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THE WITT RING OF A SPACE OF ORDERINGS

BY

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ABSTRACT. The theory of "space of orderings" generalizes the reduced theory of quadratic forms over fields (or, more generally, over semilocal rings). The category of spaces of orderings is equivalent to a certain category of "abstract Witt rings". In the particular case of the space of orderings of a formally real field K, the corresponding abstract Witt ring is just the reduced Witt ring of K. In this paper it is proved that if $X = (X, G)$ is any space of orderings with Witt ring $W(X)$, and g: $X \to \mathbf{Z}$ is any continuous function, then g is represented by an element of $W(X)$ if and only if $\Sigma_{\sigma \in V} g(\sigma) \equiv 0 \bmod |V|$ holds for all finite fans $V \subseteq X$. This generalizes a recent field theoretic result of Becker and Bröcker. Following the proof of this, applications are given to the computation of the stability index of X, and to the representation of continuous functions g: $X \to \pm 1$ by elements of G.

The major result in this paper is Theorem 5.5 (the representation theorem). Theorems 6.4 and 7.2 and Corollary 7.5 are applications of this theorem. The first three sections of the paper are of an introductory nature. Most of the material in these sections is either implicit in [**M2**] or [**M3**] or is contained in the unpublished [**M1**]. The application of the abstract theory presented here to the Witt ring of a field is explained in some detail in Theorems 1.3 and 2.6. Theorems 3.6 and 4.5 are included to establish that Theorem 5.5 does, in fact, yield [**BB**, 5.3] as a special case. For the more complicated application of this theory to the Witt ring of a semilocal ring, the reader is referred to [**KR**, 6.7, 2.24, 2.30].

1. Spaces of orderings. Throughout $X = (X, G)$ denotes a *space of orderings* in the terminology of [**M2**] or [**M3**]. That is, G is an Abelian group of exponent 2 (i.e. $x^2 = 1\ \forall\, x \in G$), and X is a (nonempty) subset of the character group $\chi(G) = \operatorname{Hom}(G, \{1, -1\})$ satisfying:

O_1 : X is a closed subset of $\chi(G)$.

O_2 : $\exists$ an element $e \in G$ such that $\sigma(e) = -1\ \forall\, \sigma \in X$.

O_3 : $X^{\perp} := \{a \in G | \sigma(a) = 1\ \forall\, \sigma \in X\} = 1$.

O_4 : If f and g are forms over G and if $x \in D(f \oplus g)$ then $\exists y \in D(f)$ and $z \in D(g)$ such that $x \in D\langle y, z\rangle$.

Observe, by O_3, that the element $e \in G$ whose existence is asserted by O_2 is unique. Also $e \neq 1$ (since $\sigma(1) = 1\ \forall\, \sigma \in X$). We will denote the element e by -1, and for $x \in G$, $(-1)(x)$ will be denoted by $-x$.

The topology on $\chi(G)$ is the standard one from the theory of locally compact Abelian groups (viewing G as discrete) [**R**, 1.2.6]. For $a \in G$ let $\hat{a}$: $\chi(G) \to \{1, -1\}$

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be the character defined by $\hat{a}(\sigma) = \sigma(a)\ \forall\ \sigma \in \chi(G)$. The topology on $\chi(G)$ is the weakest such that the maps $\hat{a}$, $a \in G$, are continuous (view $\{1, -1\}$ as discrete). Note that $\chi(G)$ is a compact totally disconnected topological group; in fact, $\chi(G)$ is naturally identified with a closed subgroup of the direct product $\{1, -1\}^G$, and the latter is compact by Tychonoff's Theorem. Thus X, with the induced topology is, using O_1, a compact totally disconnected topological space. Of course, if G is finite, then the topology on $\chi(G)$ is discrete.

Note that for $a \in G$, the set $X(a) := \{\sigma \in X | \sigma(a) = 1\}$ is *clopen* (i.e. both closed and open) in X. Moreover, $\sigma(a) = -1 \Leftrightarrow \sigma(-a) = 1$ holds for any $\sigma \in X$ (by O_2), so the sets $X(a)$, $a \in G$, are a subbasis for the topology on X. Thus, the sets $X(a_1, \dots, a_n) := \bigcap_{i=1}^n X(a_i)$, $a_1, \dots, a_n \in G$, $n \geq 1$, are clopen in X and are a basis for the topology on X. These sets will be referred to as *Harrison basic sets*.

A *form of dimension n* is just an n-tuple $f = \langle a_1, \dots, a_n \rangle$, $a_1, \dots, a_n \in G$. Let $g = \langle b_1, \dots, b_m \rangle$ be another form. The *sum* and *product* is defined by $f \oplus g = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$, $f \otimes g = \langle a_1 b_1, \dots, a_1 b_m, \dots, a_n b_1, \dots, a_n b_m \rangle$. For $l \geq 1$, $l \times f := f \oplus \cdots \oplus f$ (l terms), and for $r \in G$, $rf := \langle r \rangle \otimes f = \langle ra_1, \dots, ra_n \rangle$. $-f := \langle -a_1, \dots, -a_n \rangle = \langle -1 \rangle \otimes f$. The *discriminant* of f is $d(f) := a_1 a_2 \cdots a_n \in G$. The *signature of f at* $\sigma \in X$ is $\sigma(f) := \sigma(a_1) + \cdots + \sigma(a_n) \in \mathbf{Z}$. Note $\sigma(f \oplus g) = \sigma(f) + \sigma(g)$ and $\sigma(f \otimes g) = \sigma(f)\sigma(g)$. Two forms f and g are said to be *isometric* (over X) if they have the same dimension and $\sigma(f) = \sigma(g)\ \forall\ \sigma \in X$. This is denoted by writing $f \cong g$ or $f \cong g$ (over X). Observe the following elementary result.

(1.1) Lemma. *If $f \cong g$, then $d(f) = d(g)$.*

Proof. Let $f = \langle a_1, \dots, a_n \rangle$, and let $\sigma \in X$. Then $\sigma(d(f)) = \sigma(a_1) \cdots \sigma(a_n) = (-1)^k$, where k denotes the number of $i \leq n$ satisfying $\sigma(a_i) = -1$. Note also that $\sigma(f) = (n - k) - k = n - 2k$, so $k = (1/2)(n - \sigma(f))$. It follows that if $f \cong g$, then $\sigma(d(f)) = \sigma(d(g))\ \forall\ \sigma \in X$, so $d(f) = d(g)$ by O_3. □

A form f is said to *represent* the element $x \in G$ (over X) if $\exists$ elements $x_2, \dots, x_n \in G$ such that $f \cong \langle x, x_2, \dots, x_n \rangle$. $D(f)$ or $D(f, X)$ will be used to denote the set of elements of G which are represented by f in this sense. This explains the terminology in O_4. Observe that for $a \in G$, $D\langle a \rangle = \{a\}$ (using O_3). Also, since $f \cong g \Leftrightarrow af \cong ag$, we have $D(af) = aD(f)$. O_4 generalizes as follows.

(1.2) Lemma. *Let $f_1, \dots, f_n$ be forms ($n \geq 1$), and suppose $x \in G$. Then $x \in D(f_1 \oplus \cdots \oplus f_n)$ if and only if $\exists\ x_i \in D(f_i)$, $i = 1, \dots, n$, such that $x \in D\langle x_1, \dots, x_n \rangle$.*

Proof. One implication is trivial. The other follows easily using O_4 and induction. □

Let us take a moment to look at the classical example of a space of orderings. Let F be a field of characteristic not two, and let Q denote the subset of F consisting of finite sums $\Sigma\, x_i^2$, $x_i \in F$. Observe that $Q^\cdot = Q \setminus \{0\}$ is a subgroup of the multiplicative group $F^\cdot$ (note $x^{-1} = (x^{-1})^2 \cdot x \in Q^\cdot$ if $x \in Q$), and that $F^\cdot/Q^\cdot$

is a group of exponent 2. An *ordering* on F is a subset $P \subseteq F$ satisfying $P + P \subseteq P$, $PP \subseteq P$, $P \cup -P = F$ and $P \cap -P = 0$. Observe that if P is an ordering of F, then $P^{\cdot} = P \setminus \{0\}$ is a subgroup of index 2 in $F^{\cdot}$ and $Q^{\cdot} \subseteq P^{\cdot}$. Thus each ordering P can be identified with the character σ_P of $F^{\cdot}/Q^{\cdot}$ given by $\sigma_P(\bar{a}) = 1$ or -1 depending on whether $a \in P$ or $-a \in P$. (Here $\bar{a}$ denotes the canonical image in $F^{\cdot}/Q^{\cdot}$ of $a \in F^{\cdot}$.) Thus, the set X of all orderings of F can be identified with a certain subset of the character group of $F^{\cdot}/Q^{\cdot}$.

(1.3) THEOREM. *Let F be a field of characteristic not 2. Then the pair $(X, F^{\cdot}/Q^{\cdot})$, with notations as above, is a space of orderings (possibly $X = \varnothing$).*

This result is fairly well known and follows essentially from Pfister's local-global principle [**L**, p. 241, 4.2]. A proof is given in [**M1**]. The generalization of this to semilocal rings appears to have first been remarked by Knebusch and Becker [**M2**, Example 2], and the details of this are to be found in [**KR**, 6.7, 2.24, 3.30]. On the other hand, since [**M1**] is not published, and the proofs in the generality given in [**KR**] are quite complicated, an elementary proof will now be given.

