let $\sigma^1, \sigma^2, \ldots$ be a countable dense set in Σ. Then, for each $t \in T$, $\{f(t, \sigma^n)\colon n \geqslant 1\}$ is dense in $F(t)$. Put $f_n = f(\cdot, \sigma^n)$, $n \geqslant 1$.

REMARK 1. In [16], it is proved that Theorem 3.2 remains valid if the condition "$f(t, \cdot)$ is open" is replaced by "$f(t, \cdot)$ is closed".

REMARK 2. Let Y be a Polish space and h: $T \times Y \to X$ be a map such that for each $t \in T$, $h(t, \cdot)$ is continuous and open and for each $y \in Y$, $h(\cdot, y)$ is $\mathfrak{T}$-measurable. Define a multifunction H: $T \to X$ by $H(t) = h(t, Y)$, $t \in T$. By a result of Hausdorff [3], $H(t)$ is a G_δ in X for each $t \in T$. Let $\{y_n: n \geqslant 1\}$ be a countable dense set in Y. For $n \geqslant 1$, define f_n: $T \to X$ by $f_n(t) = f(t, y_n)$, $t \in T$. Let $V \subseteq X$ be open. Then

$$\{t \in T: H(t) \cap V \neq \varnothing\} = \bigcup_{n \geqslant 1} f_n^{-1}(V).$$

It follows that the multifunction H is $\mathfrak{T}$-measurable. The question now arises: Is $\mathrm{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_X$? We do not know the answer. In [16], it is proved that the answer to this question is 'yes' if the condition "$h(t, \cdot)$ is open" is replaced by "$h(t, \cdot)$ is closed".

4. Multifunctions of Souslin type.

DEFINITION. Let $(L, \mathfrak{L})$ be a measurable space and Z a metric space. A multifunction H: $L \to Z$ is said to be of *Souslin type* if there is a Polish space P, a continuous map β: $P \to Z$ and a $\mathfrak{L}$-measurable, closed-valued multifunction W: $L \to P$ such that $H(t) = \beta(W(t))$, for each $t \in L$.

REMARK 1. Our definition of multifunctions of Souslin type is slightly different from the one given in [8].

REMARK 2. By a representation theorem for closed valued multifunctions proved in [14], we get the following. *If $(L, \mathfrak{L})$ is a measurable space, Z a metric space and H: $L \to Z$ a multifunction of Souslin type then there is a map h: $L \times \Sigma \to Z$ such that for each $t \in L$, $h(t, \cdot)$ is a continuous map from Σ onto $H(t)$ and for each $\sigma \in \Sigma$, $h(\cdot, \sigma)$ is $\mathfrak{L}$-measurable.*

Now we prove the main result of this section.

THEOREM 4.1. *The multifunction F is of Souslin type.*

PROOF. We first assume that X is a compact, zero-dimensional metric space and basic open sets $V_1, V_2, \ldots$ are clopen. By Lemma 2.2, the sets $T_{nm} = \{t \in T: V_m \subseteq G_n^t\}$, $m \geqslant 1$, $n \geqslant 1$, belong to $\mathfrak{T}$. As G_n^t is open for $t \in T$ and $n \geqslant 1$, $G_n = \bigcup_{m=1}^{\infty} (T_{nm} \times V_m)$. Put $P = X \times \Sigma$ and $\beta = \Pi_X$. Let

$$B = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (T_{nm} \times V_m \times \Sigma_m^n),$$

where $\Sigma_m^n = \{\sigma \in \Sigma: \sigma_n = m\}$, $n \geqslant 1$, $m \geqslant 1$. Define W: $T \to P$ by $W(t) = B^t$, $t \in T$, so that $W(t) = \bigcap \bigcup (V_m \times \Sigma_m^n)$, where the inner union is taken over all $m \geqslant 1$ such that $t \in T_{nm}$ and the outer intersection is over all n. For each $n \geqslant 1$, $\{V_m \times \Sigma_m^n: m \geqslant 1\}$ is a discrete family of closed sets in P. It follows that the inner union is closed in P for each n. Therefore, $W(t)$ is closed in P for each $t \in T$. Also, it is easily checked that $\beta(W(t)) = F(t)$, $t \in T$.

