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THE FIXED POINT PROPERTY AND UNBOUNDED SETS IN HILBERT SPACE

BY

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ABSTRACT. It is shown that a closed convex subset K of a real Hilbert space H has the fixed point property for nonexpansive mappings if and only if K is bounded.

1. Introduction. A closed convex subset K of a Hilbert space H is said to have the *fixed point property* for nonexpansive mappings if, whenever $T\colon K \to K$ satisfies $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in K$, then T has a fixed point in K. While it is well known that closed, convex and *bounded* subsets of Hilbert space must have the fixed point property (see, for example, **[1]**, **[3]** or **[4]**), whether or not boundedness of K is a *necessary* condition (a question initially raised by W. A. Kirk) has remained an open problem for several years. It is our purpose in this paper to resolve this question in Hilbert space with the following theorem.

THEOREM 1. *Let K be a closed and convex subset of a real Hilbert space H. Then K has the fixed point property for nonexpansive mappings if and only if K is bounded.*

Theorem 1 is remarkable in that broad classes of *unbounded* convex sets K in both Hilbert space and the spaces l_p $(1 < p < \infty)$ have been shown to possess the *almost fixed point property*, i.e., $\inf\{\|x - Tx\|\colon x \in K\} = 0$ whenever $T\colon K \to K$ and is nonexpansive. In particular, K. Goebel and T. Kuczumow show in **[2]** that if $K \subseteq l_2$ is *block*, i.e., a set of the form $K = \{x \in l_2\colon |\langle x, e_i\rangle| \leqslant M_i\}$ where $\{e_i\}$ is an orthonormal set in l_2, then K has the almost fixed point property. This result was subsequently extended by the author in **[5]** to include all linearly bounded subsets of l_p $(1 < p < \infty)$. (A subset K of a normed space E is *linearly bounded* if K has bounded intersection with all lines in E.) Since it is known that the almost fixed point property implies the fixed point property for bounded convex sets in these spaces (see, for example, **[3]**), Theorem 1 together with the results of **[2]** and **[5]** constitute a rather surprising concatenation. Moreover, since it is clear that if $K \subseteq l_2$ is *not* linearly bounded, then K does not have the fixed point property, Theorem 1 shows that the results of **[5]**, as they apply to Hilbert space, are sharp.

While blocks possess the *almost* fixed point property, a relatively easy example shows that blocks need *not* possess the fixed point property. Take $K = \{x \in l_2\colon |\langle x, e_i\rangle| \leqslant 1\}$, and, for $x = \sum_1^\infty \langle x, e_i\rangle e_i \in K$, define $T(x) = e_1 + \sum_1^\infty \langle x, e_i\rangle e_{i+1}$.

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Then T is clearly a fixed-point free nonexpansive mapping of K into K. Our proof of Theorem 1 is an extension of the above simple example, and makes use of the following theorem.

THEOREM 2. *Let K be a closed, convex, linearly bounded and unbounded subset of a Hilbert space H. If $0 \in K$, then there is an orthonormal set $\{e_n\} \subseteq H$ such that, for each n, $\sum_{j=1}^{n} e_j \in K$.*

The proof of Theorem 2, which is motivated by the classical Gram-Schmidt procedure, is based on a sequence of geometric lemmas to which we now turn.

2. Geometric lemmas. We begin with some definitions which will facilitate the statements and proofs of the lemmas. Throughout H denotes a (real) Hilbert space.

(2.1) A subset $K \subseteq H$ will be called *admissible* if K is nonempty, unbounded, closed, convex and linearly bounded.

(2.2) For a subset $K \subseteq H, y \in H$ and $\mu \in (0, 1)$, set

$$C(y, \mu, K) = \{w \in K: \mu\|y\|^2 < \langle w, y\rangle < \|y\|^2\}$$

and

$$E(y, \mu, K) = \{w \in K: \mu\|y\|^2 = \langle w, y\rangle\}.$$

LEMMA 1. *Let K be an admissible subset of H and let $y \in K$. If y is not the (unique) element of K of minimal norm, then $C(y, \mu, K)$ is admissible for each $\mu \in (0, 1)$.*

PROOF. It is clear that $C(y, \mu, K)$ is closed, convex, linearly bounded and nonempty; therefore it suffices to show $C(y, \mu, K)$ is unbounded.