PROOF OF 1.3. Regarding O_1, one simply checks that for $\sigma \in \chi(F^{\cdot}/Q^{\cdot})$, $\sigma \in X \Leftrightarrow \sigma(-\bar{1}) = -1$ and $\sigma(\bar{a}) = 1$ $\forall$ $a, b, c \in F^{\cdot}$ s.t. $[b + c = a$ and $\sigma(\bar{b}) = \sigma(\bar{c}) = 1]$. The latter condition clearly defines a closed set in $\chi(F^{\cdot}/Q^{\cdot})$. The element $-\bar{1}$ clearly satisfies O_2. Observe that O_3 is equivalent to the assertion $\bigcap_{P \in X} P = Q$. This is the Artin-Schreier result [**L**, p. 227, 1.12]. Now let us prove O_4. For $f = \langle a_1, \ldots, a_n\rangle$, $a_1, \ldots, a_n \in F^{\cdot}$, let $\bar{f}$ denote the form $\langle \bar{a}_1, \ldots, \bar{a}_n\rangle$. Suppose also $g = \langle b_1, \ldots, b_m\rangle$ and that $\bar{f} \oplus \bar{g} \cong \bar{h}$ where x appears in the diagonalization of h. Note $\overline{f \oplus g} = \bar{f} \oplus \bar{g}$. Thus $\sigma_P(\overline{f \oplus g}) = \sigma_P(\bar{h})$ $\forall$ $P \in X$, so by [**L**, p. 241, 4.3; p. 237, 3.5] $\exists$ $k \geqslant 0$ such that $2^k \times f \oplus 2^k \times g \cong 2^k \times h$ (classical isometry). Thus $x = \sum s_i a_i + \sum t_j b_j$ with $s_i, t_j \in Q$. Assume first $\sum s_i a_i \neq 0$, $\sum t_j b_j \neq 0$. Take $y = \sum s_i a_i$, $z = \sum t_j b_j$. We claim $\bar{x} \in D\langle \bar{y}, \bar{z}\rangle$, $\bar{y} \in D(\bar{f})$, $\bar{z} \in D(\bar{g})$. The first is clear since even $x \in D\langle y, z\rangle$. For the second, define a_i^* to be either $s_i a_i$ or a_i depending on whether $s_i \neq 0$, or $s_i = 0$, and let $f^* = \langle a_1^*, \ldots, a_n^*\rangle$. Then clearly $y \in D(f^*)$. Since $\bar{f}^* = \bar{f}$, this implies $\bar{y} \in D(\bar{f})$. Similarly $\bar{z} \in D(\bar{g})$. If either $\sum s_i a_i$ or $\sum t_j b_j$ is zero, the result is trivial (for example, if $\sum s_i a_i = 0$, take $\bar{z} = \bar{x}$, and $\bar{y}$ arbitrary in $D(\bar{f})$). $\square$

2. The Witt ring $W(X)$. A form f is said to be *isotropic* if $\exists$ $x_3, \ldots, x_n \in G$ such that $f \cong \langle 1, -1, x_3, \ldots, x_n\rangle$. Note, in particular, this implies $\dim(f) \geqslant 2$. A form which is not isotropic is said to be *anisotropic*. For any $x \in G$, $\langle x, -x\rangle \cong \langle 1, -1\rangle$ (just compare signatures). Any such form will be called a *hyperbolic plane*. For future reference we include the following lemmas.

(2.1) LEMMA. *The following are equivalent*:

(i) *f is isotropic*;

(ii) $D(f) = G$;

(iii) $\exists$ $x \in G$ *such that* $x, -x \in D(f)$.

PROOF. Since $\langle -1, 1\rangle \cong \langle x, -x\rangle$ $\forall\, x \in G$, (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii) is immediate. Now assume (iii). Note f cannot be 1-dimensional, for then $f \cong \langle x\rangle \cong \langle -x\rangle$, so, comparing signatures, $\sigma(-1) = 1$ $\forall\, \sigma \in X$. This contradicts O_2 and $X \neq \varnothing$. Thus $\exists$ a form g such that $f \cong \langle x\rangle \oplus g$. By O_4, $\exists\, y \in D(g)$ such that $-x \in D\langle x, y\rangle$. Thus $\exists\, z \in G$ such that $\langle x, y\rangle \cong \langle -x, z\rangle$. By 1.1, $xy = -xz$, i.e. $z = -y$. Thus $\langle x, y\rangle \cong \langle -x, -y\rangle$. Comparing signatures, this yields $\sigma(x) + \sigma(y) = -\sigma(x) - \sigma(y)$, i.e. $2\sigma(x) = -2\sigma(y)$, i.e. $\sigma(x) = -\sigma(y)$, i.e. $\sigma(-xy) = 1$ $\forall\, \sigma \in X$. Thus, by O_3, $-xy = 1$, i.e. $y = -x$. Thus $f \cong \langle x\rangle \oplus g \cong \langle x, y, \ldots\rangle \cong \langle x, -x, \ldots\rangle \cong \langle 1, -1, \ldots\rangle$ is isotropic. □

(2.2) LEMMA. *If $f \oplus g$ is isotropic, then $\exists\, x \in G$ such that $x \in D(f)$, $-x \in D(g)$.*

For the proof, see [M3, 1.3] where it is proved that, in the presence of O_1, O_2, O_3, this condition is, in fact, equivalent to O_4.

Clearly any form f decomposes as

$$f \cong k \times \langle 1, -1\rangle \oplus f_{\mathrm{an}}$$

with $k \geq 0$, and f_{an} anisotropic (one must allow the possibility that f_{an} is zero-dimensional). Observe that k is uniquely determined by f, and f_{an} is uniquely determined (up to isometry) by f. This may be proved used *Witt cancellation*:

$$f \oplus g \cong f \oplus h \Rightarrow g \cong h.$$

Although the latter is a deep result in the classical theory of quadratic forms, it is a triviality here (just compare the signatures and dimensions). k and f_{an} are called the *Witt index* of f, and the *anisotropic part* of f, respectively. We say two forms f, g are *Witt-equivalent* if their anisotropic parts are isometric. This is denoted by writing $f \sim g$. Denote by $W(X) = W(X, G)$ the set of all equivalence classes of forms with respect to $\sim$. One sees that $\oplus$ and $\otimes$ induce binary operations on $W(X)$ making it a commutative ring with unity $\langle 1\rangle$. (The fact that the multiplication on $W(X)$ is well defined requires $\langle x, -x\rangle \cong \langle 1, -1\rangle$. The remaining ring properties are checked in a straightforward way.) This ring will be called the *Witt ring* of the space of orderings X. Of course, this construction is motivated by the classical construction of Witt [L, p. 36, 1.4].

(2.3) LEMMA. *$f \sim g \Leftrightarrow \sigma(f) = \sigma(g)$ $\forall\, \sigma \in X$.*

PROOF. The implication ($\Rightarrow$) is easy. Note that for $f = \langle a_1, \ldots, a_n\rangle$, and $\sigma \in X$, $\sigma(f) = \sigma(a_1) + \cdots + \sigma(a_n) \equiv n \pmod 2$. Thus $\dim(f) \equiv \sigma(f) \pmod 2$. Now suppose $\sigma(f) = \sigma(g)$ $\forall\, \sigma \in X$. Since $X \neq \varnothing$, it follows that $\dim(f) \equiv \dim(g) \pmod 2$. We may assume $\dim(f) \geq \dim(g)$. Then it is clear that $f \cong g \oplus r \times \langle 1, -1\rangle$, where $r = (1/2)(\dim(f) - \dim(g))$. $f \sim g$ is immediate from this.

Using 2.3, we can represent $W(X)$ as a subring of $C(X, \mathbf{Z})$, the ring of all continuous functions from X to $\mathbf{Z}$ (giving $\mathbf{Z}$ the discrete topology). Namely, for $f = \langle a_1, \ldots, a_n\rangle$, define $\hat{f}\colon X \to \mathbf{Z}$ by $\hat{f}(\sigma) = \sigma(f)$ $\forall\, \sigma \in X$. Note $\hat{f}$ is continuous, since it is the sum of the continuous functions $\hat{a}_i$, $i = 1, \ldots, n$. (Recall these latter functions are continuous by the definition of the topology on X.) Thus, we have a canonical map $W(X) \to C(X, \mathbf{Z})$. This is a ring homomorphism. It is injective by 2.3. To simplify our notation *we identify $W(X)$ with its image in $C(X, \mathbf{Z})$.*

We can recover the concepts of isometry and isotropy from $W(X)$ using:

(2.4) LEMMA. (i) $f \cong g \Leftrightarrow f \sim g$ *and* $\dim(f) = \dim(g)$.
(ii) $f \sim g$, $\dim(f) > \dim(g) \Rightarrow f$ *is isotropic*.

PROOF. Clear. □

Note that the dimension and discriminant are not well-defined functions on $W(X)$. However, if we define the *signed discriminant* of a form f by $d_\pm(f) = (-1)^{(1/2)n(n-1)}d(f)$, where $n = \dim(f)$, then we have the following.

(2.5) LEMMA. *Suppose* $f \sim g$. *Then* $\dim(f) \equiv \dim(g) \pmod 2$, *and* $d_\pm(f) = d_\pm(g)$.

PROOF. The first assertion is obvious. For the second, we may assume $\dim(f) > \dim(g)$, so $f \cong g \oplus k \times \langle 1, -1\rangle$, where $k = (1/2)(\dim(f) - \dim(g))$. Thus, by (1.1), $d(f) = (-1)^k d(g)$ and $\dim(f) = \dim(g) + 2k$. The result follows easily from this. □

We will pursue the consequences of 2.5 a little further at the beginning of §6. We close this section by examining $W(X)$ in the case that X is the space of orderings of a field.