We now check that W is $\mathfrak{T}$-measurable. Let $i \in N$ and $s \in S_k$, $k > 0$. It is enough to show that

$$\Pi_T^{T\times P}((T \times V_i \times \Sigma_s) \cap B) \in \mathfrak{T}.$$

Now,

$$\begin{aligned}
(T \times V_i \times \Sigma_s) \cap B &= \bigcap_{n\geq 1} \bigcup_{m\geq 1} (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s)) \\
&= \left(\bigcap_{j=1}^{k} (T_{js_j} \times (V_{s_j} \cap V_i) \times \Sigma_s) \right) \\
&\quad \cap \left(\bigcap_{n>k} \bigcup_{m\geq 1} (T_{nm} \times (V_m \cap V_i) \times (\Sigma_m^n \cap \Sigma_s)) \right) \\
&= \left(\bigcap_{j=1}^{k} (T_{js_j} \times P) \right) \\
&\quad \cap \left(\bigcap_{n>k} \bigcup_{m\geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j} \right) \times (\Sigma_m^n \cap \Sigma_s) \right) \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Pi_T^{T\times P}((T \times V_i \times \Sigma_s) \cap B) \\
&\quad = \Pi_T^{T\times P}\Bigg[\left(\bigcap_{j=1}^{k} (T_{js_j} \times P) \right) \\
&\qquad\quad \cap \left(\bigcap_{n>k} \bigcup_{m\geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j} \right) \times (\Sigma_m^n \cap \Sigma_s) \right) \right) \Bigg], \\
&\quad = \bigcap_{j=1}^{k} T_{js_j} \cap \Pi_T^{T\times X}\left(\bigcap_{n>k} \bigcup_{m\geq 1} \left(T_{nm} \times \left(V_m \cap V_i \cap \bigcap_{j=1}^{k} V_{s_j} \right) \right) \right) \\
&\quad = \bigcap_{j=1}^{k} T_{js_j} \cap \Pi_T^{T\times X}\left(\left(T \times \left(V_i \cap \bigcap_{j=1}^{k} V_{sj_j} \right) \right) \cap \left(\bigcap_{n>k} \bigcup_{m\geq 1} (T_{nm} \times V_m) \right) \right) \\
&\quad = \bigcap_{j=1}^{k} T_{js_j} \cap \Pi_T^{T\times X}\left(\left(T \times \left(V_i \cap \bigcap_{j=1}^{k} V_{sj} \right) \right) \cap G \right).
\end{aligned}$$

As F is $\mathfrak{T}$-measurable, it follows that W is $\mathfrak{T}$-measurable.

Now, let X be a zero-dimensional Polish space. Let Y be a zero-dimensional compact metric space containing a homeomorph of X. We consider F as a multifunction with values subsets of Y. By the previous case, we get a Polish space Q, a continuous map $g: Q \to Y$ and a $\mathfrak{T}$-measurable, closed set-valued multifunction $H: T \to Q$ such that $F(t) = g(H(t))$, $t \in T$. Put $P = g^{-1}(X)$ and β the restriction of g to P. As X is a G_δ in Y, P is Polish. Note that $H(t) \subseteq P$, $t \in T$. Put $W = H$.

Finally, let X be an arbitrary Polish space. Let $g: \Sigma \to X$ be a continuous, open and onto map. Define a multifunction $H: T \to \Sigma$ by $H(t) = g^{-1}(F(t))$, $t \in T$. Clearly, $H(t)$ is a G_δ in Σ for each $t \in T$. Let $U \subseteq \Sigma$ be open, then

$$\{t \in T: H(t) \cap U \neq \varnothing\} = \{t \in T: F(t) \cap g(U) \neq \varnothing\}.$$

As F is $\mathfrak{T}$-measurable and g open, H is $\mathfrak{T}$-measurable. To show $\mathrm{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_\Sigma$, we define $h\colon T \times \Sigma \to T \times X$ by

$$h(t, \sigma) = (t, g(\sigma)), \qquad t \in T, \sigma \in \Sigma.$$

Then h is continuous and $\mathrm{Gr}(H) = h^{-1}(G)$ so that $\mathrm{Gr}(H)$ is Borel in $T \times \Sigma$. Note that whenever t and t' belong to the same $\mathfrak{T}$-atoms, $G^t = G^{t'}$ and consequently, $(\mathrm{Gr}(H))^t = (\mathrm{Gr}(H))^{t'}$. This implies that $\mathrm{Gr}(H)$ is a union of $\mathfrak{T} \otimes \mathfrak{B}_\Sigma$-atoms. Therefore, by Lemma 2.1, $\mathrm{Gr}(H) \in \mathfrak{T} \otimes \mathfrak{B}_\Sigma$. By the previous case, we get a Polish space P, a closed set-valued, $\mathfrak{T}$-measurable multifunction $W\colon T \to P$ and a continuous map $f\colon P \to \Sigma$ such that $H(t) = f(W(t))$, $t \in T$. Put $\beta = g \circ f$. The desired properties are easy to verify.

Remark. A close examination of the various cases in the above proof reveals that the map $\beta\colon P \to X$ is obtained to be continuous, open and onto.