(2.6) THEOREM. *Let F be a field of characteristic not* 2, *and let* $X = (X, F^\cdot/Q^\cdot)$ *be the asociated space of orderings in the terminology of* 1.3. *Assume* $X \neq \varnothing$. *Then* $W(X)$ *is the reduced Witt ring of F.*

Recall, the *reduced Witt ring* of F is the classical Witt ring $W(F)$ factored by its nilradical.

PROOF OF 2.6. For $f = \langle a_1, \dots, a_n\rangle$, $a_1, \dots, a_n \in F^\cdot$, consider the associated form $\bar{f} = \langle \bar{a}_1, \dots, \bar{a}_n\rangle$ over $F^\cdot/Q^\cdot$. The correspondence $f \mapsto \bar{f}$ defines a ring homomorphism from $W(F)$ onto $W(X)$. By Pfister's local-global principle [**L**, p. 241, Remark 4.3], the kernel is $W_t(F)$ the torsion subgroup of $W(F)$. On the other hand, by another result of Pfister's, $\operatorname{nil}(W(F)) = W_t(F)$ [**L**, p. 248, Theorem 6.1], since $X \neq \varnothing$. This completes the proof. □

Note that if $X = \varnothing$, then the reduced Witt ring of F is $\mathbf{Z}/2$. (See [**L**, p. 248, Theorem 6.1].) One can round out the theory by viewing this to be $W(\varnothing)$.

3. Subspaces. A form of the type $p = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$, $a_1, \dots, a_n \in G$, will be called an (*n-fold*) *Pfister form*.

(3.1) LEMMA. *Let* $p = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$, $a \in G$. *Then the following assertions are equivalent.*
(i) $\sigma(a) = 1 \ \forall\, \sigma \in X(a_1, \dots, a_n)$.
(ii) $ap \cong p$.
(iii) $a \in D(p)$.

(Recall: $X(a_1, \dots, a_n) = \bigcap_{i=1}^n X(a_i) = \{\sigma \in X \mid \sigma(a_i) = 1, i = 1, \dots, n\}$.)

PROOF. Note that for $\sigma \in X$, $\sigma(p) = 2^n$ or 0 depending on whether $\sigma \in X(a_1, \dots, a_n)$ or not. Thus, if $\sigma(a) = 1 \ \forall\, \sigma \in X(a_1, \dots, a_n)$, then $\sigma(ap) = \sigma(a)\sigma(p) = \sigma(p) \ \forall\, \sigma \in X$, so $ap \cong p$. This, in turn, implies $D(p) = D(ap) = aD(p)$

so $a \in D(p)$ (since $1 \in D(p)$). Now suppose $a \in D(p)$. Thus $p \cong \langle a, \cdots \rangle$; and suppose $\sigma \in X(a_1, \ldots, a_n)$. Then $\sigma(p) = 2^n = \dim(p)$. This implies *all* the entries in $\langle a, \cdots \rangle$ are positive at σ, so in particular, $\sigma(a) = 1$. $\square$

(3.2) COROLLARY [**M2**, 2.1]. *Let* $Y = X(a_1, \ldots, a_n)$, *and* $\Delta = D(p)$, *where* $p = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$. *Then*

$$\Delta = Y^{\perp} \quad \textit{and} \quad Y = \Delta^{\perp} \cap X. \tag{$*$}$$

PROOF. Let $\sigma \in Y$, $a \in \Delta$. Then $\sigma(a) = 1$ by 3.1 so $\Delta \subseteq Y^{\perp}$, $Y \subseteq \Delta^{\perp} \cap X$. Let $\sigma \in \Delta^{\perp} \cap X$. Since $a_1, \ldots, a_n \in \Delta$, it follows that $\sigma(a_i) = 1$, $i = 1, \ldots, n$, so $\sigma \in Y$. Thus $Y = \Delta^{\perp} \cap X$. Let $a \in Y^{\perp}$. Then $a \in \Delta$ by 3.1. Thus $\Delta = Y^{\perp}$. $\square$

Recall [**M3**, 2.1] a *subspace* of (X, G) is a pair $(Y, G/\Delta)$ where $Y \subseteq X$, $\Delta \subseteq G$ satisfy condition $(*)$ of 3.2. Thus 3.2 asserts that, for $p = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$, $(X(a_1, \ldots, a_n), G/D(p))$ is a subspace of X. Note if $(Y, G/\Delta)$ is a subspace of X, then Δ is a subgroup of $G(\Delta = Y^{\perp})$, so G/Δ makes sense. Note also that $\Delta^{\perp}$ is a closed subgroup of $\chi(G)$ which is canonically isomorphic to $\chi(G/\Delta)$ (in the sense of topological groups). Thus Y may be viewed as a closed subset of $\chi(G/\Delta)$.

(3.3) THEOREM. *Every* (*nonempty*) *subspace of a space of orderings is again a space of orderings.*

PROOF. See [**M3**, 2.2].

The concept of a subspace is very important in the theory of spaces of orderings as it allows proofs by Zorn's lemma. Here are some notations that will be used. Suppose $Y = (Y, G/\Delta)$ is a subspace of X. Then for each form $f = \langle a_1, \ldots, a_n\rangle$, $a_1, \ldots, a_n \in G$, one can associate the form $\bar{f} = \langle \bar{a}_1, \ldots, \bar{a}_n\rangle$ over G/Δ. Here, $\bar{a}$ denotes the coset of a in G/Δ. For f, g forms (over G) we write $f \cong g$ (over Y) (respectively $f \sim g$ (over Y)) to indicate that $\bar{f} \cong \bar{g}$ (respectively $\bar{f} \sim \bar{g}$). We use $D(f, Y)$ to indicate the set $\{a \in G \mid \bar{a} \in D(\bar{f})\}$. There is an obvious commutative diagram of rings and ring homomorphisms:

$$\begin{array}{ccc} W(Y) & \hookrightarrow & C(Y, \mathbf{Z}) \\ \uparrow & & \uparrow \\ W(X) & \hookrightarrow & C(X, \mathbf{Z}). \end{array}$$

We will say $g \in C(X, \mathbf{Z})$ is *represented over* Y to indicate that its restriction $g|Y$ is in $W(Y)$. This is equivalent to asserting the existence of a form f over G such that $\sigma(f) = g(\sigma)\ \forall\ \sigma \in Y$. Such a form is said to *represent the function g over Y*.

The following lemma shows how to transfer results between X and the subspace $X(a_1, \ldots, a_n)$.

(3.4) LEMMA. *Suppose* $p = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_n\rangle$, *and* $X(a_1, \ldots, a_n) \neq \varnothing$. *Then for any forms f, g over G*:

(i) $f \cong g$ (*over* $X(a_1, \ldots, a_n)$) $\Leftrightarrow f \otimes p \cong g \otimes p$ (*over* X);

(ii) $f \sim g$ (*over* $X(a_1, \ldots, a_n)$) $\Leftrightarrow f \otimes p \sim g \otimes p$ (*over* X);

(iii) $D(f, X(a_1, \ldots, a_n)) = D(f \otimes p, X)$;

(iv) *f is isotropic over* $X(a_1, \ldots, a_n) \Leftrightarrow f \otimes p$ *is isotropic over* X.

PROOF. Let $\sigma \in X$. Since $\sigma(p) = 2^n$ or 0 according as $\sigma \in X(a_1, \ldots, a_n)$ or not, and since $\sigma(f \otimes p) = \sigma(f)\sigma(p)$ and $\sigma(g \otimes p) = \sigma(g)\sigma(p)$, we see that

$$\begin{gathered}\sigma(f \otimes p) = \sigma(g \otimes p) \quad \forall\, \sigma \in X \Leftrightarrow \\ 2^n\sigma(f) = 2^n\sigma(g) \quad \forall\, \sigma \in X(a_1, \ldots, a_n) \Leftrightarrow \\ \sigma(f) = \sigma(g) \quad \forall\, \sigma \in X(a_1, \ldots, a_n)\end{gathered}$$

This proves (i). By 2.3 applied to X and to $X(a_1, \ldots, a_n)$ it proves (ii). The proof of (iii) is in [**M2**, 2.3]. (Note the proof in [**M2**, 2.3] already proves (i).) Finally (iv) follows from 2.1, applied to X and to $X(a_1, \ldots, a_n)$, together with (iii). $\square$

(3.5) COROLLARY. *For any form f and any $k \geq 1$,*
(i) $D(k \times f) = D(f)$; *and*
(ii) *f is isotropic $\Leftrightarrow$ $k \times f$ is isotropic.*

PROOF. (i) Choose n so large that $2^n > k$. Then clearly $D(f) \subseteq D(k \times f) \subseteq D(2^n \times f)$. On the other hand, $2^n \times f \cong \langle 1, 1\rangle^n \otimes f$, and $X(1, 1, \ldots, 1) = X$, so by 3.4(iii), $D(f) = D(2^n \times f)$. This proves (i).

(ii) This is immediate from (i) and 2.1. $\square$

Note. It is also possible to give a direct elementary proof of 3.5(i) using 1.2.

In order to be able to compare the results in this paper with the field-theoretic results in [**BB**], we need the following.

(3.6) THEOREM. *Let F be a field of characteristic not* 2, *and let $X = (X, F^{\cdot}/Q^{\cdot})$ be its associated space of orderings as in* 1.3. *Let T be a preorder of F, and let $X/T = \{P \in X \mid P \supseteq T\}$. Then $(X/T, F^{\cdot}/T^{\cdot})$ is a subspace of X whose Witt ring is the ring denoted $W(F/T)$ in the terminology of* [**BB**]. *Conversely, every nonempty subspace of X has this form, for some preorder T.*

PROOF. Recall that a preorder of F is a subset $T \subseteq F$ satisfying $T + T \subseteq T$, $T \cdot T \subseteq T$, $x^2 \in T\ \forall\, x \in F$, and $-1 \notin T$. To verify $(X/T, F^{\cdot}/T^{\cdot})$ is a subspace of X, we need only verify $X/T = (T^{\cdot}/Q^{\cdot})^{\perp} \cap X$ and $T^{\cdot}/Q^{\cdot} = (X/T)^{\perp}$. The first of these is just the definition of X/T translated into our terminology. The second follows from $\bigcap_{P \in X/T} P = T$ [**BB**, §1]. Note that $W(F/T)$ and $W(X/T)$ are both identified with the same subring of $C(X/T, \mathbf{Z})$ (see [**BB**, 2.1]).

Now let $(Y, F^{\cdot}/\Delta)$ be a subspace of X. Thus $\Delta/Q^{\cdot} = Y^{\perp}$ and $Y = (\Delta/Q^{\cdot})^{\perp} \cap X$. The first condition asserts that $T := \Delta \cup \{0\}$ satisfies $T = \bigcap_{P \in Y} P$ and hence is a preorder (any intersection of orderings is a preorder). The second asserts $Y = X/T$. $\square$

4. Fans. We begin this section by generalizing a result proved in [**M2**].

(4.1) LEMMA. *Suppose $\sigma \in \chi(G)$, $\sigma \neq 1$, satisfies the following condition*: $x \in \ker(\sigma) \Rightarrow D\langle 1, x\rangle \subseteq \ker(\sigma)$. *Then $\sigma \in X$.*

PROOF. This is proved in the case X is finite in [**M2**, 4.1]. In any case, we see, by the *proof* of [**M2**, 4.1], that for each finite set $a_1, \ldots, a_k \in G$ satisfying $a_1, \ldots, a_k \in \ker(\sigma)$, $\exists\, \sigma' \in X$ such that $a_1, \ldots, a_k \in \ker(\sigma')$. Note, in particular, this implies

$-1 \notin \ker(\sigma)$, i.e. $\sigma(-1) = -1$. Now let $b_1, \ldots, b_k \in G$ be arbitrary, and let $\epsilon_i = \sigma(b_i)$, and $a_i = \epsilon_i b_i$, $i = 1, \ldots, k$. Then $\sigma(a_i) = 1$ (since $\sigma(-1) = -1$) $\forall\, i = 1, \ldots, k$. Thus, by the above, $\exists\, \sigma' \in X$ such that $\sigma'(a_i) = 1$, i.e. $\sigma'(b_i) = \epsilon_i$ $\forall\, i = 1, \ldots, k$. This shows σ lies in the closure of X. Thus, by O_1, $\sigma \in X$. $\square$

Using 4.1, we can now prove the following.

(4.2) THEOREM. *The following are equivalent.*

(i) $\forall\, x \in G, x \neq -1 \Rightarrow D\langle 1, x\rangle = \{1, x\}$.

(ii) $X = \{\alpha \in \chi(G) | \alpha(-1) = -1\}$.

PROOF. (i) $\Rightarrow$ (ii). Clearly $X \subseteq \{\alpha \in \chi(G) | \alpha(-1) = -1\}$ by O_2. Let $\alpha \in \chi(G)$, $\alpha(-1) = -1$. Let $x, y \in G$ be such that $y \in D\langle 1, x\rangle$, $\alpha(x) = 1$. Then $x \neq -1$, since $\alpha(-1) = -1$. Thus, by (i), $y = 1$ or x, so $\alpha(y) = 1$. It follows from 4.1 that $\alpha \in X$.

(ii) $\Rightarrow$ (i). Let $x \in G$, $x \neq -1$, and let $y \in D\langle 1, x\rangle$. Suppose $y \neq 1, x$. Then the subgroup $C = \{1, x, -y, -xy\}$ of G satisfies $-1 \notin C$. Thus (viewing G as a vector space over $\mathbf{Z}/2$ and using elementary linear algebra) $\exists\, \alpha \in \chi(G)$ such that $\alpha(C) = 1$, $\alpha(-1) = -1$. By (ii), $\alpha \in X$. Since $\alpha(C) = 1$, it follows $\alpha(x) = 1$, $\alpha(y) = -1$. This contradicts $y \in D\langle 1, x\rangle$ (see 3.1). $\square$

A space of orderings satisfying either of the equivalent conditions in 4.2 will be referred to as a *fan*. Fans play the role of the "local objects" in the representation theorem 5.5. The following two corollaries present some elementary properties of fans.

(4.3) COROLLARY. *Suppose X is a fan. Then every subspace of X is also a fan.*

PROOF. Let $(Y, G/\Delta)$ be a subspace of X, and suppose $\sigma \in \chi(G/\Delta)$, $\sigma(-1) = -1$. Note the distinguished element $-1 \in G/\Delta$ is just the coset of $-1 \in G$. It follows that σ, viewed as an element of $\Delta^\perp \subseteq \chi(G)$, satisfies $\sigma(-1) = -1$, so $\sigma \in X$. Thus $\sigma \in \Delta^\perp \cap X = Y$. $\square$

Recall, a space of orderings (X, G) is said to be *finite* if X (or equivalently G) is a finite set.

(4.4) COROLLARY. *Suppose (X, G) is finite. Then*: (i) *X is a fan if and only if $|X| = (1/2)|G|$*;

(ii) *if X is a fan, and $a \in G$, $a \neq \pm 1$, then $|X(a)| = (1/2)|X|$.*

PROOF. (i). Let $X = \{\sigma_1, \ldots, \sigma_k\}$. Thus $1 = \bigcap_{i=1}^k \ker(\sigma_i)$ by O_3. Since each $\ker(\sigma_i)$ has index 2 in G it follows that $|G| \leq 2^k$. Thus G is finite, so $|G| = |\chi(G)| = 2^n$, where n is the dimension of G (or $\chi(G)$) as a $\mathbf{Z}/2$-vector space. We have $X \subseteq \{\alpha \in \chi(G) | \alpha(-1) = -1\}$ (by O_2), with equality if and only if X is a fan. Thus, it is enough to verify that exactly half the characters on G satisfy $\alpha(-1) = -1$. But this is clear, since the remaining characters on G satisfy $\alpha(-1) = 1$, and this condition defines a hyperplane in $\chi(G)$.

(ii). Since X is a fan $(X(a), G/D\langle 1, a\rangle)$ is also a fan by 4.3. Thus, by (i), $|X| = (1/2)|G|$, and $|X(a)| = (1/2)|G/D\langle 1, a\rangle|$. Since $a \neq -1$, $D\langle 1, a\rangle = \{1, a\}$. Since $a \neq 1$, this is a group of order 2. The result is now clear. $\square$

To see that 5.5 generalizes **[BB**, 5.3], the following result is required.

(4.5) THEOREM. *Let F be a field of characteristic not* 2, *and let T be a preorder of F* (*see* 3.6). *Then $(X/T, F^\cdot/T^\cdot)$ is a fan if and only if T is a fan in the terminology of* **[BB]**, **[B2]**.

PROOF. Recall that T is said to be a fan if and only if

$$a \not\equiv -1 \pmod{T^\cdot} \Rightarrow T + Ta = T \cup Ta \qquad \forall\, a \in F^\cdot.$$

By 4.2, $(X/T, F^\cdot/T^\cdot)$ is a fan if and only if $\forall\, a \in F^\cdot$,

$$a \not\equiv -1 \pmod{T^\cdot} \Rightarrow D\langle \bar{1}, \bar{a}\rangle = \{\bar{1}, \bar{a}\}.$$

Here $\bar{x}$ denotes the coset of $x \in F^\cdot$ in the group $F^\cdot/T^\cdot$. Thus, it is only necessary to remark that $x \in (T + aT)^\cdot \Leftrightarrow \bar{x} \in D\langle \bar{1}, \bar{a}\rangle$. But $T + aT$ is a preorder, so

$$T + aT = \bigcap \{P \mid P \in X/T + aT\} = \bigcap \{P \mid P \in X/T, a \in P\}.$$

On the other hand, by 3.3

$$\bar{x} \in D\langle \bar{1}, \bar{a}\rangle \Leftrightarrow \sigma_p(\bar{x}) = 1 \qquad \forall\, P \in X/T$$

satisfying $\sigma_p(\bar{a}) = 1$. By the above, this is equivalent to $x \in (T + aT)^\cdot$. $\square$

5. The representation theorem. The main goal of this section is the proof of Theorem 5.3. This is accomplished with the aid of Lemmas 5.1 and 5.2. Once this is done, Theorem 5.5 (the representation theorem) follows by an elementary argument.

(5.1) LEMMA. *Suppose $x_1, x_2 \in G$ and f_1, f_2 are forms over G such that*

$$f_1 \otimes \langle 1, x_1\rangle \cong f_2 \otimes \langle 1, x_2\rangle. \tag{1}$$

Then $\exists$ a form f over G such that $f \cong f_i$ over $X(x_i)$, $i = 1, 2$. (*Recall the terminology of* 3.4.)