5. Decomposition of $\mathrm{Gr}(F)$ **into graphs of measurable selectors.** In this section, X and $F(t)$, $t \in T$, are all assumed to be uncountable. We prove

Theorem 5.1. *If for every $t \in T$, $F(t)$ is dense-in-itself then there is a $\mathfrak{T} \otimes \mathfrak{B}_X$-measurable map $f\colon T \times X \to X$ such that for each $t \in T$, $f(t, \cdot)$ is a Borel isomorphism of X onto $F(t)$.*

This result is analogous to a result of Mauldin **[11]** and we follow some of his ideas. We first show by an example that the condition "$F(t)$ is dense in itself" cannot be dropped from Theorem 5.1.

Example. Let X be an uncountable Polish space containing a countable, dense, open set U. (Union of the Cantor set and the mid-points of the removed intervals is such a Polish space.) Let $T = \Sigma$, $\mathfrak{T} = \mathfrak{B}_T$ and let $Y = X \setminus U$. Let E be a G_δ set in $\Sigma \times Y$ such that $\Pi_\Sigma(E) = \Sigma$ and E does not admit a Borel uniformization **[9**, p. 265]. Let $G\colon Y \to Y$ be a continuous, onto map such that $g^{-1}(y)$ is uncountable for each $y \in Y$. Let $B = \{(t, y) \in T \times Y\colon (t, g(y)) \in E\}$. Then B is a G_δ set in $\Sigma \times Y$ such that every section of B is uncountable and B does not admit a Borel uniformization. Let $F\colon T \to X$ be defined by $F(t) = B^t \cup U$, $t \in T$. It is clear that $F(t)$ is a G_δ in X for each $t \in T$ and $\mathrm{Gr}(F) \in \mathfrak{T} \otimes \mathfrak{B}_X$. As U is dense in X, F is $\mathfrak{T}$-measurable. If F satisfies the conclusions of Theorem 5.1, by a result of Mauldin **[11]**, there is a Borel set $M \subseteq \mathrm{Gr}(F)$ such that for each $t \in T$, M^t is nonempty and perfect. Let $H = M \setminus (T \times U) \subseteq B$. The sections of H are nonempty and compact so that H, and therefore, B admits a Borel uniformization.

Lemma 5.2. *Let X be compact. Then for each $t \in T$, there is a system $\{n_d^t\colon d \in D\}$ of positive integers such that for $d \in D_k$, $k > 0$, and $t \in T$,*

(i) *the map $t' \to n_d^{t'}$ is $\mathfrak{T}$-measurable,*
(ii) $\mathrm{diam}(V_{n_d^t}) < 2^{-k}$,
(iii) $d' \in D_k, d \neq d' \Rightarrow \overline{V}_{n_d^t} \cap \overline{V}_{n_{d'}^t} = \varnothing$,
(iv) $F(t) \cap V_{n_d^t} \neq \varnothing$,
(v) $\overline{V}_{n_{di}^t} \subseteq G_{k+1}^t \cap V_{n_d^t}$, $i = 0$ *or* $i = 1$.

Proof. We use induction on $|d|$. Let $n_e^t = 1$ for all t. (i)–(v) are satisfied for

$d = e$. Suppose for some $k \geqslant 0$, n_d^t is defined for all $d \in \bigcup_{l<k} D_l$ and for all $t \in T$ satisfying (i)–(v). Fix a $d \in D_k$. Put

$$T^m = \{t \in T: n_d^t = m\}, \qquad m \geqslant 1.$$

By the induction hypothesis, the sets T^m, $m \geqslant 1$, belong to $\mathfrak{T}$, are pairwise disjoint and $T = \bigcup_{m \geqslant 1} T^m$. Now, for any pair (u, v) of positive integers define T_{uv}^m, $m \geqslant 1$, as follows.

If $\operatorname{diam}(V_u) < 2^{-(k+1)}$, $\operatorname{diam}(V_v) < 2^{-(k+1)}$, $\bar{V}_u \cap \bar{V}_v = \varnothing$, $\bar{V}_u \subseteq V_m$, $\bar{V}_v \subseteq V_m$ then

$$\begin{aligned} T_{uv}^m &= \{t \in T^m: \bar{V}_u \subseteq G_{k+1}^t, \bar{V}_v \subseteq G_{k+1}^t, V_u \cap F(t) \neq \varnothing \text{ and } V_v \cap F(t) \neq \varnothing\}; \\ &= \varnothing \quad \text{otherwise.} \end{aligned}$$

As F is $\mathfrak{T}$-measurable, by Lemma 2.2, $T_{uv}^m \in \mathfrak{T}$. Also $T^m = \bigcup_{(u,v)} T_{uv}^m$.

Let $\alpha: N \to N \times N$ be a one-one, onto function.