PROOF. By (1), and 3.4(iii),

$$\begin{aligned} D(f_1, X(x_1)) &= D(f_1\langle 1, x_1\rangle, X) = D(f_2\langle 1, x_2\rangle, X) \\ &= D(f_2, X(x_2)) := D. \end{aligned}$$

Pick $p \in D$ and decompose $f_i \cong \langle p\rangle \oplus f_i'$ over $X(x_i)$. Thus, by 3.4(i), $f_i\langle 1, x_i\rangle \cong \langle p, px_i\rangle \oplus f_i'\langle 1, x_i\rangle$ over X, $i = 1, 2$. Rewriting (1) using this, and cancelling the 1-dimensional form $\langle p\rangle$, we obtain $\langle px_1\rangle \oplus f_1'\langle 1, x_1\rangle \cong \langle px_2\rangle \oplus f_1'\langle 1, x_2\rangle$. Multiplying this by x_2 and adding $\langle -px_1x_2\rangle$ to each side yields $\langle px_1x_2\rangle \oplus x_2f_1'\langle 1, x_1\rangle \cong \langle p\rangle \oplus f_2'\langle 1, x_2\rangle$ and

$$\langle 1, -1\rangle \oplus x_2f_1'\langle 1, x_1\rangle \cong p\langle 1, -x_1x_2\rangle \oplus f_2'\langle 1, x_2\rangle \quad (\text{over } X). \tag{2}$$

It follows that the right side of (2) is isotropic so by 2.2, $\exists\, s \in D(\langle 1, -x_1x_2\rangle, X)$ such that $-ps \in D(f_2'\langle 1, x_2\rangle, X) = D(f_2', X(x_2))$. Thus $f_2' \cong \langle -ps\rangle \oplus f_2''$ over $X(x_2)$, so $f_2'\langle 1, x_2\rangle \cong \langle -ps, -psx_2\rangle \oplus f_2''\langle 1, x_2\rangle$ over X. Also, by choice of s and 1.1, $\langle 1, -x_1x_2\rangle \cong \langle s, -sx_1x_2\rangle$ over X. Rewriting (2) using these last two relations, we obtain

$$\langle 1, -1\rangle \oplus x_2f_1'\langle 1, x_1\rangle \cong \langle ps, -psx_1x_2, -ps, -psx_2\rangle \oplus f_2''\langle 1, x_2\rangle.$$

Cancelling the hyperbolic planes, $\langle 1, -1\rangle \cong \langle ps, -ps\rangle$, and multiplying by x_2, this yields

$$f_1'\langle 1, x_1\rangle \cong \langle -psx_1, -ps\rangle \oplus f_2''\langle 1, x_2\rangle \quad (\text{over } X). \tag{3}$$

It follows that $-ps \in D(f_1'\langle 1, x_1\rangle, X) = D(f_1', X(x_1))$, so $f_1' \cong \langle -ps\rangle \oplus f_1''$ over $X(x_1)$, i.e. $f_1'\langle 1, x_1\rangle \cong \langle -ps, -psx_1\rangle \oplus f_1''\langle 1, x_1\rangle$ over X. Rewriting (3) using this, and cancelling, we obtain $f_1''\langle 1, x_1\rangle \cong f_2''\langle 1, x_2\rangle$ over X. Since $f_i \cong \langle p\rangle \oplus f_i' \cong \langle p, -ps\rangle \oplus f_i''$ over $X(x_i)$, $i = 1$, we are done by induction on the dimension. □

(5.2) LEMMA. *Suppose $x_1, x_2, x_3 \in G$ satisfy $\langle x_1, x_2, x_3\rangle \sim \langle 1\rangle$. Suppose $g \in C(X, \mathbf{Z})$ is represented over $X(x_i)$, $i = 1, 2, 3$. Then g is represented over X.*

PROOF. Observe, using 2.3, that $\langle x_1, x_2, x_3\rangle \sim \langle 1\rangle$ is equivalent to

$$\forall\, \sigma \in X, \quad \textit{exactly one} \text{ of } \sigma(x_1), \sigma(x_2), \sigma(x_3) \text{ is } -1. \tag{$*$}$$

In particular, $X(x_i) \cup X(x_j) = X\ \forall\, i \neq j$. Thus we assume $X(x_i) \neq \varnothing$ (otherwise g is represented over $X(x_j) = X$) and that g is not identically zero on both $X(x_i)$ and $X(x_j)$ (otherwise g is identically zero on X, so is trivially represented). Now let f_i be a form over G representing g over $X(x_i)$, $i = 1, 2, 3$. We may assume $f_3 = 0$ (replace g by $g - f_3$ if necessary), and that f_i is anisotropic over $X(x_i)$, $i = 1, 2$. Thus by 3.4(iv) $f_i\langle 1, x_i\rangle$ is anisotropic over X, $i = 1, 2$. Consider these two forms carefully. Let $\sigma \in X$. By ($*$), there are three possibilities. If $\sigma(x_1) = \sigma(x_2) = 1$, then $\sigma f_i = g(\sigma)$, $i = 1, 2$, so $\sigma(f_1\langle 1, x_1\rangle) = g(\sigma)\cdot 2 = \sigma(f_2\langle 1, x_2\rangle)$. If $\sigma(x_1) = 1$, $\sigma(x_2) = -1$, $\sigma(x_3) = 1$ then $\sigma f_1 = g(\sigma) = \sigma f_3 = 0$, and

$$\sigma(f_1\langle 1, x_1\rangle) = 0\cdot 2 = 0 = \sigma(f_2)\cdot 0 = \sigma(f_2\langle 1, x_2\rangle).$$

Similarly, if $\sigma(x_1) = -1$, $\sigma(x_2) = 1$, $\sigma(x_3) = 1$ then $\sigma(f_1\langle 1, x_1\rangle) = 0 = \sigma(f_2\langle 1, x_2\rangle)$. Thus $f_1\langle 1, x_1\rangle$ and $f_2\langle 1, x_2\rangle$ have the same signature at all $\sigma \in X$, i.e. $f_1\langle 1, x_1\rangle \sim f_2\langle 1, x_2\rangle$. Since both forms are anisotropic they must, by 2.4(ii), have the same dimension. Thus $f_1\langle 1, x_1\rangle \cong f_2\langle 1, x_2\rangle$ by 2.4(i). By 5.1 $\exists$ a form f such that $f \cong f_i$ over $X(x_i)$, $i = 1, 2$. Thus f represents g over $X(x_i)$, $i = 1, 2$. Since $X = X(x_1) \cup X(x_2)$, f represents g over X. □

(5.3) THEOREM. *Let $X_0 = (X_0, G_0)$ be a space of orderings. Let $g \in C(X_0, \mathbf{Z})$, and suppose g is represented over each fan $V \subseteq X_0$. Then g is represented over X_0.*

(*Note.* By a fan in X_0 is meant a subspace of X_0 which is a fan.)

PROOF. Suppose, to the contrary, that g is not represented over X_0. By Zorn's lemma one can prove the existence of a nonempty subspace $X \subseteq X_0$ minimal subject to: ($*$) g is not represented over X. For let $\{X_i\}$ be a collection of nonempty subspaces of X_0 each satisfying ($*$) and linearly ordered by inclusion, and let $X' = \bigcap X_i$. Note that $X'^{\perp\perp} \cap X \subseteq X_i^{\perp\perp} \cap X = X_i$, so $X'^{\perp\perp} \cap X \subseteq X'$, i.e. $X'^{\perp\perp} \cap X = X'$. Thus X' is a subspace of X_0. $X' \neq \varnothing$ by compactness (each X_i is closed). Suppose now that g is represented by a form f over X'. By continuity $U = \{\sigma \in X_0 \mid g(\sigma) = \sigma f\}$ is open in X_0 (even clopen), and by assumption, $X' \subseteq U$. Note that each $X_i \setminus U$ is closed in the compact set $X_0 \setminus U$, $\{X_i \setminus U\}$ is linearly ordered by inclusion, and $\bigcap (X_i \setminus U) = \varnothing$. Thus $\exists i$ such that $X_i \setminus U = \varnothing$, i.e. $X_i \subseteq U$. But then g is represented (by f) over X_i, a contradiction. Thus X' does indeed satisfy ($*$).

Now consider the subspace $X = (X, G)$ whose existence has just been proved. Clearly X is not a fan (g *is* represented over fans). Thus by 4.2, $\exists\, x \in G$, $x \neq -1$, $D\langle 1, x\rangle \neq \{1, x\}$. Thus $\exists\, y \in D\langle 1, x\rangle$, $y \neq 1, x$. Now using 1.1, $\langle 1, x\rangle \cong \langle y, xy\rangle$, i.e., $\langle -x, y, xy\rangle \sim \langle 1\rangle$. Take $x_1 = -x$, $x_2 = y$, $x_3 = xy$. Note $x_i \neq 1$, $i = 1, 2, 3$ so $X(x_i)$ is a proper subspace of X. By the minimal choice of X, g is represented over $X(x_i)$, $i = 1, 2, 3$. But then by 5.2, g is represented over X, a contradiction. $\square$

(5.4) LEMMA. *For $\overline{G}$ a given subgroup of G, denote by $\bar{\sigma}$ the restriction of $\sigma \in X$ to $\overline{G}$, and by $\overline{X}$ the set $\{\bar{\sigma} | \sigma \in X\}$. Let $g \in C(X, \mathbf{Z})$. Then*

(i) *$\exists$ a pair $(\bar{g}, \overline{G})$ where $\overline{G}$ is a finite subgroup of G such that $-1 \in \overline{G}$, and $\bar{g} \in C(\overline{X}, \mathbf{Z})$ is such that $g(\sigma) = \bar{g}(\bar{\sigma})\ \forall\, \sigma \in X$.*

(ii) *Let $(\bar{g}, \overline{G})$ be any pair as in* (i). *Let $-1, a_1, \ldots, a_n$ be a $\mathbf{Z}/2$-basis of $\overline{G}$ and let the Pfister form $p_{\bar{\sigma}}$ be defined by*

$$p_{\bar{\sigma}} = \prod_{i=1}^{n} \langle 1, \sigma(a_i)a_i\rangle \qquad \forall\, \bar{\sigma} \in \overline{X}.$$

Then

$$g = \left(\frac{1}{2}\right)^n \sum_{\bar{\sigma} \in \overline{X}} \bar{g}(\bar{\sigma})p_{\bar{\sigma}}.$$

PROOF. Let $\sigma \in X$. Since g is continuous, and $\mathbf{Z}$ is discrete, $g^{-1}(g(\sigma))$ is open so $\exists$ a Harrison basic set $X(a_1, \ldots, a_v)$ such that $\sigma \in X(a_1, \ldots, a_v) \subseteq g^{-1}(g(\sigma))$. Note g is constant on $g^{-1}(g(\sigma))$ and hence on the clopen set $X(a_1, \ldots, a_v)$. By compactness, $\exists$ elements $a_{ij} \in G$, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant v_i$ such that

$$X = \bigcup_{i=1}^{k} X(a_{i1}, \ldots, a_{iv_i}),$$

and g is constant on each $X(a_{i1}, \ldots, a_{iv_i})$. Take $\overline{G}$ to be any finite subgroup of G which contains -1 and the elements a_{ij}. Note if $\bar{\sigma} = \bar{\tau}$, then $\sigma(a_{ij}) = \tau(a_{ij})\ \forall\, i, j$ so σ, τ both lie in the same $X(a_{i1}, \ldots, a_{iv_i})$ i.e. $g(\sigma) = g(\tau)$. Hence $\bar{g}$ defined as in (i) is well defined. This proves (i). As for (ii), one verifies directly that $\forall\, \sigma, \tau \in X$, $\sigma(p_{\bar{\tau}}) = 2^n$, if $\bar{\sigma} = \bar{\tau}$, and $\sigma(p_{\bar{\tau}}) = 0$, if $\bar{\sigma} \neq \bar{\tau}$. (ii) is immediate from this. $\square$

(5.5) THEOREM. *Let $X = (X, G)$ be an arbitrary space of orderings, $g \in C(X, \mathbf{Z})$. Then the following are equivalent*:

(i) $g \in W(X)$;

(ii) $g|Y \in W(Y)\ \forall$ *finite fans* $Y \subseteq X$;

(iii) $\Sigma_{\sigma \in Y}\, g(\sigma) \equiv 0 \bmod |Y|\ \forall$ *finite fans* $Y \subseteq X$;

(iv) $\Sigma_{\sigma \in Y}\, g(\sigma)\sigma(a) \equiv 0 \bmod |Y|\ \forall$ *finite fans* $Y \subseteq X$, *and* $\forall\, a \in G$.

PROOF. (i) $\Rightarrow$ (ii) is a triviality.

(ii) $\Rightarrow$ (iii). For a finite fan $Y = (Y, G/\Delta)$ in X, and a form f, we wish to show $\Sigma_{\sigma \in Y}\, \sigma(f) \equiv 0 \bmod |Y|$. This reduces to the 1-dimensional case, say $f = \langle a\rangle$. In this case $\Sigma_{\sigma \in Y}\, \sigma(f) = \Sigma_{\sigma \in Y}\, \sigma(a) = 0$ of $\pm|Y|$ according as $a \not\equiv \pm 1 \pmod{\Delta}$ or

$a \equiv \pm 1 \pmod{\Delta}$. (Note since Y is a fan, by 4.4(ii), half the orderings in Y make a positive if $a \not\equiv \pm 1 \pmod{\Delta}$.)

(iii) $\Rightarrow$ (iv). Let $Y = (Y, G/\Delta)$ be a finite fan in X, $a \in G$. We wish to show $\sum_{\sigma \in Y} g(\sigma)\sigma(a) \equiv 0 \bmod |Y|$. This is clear by (iii) if $a \equiv \pm 1 \pmod{\Delta}$. Otherwise, by 4.3, $Y(a)$ is a fan, $|Y(a)| = (1/2)|Y|$ by 4.4(ii), and since

$$\sum_{\sigma \in Y} g(\sigma)\sigma(a) = \sum_{\sigma \in Y(a)} g(\sigma) - \sum_{\sigma \in Y(-a)} g(\sigma)$$
$$= 2\Bigg(\sum_{\sigma \in Y(a)} g(\sigma)\Bigg) - \sum_{\sigma \in Y} g(\sigma),$$

the result follows from (iii) applied to the fans Y and $Y(a)$.

(iv) $\Rightarrow$ (i). By 5.3, we may assume X is itself a fan. Define $\overline{G}$ as in 5.4 and choose any subgroup Δ of G such that $G = \Delta \times \overline{G}$ (direct product), and let $Y = \Delta^{\perp} \cap X$. Let $\bar{\sigma} \in \overline{X} \subseteq \chi(\overline{G})$ (notations as in 5.4). By choice of Δ, $\bar{\sigma}$ extends uniquely to a character $\tau \in \chi(G)$ which is 1 on Δ. Since $-1 \in \overline{G}$, $\tau(-1) = \sigma(-1) = -1$ it follows that $\tau \in X$ (X is a fan). Thus $\tau \in \Delta^{\perp} \cap X = Y$. Thus $\tau \to \bar{\tau}$ defines a bijection between Y and $\overline{X}$. Now let $b \in Y^{\perp}$, say $b = b_1 b_2$, $b_1 \in \overline{G}$, $b_2 \in \Delta$. Since $\Delta \subseteq Y^{\perp}$, this yields $b_1 \in Y^{\perp}$. By virtue of the bijection between Y and $\overline{X}$ this yields $\sigma(b_1) = \bar{\sigma}(b_1) = 1\ \forall\, \sigma \in X$. Thus $b_1 = 1$ by O_3, so $b = b_2 \in \Delta$. Thus $Y^{\perp} = \Delta$, so $(Y, G/\Delta)$ is a subspace of X and hence a fan, by 4.3. Also $G/\Delta \cong \overline{G}$, so by 4.4(i), $|Y| = 2^n$, with n as in 5.4. Thus the formula of 5.4(ii) can be written: $g = (1/2)^n \sum_{\sigma \in Y} g(\sigma) p_{\bar{\sigma}}$. Now expand each $p_{\bar{\sigma}}$ as $p_{\bar{\sigma}} = \sum_S \sigma(a_S)\langle a_S \rangle$, where S runs through all finite subsets of $\{1, 2, \ldots, n\}$, and $a_S = \prod_{i \in S} a_i$. Then

$$g = \left(\frac{1}{2}\right)^n \sum_{\sigma \in Y} g(\sigma)\Big(\sum_S \sigma(a_S)\langle a_S \rangle\Big) = \left(\frac{1}{2}\right)^n \sum_S \Big(\sum_{\sigma \in Y} g(\sigma)\sigma(a_S)\Big)\langle a_S \rangle.$$

This lies in $W(X)$ by assumption (iv). $\square$

(5.6) Remark. Theorem 5.5 yields the representation theorem of Becker and Bröcker [**BB**, 5.3] as a special case. To see this refer to 3.8 and 4.5. Moreover, by [**KR**, 6.7, 2.24, 2.30], it also gives a representation theorem for the reduced Witt ring of a (connected) semilocal ring A in the case $2 \in A$.

6. Stability. Let $g \in C(X, \mathbf{Z})$. By 5.4, $\exists\, n \geqslant 0$ (possibly depending on g) such that $2^n g \in W(X)$. In other words the factor group $C(X, \mathbf{Z})/W(X)$ is 2-primary torsion (also see [**KRW**, 3.18]). It is natural to consider the case in which there exists an integer $k \geqslant 0$ such that $2^k g \in W(X)$ *for all* $g \in C(X, \mathbf{Z})$. (See [**EL**], [**B1**], [**B2**], [**C2**].)

(6.1) Definition. A space of orderings X is said to be *k-stable* ($k \geqslant 0$) if $2^k C(X, \mathbf{Z}) \subseteq W(X)$. The *stability index* of X (denoted $\mathrm{st}(X)$) is defined to be the least integer $k \geqslant 0$ such that X is k-stable (or ∞ if no such integer exists).

Stability can be described in terms of the augmentation ideal $I(X)$ of $W(X)$ (see [**M1**]). Recall that $I(X)$ is the ideal of $W(X)$ consisting of the even dimensional forms. It can be viewed as the kernel of the ring homomorphism $f \to \dim f \pmod 2$ from $W(X)$ to $\mathbf{Z}/2\mathbf{Z}$ (see 2.5), so $W(X)/I(X) \cong \mathbf{Z}/2\mathbf{Z}$. Note that in $W(X)$, $\langle a, b \rangle = \langle 1, a \rangle - \langle 1, -b \rangle$ so $I(X)$ is generated additively by the 1-fold Pfister forms. Thus its kth power $I^k(X)$ is generated additively by the k-fold Pfister forms $p = \langle 1, a_1\rangle \otimes \cdots \otimes\langle 1, a_k\rangle$, $a_1, \ldots, a_k \in G$. It follows from $\sigma(p) = 2^k$ or 0 (for any such p and any $\sigma \in X$) that

$$I^k(X) \subseteq 2^k C(X, \mathbf{Z}). \tag{4}$$

Since $\langle 1, -a\rangle \oplus \langle 1, -b\rangle \sim \langle 1, -ab\rangle \oplus \langle 1, -a\rangle\langle 1, -b\rangle$, the map $a \to \langle 1, -a\rangle + I^2(X)$ defines a group homomorphism of G into $I(X)/I^2(X)$. Since $I(X)$ is generated by 1-fold Pfister forms, this map is surjective. It is also injective, since by (4),

$$\langle 1, -a\rangle \in I^2(X) \Rightarrow \sigma(a) \equiv 1 \pmod 4 \Rightarrow \sigma(a) = 1 \qquad (\forall\, \sigma \in X).$$

Thus $I(X)/I^2(X) \cong G$ (canonically). The induced map from $I(X)$ to G is the signed discriminant of 2.5 (compare to [L, p. 41, 2.3]).