Put

$$\begin{aligned} S_i^m &= T_{\alpha(i)}^m \qquad \text{if } i = 1, \\ &= T_{\alpha(i)}^m \setminus \bigcup_{j<i} T_{\alpha(j)}^m \quad \text{if } i > 1. \end{aligned}$$

The sets S_i^m, $i \geqslant 1$, belong to $\mathfrak{T}$, are pairwise disjoint and $T^m = \bigcup_{i \geqslant 1} S_i^m$. We define

$$\left.\begin{aligned} n_{d0}^t &= (\alpha(j))_1 \\ n_{d1}^t &= (\alpha(j))_2 \end{aligned}\right\} \quad \text{if } t \in S_j^m \text{ for any } m.$$

This completes the definition of $\{n_d^t: d \in D_{k+1}\}$, $t \in T$. It is easy to check that (i)–(v) are satisfied.

LEMMA 5.3. *There is a map $g: T \times C \to X$ such that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism from C into $F(t)$ and for each $\varepsilon \in C$, $g(\cdot, \varepsilon)$ is $\mathfrak{T}$-measurable.*

PROOF. Without loss of generality, we assume that X is a compact metric space. For each $t \in T$, we get a system $\{n_d^t: d \in D\}$ of positive integers satisfying (i)–(v) of Lemma 5.2. Let $g(t, \varepsilon)$ be the unique point in $\bigcap_k \bar{V}_{n_{\varepsilon|k}^t}$, $t \in T$, $\varepsilon \in C$. By standard arguments, we show that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism defined on C and by (v), it is into $F(t)$. Let $t \in T$, $\varepsilon \in C$ and $U \subseteq X$ be open. Then

$$\begin{aligned} g(t, \varepsilon) \in U &\Leftrightarrow \bigcap_k \bar{V}_{n_{\varepsilon|k}^t} \subseteq U \\ &\Leftrightarrow (\exists\, k \geqslant 1)\left(\bar{V}_{n_{\varepsilon|k}^t} \subseteq U\right) \\ &\Leftrightarrow (\exists\, k \geqslant 1)(\exists\, l \geqslant 1)\left(\bar{V}_l \subseteq U\right)\left(n_{\varepsilon|k}^t = l\right) \end{aligned}$$

so that

$$g(\cdot, \varepsilon)^{-1}(U) = \bigcup \bigcup \left(\{t \in T: n_{\varepsilon|k}^t = l\}\right) \in \mathfrak{T},$$

where the inner union is taken over all l such that $\overline{V}_l \subseteq U$ and the outer union is over all k. It follows that $g(\cdot, \varepsilon)$ is $\mathfrak{T}$-measurable for every $\varepsilon \in C$.

PROOF OF THEOREM 5.1. By Lemma 5.3, we get a map g: $T \times C \to X$ such that for each $t \in T$, $g(t, \cdot)$ is a homeomorphism from C into $F(t)$ and for all $\varepsilon \in C$, $g(\cdot, \varepsilon)$ is $\mathfrak{T}$-measurable. In particular, g is $\mathfrak{T} \otimes \mathfrak{B}_C$-measurable [**6**, p. 378]. As X and C are Borel isomorphic we get a $\mathfrak{T} \otimes \mathfrak{B}_X$-measurable map h: $T \times X \to X$ such that for each $t \in T$, $h(t, \cdot)$ is a Borel isomorphism from X into $F(t)$. Let k: $T \times X \to T \times X$ be defined by

$$k(t, x) = (t, h(t, x)), \qquad t \in T, x \in X,$$

and let

$$B = \{(t, x) \in T \times X: x \in h(t, X)\}.$$

Then, $B \subseteq G$ and as k is one-one, Borel, B is Borel in $T \times X$. By Lemma 2.1, $B \in \mathfrak{T} \otimes \mathfrak{B}_X$. Also, k: $(T \times X, \mathfrak{T} \otimes \mathfrak{B}_X) \to (T \times X, \mathfrak{T} \otimes \mathfrak{B}_X)$ is a measurable map such that for each $t \in T$, $k(t, \cdot)$ is a Borel isomorphism from X into $\{t\} \times F(t)$. Now, we do a Schroeder-Bernstein type argument as done by Mauldin [**11**] and get a measurable map α: $(T \times X, \mathfrak{T} \otimes \mathfrak{B}_X) \to (T \times X, \mathfrak{T} \otimes \mathfrak{B}_X)$ such that for each $t \in T$, $\alpha(t, \cdot)$ is a Borel isomorphism from X onto $\{t\} \times F(t)$. Put $f = \Pi_X \circ \alpha$.

COROLLARY 1. *Let $M \subseteq T \times X$ be a Borel set such that for every $t \in \Pi_T(M)$, M^t is dense in itself, and both a K_σ and a G_δ set in X. Then M is a union of $2^{\aleph_0}$ disjoint Borel uniformizations.*

Under the hypothesis of the above corollary, Larman [**7**] proved that M contains $2^{\aleph_0}$ disjoint Borel uniformizations. The problem of the existence of $2^{\aleph_0}$ disjoint Borel uniformizations of M when its sections are not assumed to be dense in itself remains open.

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