(6.2) THEOREM. *For $k > 0$ the following are equivalent.*

(i) *For each anisotropic $(k + 1)$-fold Pfister form p, $\exists$ a k-fold Pfister form q such that $p \cong 2 \times q$;*

(ii) $I^{k+1}(X) = 2I^k(X)$;

(iii) $2^kC(X, \mathbf{Z}) = I^k(X)$;

(iv) $\mathrm{st}(X) \leqslant k$.

PROOF. Since $I^n(X)$ is generated by (anisotropic) n-fold Pfister forms, it is clear that (i) $\Rightarrow$ (ii). Now let $g \in C(X, \mathbf{Z})$. By 5.4, $\exists\, n > 0$ such that $2^ng \in I^n(X)$. If $n \leqslant k$, this yields $2^kg \in 2^{k-n}I^n(X) \subseteq I^k(X)$. If $n > k$, then by (ii), $I^n(X) = 2^{n-k}I^k(X)$, so $2^ng \in 2^{n-k}I^k(X)$. Cancelling yields $2^kg \in I^k(X)$. This proves (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (iv) is trivial. The only nontrivial implication is the implication (iv) $\Rightarrow$ (i). This requires a lemma.

(6.3) LEMMA. (*Compare to* [S, 2.44].) *Let p be an n-fold Pfister form, $n \geqslant 1$. Suppose $x \in D(p')$ where p' is the form defined by $p \cong \langle 1\rangle \oplus p'$. Then $\exists$ an $(n - 1)$-fold Pfister form q such that $p \cong \langle 1, x\rangle q$.*

First, let us assume this lemma to complete the proof of 6.2. Let p be an anisotropic $(k + 1)$-fold Pfister form. Then by (iv) and equation (4), $(1/2)p \in 2^kC(X, \mathbf{Z}) \subseteq W(X)$. Thus $\exists$ a form f such that $p \sim 2 \times f$. We may assume f is anisotropic. Thus $2 \times f$ is anisotropic by 3.5(ii), so by 2.4, $p \cong 2 \times f$. Now $1 \in D(p) = D(2 \times f) = D(f)$ by 3.5(i), so $f \cong \langle 1\rangle \oplus f'$. Thus $p \cong \langle 1, 1\rangle \oplus 2 \times f'$, so $p' \cong \langle 1\rangle \oplus 2 \times f'$, i.e. $1 \in D(p')$. By 6.3, $\exists$ a k-fold Pfister form q such that $p \cong \langle 1, 1\rangle q \cong 2 \times q$ (and thus $f \cong q$). $\square$

PROOF OF 6.3. Since p is a Pfister form, p factors as $p \cong \langle 1, a\rangle q \cong q \oplus aq$ with $a \in G$, and q an $(n - 1)$-fold Pfister form. Thus $p' \cong q' \oplus aq$, so by O_4, $\exists\, y \in D(q')$, $z \in D(q)$ such that $x \in D\langle y, az\rangle$. Since q is a Pfister form, $z \in D(q)$ it follows that $zq \cong q$ (by 3.1), so $p \cong q \oplus aq \cong q \oplus azq \cong \langle 1, az\rangle q$. Thus, we may assume $z = 1$ (replace a by az). Also, by induction on n, $q \cong \langle 1, y\rangle \otimes \cdots$. Thus, $p \cong \langle 1, a\rangle \otimes \langle 1, y\rangle \otimes \cdots \cong \langle 1, a, y, ay\rangle \otimes \cdots$. But $x \in D\langle a, y\rangle$, i.e. $\langle a, y\rangle \cong \langle x, xay\rangle$ (using 1.1), so this yields

$$p \cong \langle 1, x, xay, ay\rangle \otimes \cdots \cong \langle 1, x\rangle \otimes \langle 1, ay\rangle \otimes \cdots. \quad \square$$

Using Theorem 5.5 we get the following nice characterization of the stability index in terms of fans.

(6.4) THEOREM. *Let X be an arbitrary space of orderings. Then* $\mathrm{st}(X) = \sup\{k \mid \exists$ *a finite fan* $Y \subseteq X$ *with* $|Y| = 2^k\}$.

PROOF. Let $Y \subseteq X$ be a fan $|Y| = 2^k < \infty$. Fix $\tau \in Y$ and let $g\colon Y \to \mathbf{Z}$ denote the characteristic function of $\{\tau\}$. Because Y is finite, g is continuous. Then $\sum_{\sigma \in Y} 2^l g(\sigma) = 2^l$, so by 5.5,

$$2^l g \in W(Y) \Rightarrow \sum_{\sigma \in Y} 2^l g(\sigma) \equiv 0 \mod |Y| \Rightarrow l \geqslant k.$$

Thus $\mathrm{st}(Y) \geqslant k$. On the other hand $\mathrm{st}(X) \geqslant \mathrm{st}(Y)$ (for example, use 6.2). Thus $\mathrm{st}(X) \geqslant k$. Now let $g \in C(X, \mathbf{Z})$ be arbitrary, and suppose $\exists\, k < \infty$ such that $|Y| \leqslant 2^k$ $\forall$ finite fans $Y \subseteq X$. Then clearly, for all such fans, $\sum_{\sigma \in Y} 2^k g(\sigma) \equiv 0$ (mod 2^k) and hence (mod $|Y|$). By Theorem 5.5, this implies $2^k g \in W(X)$. This completes the proof.

(6.5) REMARK. In the special case that X is the space of orderings of a field F, $\mathrm{st}(X) = \mathrm{st}_{red}(F)$ in the terminology of [**B2**], and 6.4 is just the stability formula in [**B2**, 2.11].

7. Representation of G. Each $a \in G$ gives rise to a function $\hat{a}\colon X \to \pm 1$ by $\hat{a}(\sigma) = \sigma(a)$ $\forall\, \sigma \in X$. The map $a \to \hat{a}$ defines an injection $G \to C(X, \pm 1)$. As a corollary to the representation theorem for $W(X)$ we have a representation theorem for G. First we need a lemma.

(7.1) LEMMA. *Suppose* $g = \langle a_1, \ldots, a_n \rangle$, *and* $\sigma(g) = \pm 1$ $\forall\, \sigma \in X$. *Then* $g \sim \langle a \rangle$ *where a is the signed discriminant of g.*

PROOF. Recall $a = (-1)^{(1/2)(n)(n-1)} d(g)$, $d(g) = a_1 a_2 \cdots a_n$. Let $\sigma \in X$. By the proof of 2.3, $n \equiv \sigma(g)$ (mod 2). Since $\sigma(g) = \pm 1$, this implies n is odd, say $n = 2k + 1$. Thus $(-1)^{(1/2)n(n-1)} = (-1)^{(2k+1)k} = (-1)^k$. Note also (since $\sigma(g) = \pm 1$) there are either k or $k + 1$ values of i such that $\sigma(a_i) = -1$, depending on whether $\sigma(g) = 1$ or -1. Thus $\sigma(d(g)) = (-1)^k$ or $(-1)^{k+1}$ depending on whether $\sigma(g) = 1$ or -1. It follows that $\sigma(g) = \sigma(a)$ $\forall\, \sigma \in X$, so $g \sim \langle a \rangle$ by 2.3.

(7.2) THEOREM. *Let* $f \in C(X, \pm 1)$. *Then* $f \in G$ *if and only if* $\sum_{\sigma \in W} f(\sigma) \equiv 0$ (mod 4) *holds for all 4-element fans* $W \subseteq X$.

PROOF. Suppose $\sum_{\sigma \in W} f(\sigma) \equiv 0$ (mod 4) holds for all 4-element fans $W \subseteq X$. It is enough to show f is represented by a form $g = \langle a_1, \ldots, a_n \rangle$ over X. For then, by 7.1, $f = \hat{a}$ where a is the signed discriminant of g. By 5.5, if f is not so represented, $\exists$ a finite fan $V \subseteq X$ such that

$$\sum_{\sigma \in V} f(\sigma) \not\equiv 0 \mod |V|. \tag{6}$$

Take a minimal such V. Clearly $|V| \neq 1, 2$, and by hypothesis $|V| \neq 4$, so $|V| = 2^n \geqslant 8$. To simplify notation, assume $V = X$. Thus, by 4.4(i), $|G| = 2^{n+1}$. Pick $b_1 \in G$, $b_1 \neq \pm 1$ and let $b_2 = -b_1$. By minimality of V, $\sum_{\sigma \in V(b_i)} f(\sigma) \equiv 0$

mod 2^{n-1}, $i = 1, 2$ (see 4.4(ii)). Moreover since $f(\sigma) = \pm 1$, $\Sigma_{\sigma \in V(b_i)} f(\sigma) = 0$, 2^{n-1} or -2^{n-1}. Note also $\Sigma_{\sigma \in V} f(\sigma) = \Sigma_{\sigma \in V(b_1)} f(\sigma) + \Sigma_{\sigma \in V(b_2)} f(\sigma)$ (since $b_2 = -b_1$). Thus, for (6) to hold, one of these sums must be zero and the other $\pm 2^{n-1}$. Replacing f by $-f$ and (or) interchanging the roles of b_1 and b_2 if necessary, we may assume $\Sigma_{\sigma \in V(b_1)} f(\sigma) = 0$, $\Sigma_{\sigma \in V(b_2)} f(\sigma) = 2^{n-1}$. Pick $\sigma_1, \sigma_2 \in V(b_1)$ such that $f(\sigma_1) = 1$, $f(\sigma_2) = -1$. Pick $\sigma_3 \in V(b_2)$. Then $\sigma_4 = \sigma_1\sigma_2\sigma_3 \in V(b_2)$ since $\sigma_4(b_2) = (-1)(-1)(1) = 1$. (Note $\sigma_4 \in V$ since $\sigma_4(-1) = -1$.) Thus $f(\sigma_i) = 1$, $i = 3, 4$, so $\Sigma_{i=1}^4 f(\sigma_i) = 2 \not\equiv 0 \pmod 4$. We will have a contradiction if we show $W = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is a 4-element fan. Note, by choice, $\sigma_1, \sigma_2, \sigma_3$ are distinct. It follows they are independent. (E.g. if σ_3 is a linear combination of σ_1 and σ_2, then σ_3 is one of $\sigma_1, \sigma_2, 1, \sigma_1\sigma_2$. The first two possibilities have been excluded. The last two are also excluded, since $\sigma_3(-1) \neq 1$.) In particular, $\sigma_4 = \sigma_1\sigma_2\sigma_3$ is distinct from $\sigma_1, \sigma_2, \sigma_3$, so $|W| = 4$. Also $W^\perp = \bigcap_{i=1}^3 \ker(\sigma_i)$ has index $2^3 = 8$ in G, and $W^{\perp\perp} = \chi(G/W^\perp)$ is generated by $\sigma_1, \sigma_2, \sigma_3$. Let $\sigma \in W^{\perp\perp} \cap V$. Since $\sigma \in W^{\perp\perp}$,

$$\sigma = \sigma_1^{e_1}\sigma_2^{e_2}\sigma_3^{e_3}, \qquad e_i = 0 \quad \text{or} \quad 1.$$

Since $\sigma \in V$, $\sigma(-1) = -1$, so $\Sigma\, e_i \equiv 1 \pmod 2$, i.e. $\sigma \in W$. This proves $W = W^{\perp\perp} \cap V$ is a subspace. It is a fan by 4.3. $\square$

If $\sigma, \tau \in X$, then one verifies easily that $\{\sigma, \tau\}$ is a fan in X. Such a fan is said to be *trivial* in the sense that it contains at most two elements (we allow $\sigma = \tau$). The *subspace generated* by a set $S \subseteq X$ is defined to be $Y = S^{\perp\perp} \cap X$. (Since $S \subseteq Y \subseteq S^{\perp\perp}$, we have $S^\perp \supseteq Y^\perp \supseteq (S^{\perp\perp})^\perp = (S^\perp)^{\perp\perp} \supseteq S^\perp$. Thus $Y^\perp = S^\perp$, so $Y = (Y, G/S^\perp)$ is indeed a subspace of X.)

(7.3) Corollary. *Let $X' = (X', G')$ denote the subspace of X generated by the nontrivial fans (thus $X' = \varnothing$ if $\mathrm{st}(X) \leqslant 1$). Then $G = \{a \in C(X, \pm 1) \mid a|X' \in G'\}$.*

Proof. This is clear from 7.2, since X' contains all the 4-element fans of X. $\square$

In case every fan in X is trivial 5.5, 6.4, and 7.2 have a very simple form. In fact we have

(7.4) Theorem. *The following are equivalent.*

(i) *All fans in X are trivial*;

(ii) $W(X) = \{f \in C(X, \mathbf{Z}) \mid f(\sigma) \equiv f(\tau) \pmod 2\ \forall\, \sigma, \tau \in X\}$;

(iii) $G = C(X, \pm 1)$;

(iv) $\mathrm{st}(X) \leqslant 1$.

The equivalence of (ii), (iii), and (iv), and the fact that they imply (i) appears to be fairly well known (e.g. see [**KRW**, 3.20]). The fact that (i) implies the other three was known in the field theoretic situation (e.g. it follows from [**B2**, 2.11]) but not in the generality given here. In any case, the proof is now elementary.

Proof. (i) $\Rightarrow$ (ii). By (i), all fans in X have the form $\{\sigma, \tau\}$. Thus (ii) follows from 5.5.

(ii) $\Rightarrow$ (iii). Let $f \in C(X, \pm 1)$. By (ii) $\exists$ a form g representing f. Take $a \in G$ as in 7.1. Thus $g \sim \langle a \rangle$, i.e. $f = \hat{a}$.

(iii) $\Rightarrow$ (iv). Let p be a 2-fold Pfister form. Since $\sigma(p) = 0$ or $4\ \forall\ \sigma \in X$, it follows that $2f + 2 = p$ defines $f \in C(X, \pm 1)$. By (iii), $\exists\ c \in G$ such that $f = \hat{c}$. Thus $p \cong 2 \times \langle 1, c\rangle$. Thus $\mathrm{st}(X) \leqslant 1$, by 6.2.

(iv) $\Rightarrow$ (i). By 6.4.

Recall [M3, 2.6] we say X is the *direct sum* of the subspaces $X_1, \ldots, X_n$ (and write $X = X_1 \oplus \cdots \oplus X_n$) if $X = \bigcup_{i=1}^n X_i$, and the product $[X] = \prod_{i=1}^n [X_i]$ is direct. Recall, by the notations of [M3], that if $S \subseteq \chi(G)$, then $[S] := S^{\perp\perp}$. $[S]$ is clearly a closed subgroup of $\chi(G)$. As remarked in [M3], $[S]$ can also be characterized as the smallest closed subgroup of $\chi(G)$ containing S. This follows from Pontryagin duality, e.g. see [R, 2.1.3, 1.7.2]. Note, in particular, if $X_i = (X_i, G/\Delta_i)$, then $[X_i] = X_i^{\perp\perp} = \Delta_i^{\perp}$, and $[X] = X^{\perp\perp} = 1^{\perp} = \chi(G)$.

(7.5) Corollary. (i) *Suppose $X = X_1 \oplus \cdots \oplus X_n$. Then every nontrivial fan of X lies in some X_i, $i \leqslant n$.*

(ii) *Conversely, suppose $X_1, \ldots, X_n$ are disjoint closed subsets of X, $X = \bigcup_{i=1}^n X_i$, and each 4-element fan $V \subseteq X$ lies in some X_i. Then $X_1, \ldots, X_n$ are subspaces of X, and $X = X_1 \oplus \cdots \oplus X_n$.*

Proof. (i) By assumption, $[X] = \prod_{i=1}^n [X_i]$ (direct product). Let $V \subseteq X$ be a nontrivial fan, let $\sigma_1, \sigma_2, \sigma_3 \in V$ be any distinct elements, and let $\sigma_4 = \sigma_1\sigma_2\sigma_3$. Then $\sigma_4(-1) = -1$, so $\sigma_4 \in V$ and $\sigma_1\sigma_2\sigma_3\sigma_4 = 1$. If we view this as a relation in the direct product, we have a contradiction unless all σ_i, $i < 4$, lie in the same subspace X_j. Since $\sigma_1, \sigma_2 \in V$ are arbitrary distinct elements, this proves (i).

(ii) For each $i \leqslant n$ let $g_i\colon X \to \pm 1$ be defined by $g_i(\sigma) = -1$ if $\sigma \in X_i$, $g_i(\sigma) = 1$ if $\sigma \in X_j, j \neq i$. One verifies $\sum_{\sigma \in V} g_i(\sigma) = \pm 4$ for all 4-element fans $V \subseteq X$. Note g_i is continuous by the hypothesis. Thus by 7.2, $\exists\ a_i \in G$ representing g_i. It is clear from the definition of a_i that $X_i = X(-a_i)$. Thus X_i is indeed a subspace of X, and $X_i^{\perp} = D\langle 1, -a_i\rangle$. Now let $b_1, \ldots, b_n \in G$ be given, and let $g\colon X \to \{\pm 1\}$ be defined by $g(\sigma) = \sigma(b_i)\ \forall\ \sigma \in X_i\ \forall\ i < n$. Again g is continuous and $\sum_{\sigma \in V} g(\sigma) \equiv 0 \pmod 4$ $\forall$ 4-elements fans $V \subseteq X$ (since each such V lies in some X_i). Thus, by 7.2, $\exists\ b \in G$ such that $\hat{b} = g$, i.e. $\sigma(b) = \sigma(b_i)\ \forall\ \sigma \in X_i\ \ \forall\ i < n$. Thus $b \equiv b_i$ $(\mathrm{mod}\ D\langle 1, -a_i\rangle)$, $i \leqslant n$. Thus, the canonical homomorphism

$$G \to \prod_{i=1}^{n} G/D\langle 1, -a_i\rangle$$

is surjective. Taking duals, the natural map from the direct product $\prod_{i=1}^n [X_i]$ to $[X]$ is one-to-one. Since it is clearly onto ($X = \bigcup_{i=1}^n X_i$), this completes the proof. □

(7.6) Remark. The above corollary makes Theorem 3.3 of [M2] a triviality. It also proves Remark 2.12 of [M2]. One can employ this corollary where, in the corresponding "field case", one would use some approximation theorem from valuation theory.

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