Choose $\{x_n\} \subseteq K$ such that $\|x_n\| \to \infty$ and fix $R > 0$. Choose $\{\lambda_n\} \subseteq R^+$ so that $\|\lambda_n x_n + (1 - \lambda_n)y\| = R$. Since $\|x_n - y\| \to \infty$, it follows that $\lambda_n \to 0$, and hence, for large n, that $\lambda_n x_n + (1 - \lambda_n)y \in K$. Now $\{\lambda_n x_n + (1 - \lambda_n)y\}$ is a bounded sequence in H and thus must have a weakly convergent subsequence; reindexing if necessary, we may thus suppose that $\lambda_n x_n + (1 - \lambda_n)y \rightharpoonup w_0$. Note that since K is weakly closed and $\lambda_n x_n + (1 - \lambda_n)y \in K$ for large n, it follows that $w_0 \in K$. We next show that $w_0 = y$.

To see this, fix $\mu > 0$ and consider $\mu w_0 + (1 - \mu)y$. Choose N so large that if $n > N$ then $\mu\lambda_n < 1$. Since $\lambda_n x_n + (1 - \lambda_n)y \rightharpoonup w_0$, we see

$$w_0 \in \overline{\mathrm{CO}}\{\lambda_n x_n + (1 - \lambda_n)y: n > N\},$$

and thus

$$\begin{aligned}\mu w_0 + (1 - \mu)y \in \overline{\mathrm{CO}}&\{\mu(\lambda_n(x_n - y) + y) + (1 - \mu)y: n > N\}\\ = \overline{\mathrm{CO}}&\{\mu\lambda_n(x_n - y) + y: n > N\}\\ = \overline{\mathrm{CO}}&\{\mu\lambda_n x_n + (1 - \mu\lambda_n)y: n > N\} \subseteq K\end{aligned}$$

since $\mu\lambda_n < 1$. This shows that, if $w_0 \neq y$, then the ray from y through w_0 is contained in K, contradicting the requirement that K be linearly bounded. Consequently, $\lambda_n x_n + (1 - \lambda_n)y \rightharpoonup y$.

Thus we see that for each $R > 0$ and each $\varepsilon > 0$ we may choose $w \in K$ so that $\|w\| = R$ and $\|y\|^2 - \varepsilon < \langle w, y\rangle < \|y\|^2 + \varepsilon$. The proof will be completed if we can show that, for each $R > 0$, $\varepsilon > 0$, there is a $w \in K$ with $\|w\| > R$ and $\|y\|^2 - \varepsilon < \langle w, y\rangle \leqslant \|y\|^2$.

Let u be the element of K of minimal norm, and choose a sequence $\{\varepsilon_n\} \subseteq (0, \frac{1}{2}(\|y\|^2 - \|u\|\,\|y\|))$ so that $\varepsilon_n \to 0$. Set

$$\mu_n = \frac{\varepsilon_n}{\|y\|^2 - \|y\|\,\|u\| + \varepsilon_n};$$

note that $\mu_n \in (0, 1)$ for each n and that $\mu_n \to 0$. Now choose $\{w_n\} \subseteq K$ such that $\|w_n\| = R' > R$ and

$$\|y\|^2 - \varepsilon_n < \langle w_n, y\rangle < \|y\|^2 + \varepsilon_n,$$

and set $z_n = (1 - \mu_n)w_n + \mu_n u$. Then $z_n \in K$, $\|z_n\| \to R' > R$ and moreover

$$\begin{aligned}\langle z_n, y\rangle &= \langle (1 - \mu_n)w_n, y\rangle + \mu_n\langle u, y\rangle \\ &\leqslant (1 - \mu_n)\big(\|y\|^2 + \varepsilon_n\big) + \mu_n\|u\|\,\|y\| \\ &= \|y\|^2 - \mu_n\big(\|y\|^2 - \|u\|\,\|y\| + \varepsilon_n\big) + \varepsilon_n = \|y\|^2,\end{aligned}$$

i.e., $\langle z_n, y\rangle \leqslant \|y\|^2$. Clearly $\langle z_n, y\rangle \to \|y\|^2$. Then given $R > 0$, $\varepsilon > 0$, we can choose n sufficiently large that $\|z_n\| > R$ and $\|y\|^2 - \varepsilon < \langle z_n, y\rangle \leqslant \|y\|^2$. This completes the proof of Lemma 1.

LEMMA 2. *Let K be an admissible subset of a Hilbert space H, let $\mu \in (0, 1)$ and let u be the element of K of smallest norm. If $y \in K$ and if $\|y\| > \mu^{-1}\|u\|$, then the set $E(y, \mu, K)$ is admissible.*

REMARK. Since K is unbounded, we can always choose $y \in K$ so that $\|y\| > \mu^{-1}\|u\|$. Then Lemma 2 in particular asserts that, given μ and K, we can always find y so that $E(y, \mu, K) \neq \varnothing$.

PROOF. Set $\alpha = \langle u, y\rangle$, and observe that our hypotheses imply $\mu\|y\|^2 - \alpha > 0$. Fix $M > 0$. By Lemma 1 we may choose $w \in K$ so that

$$\|w\| > \left[\frac{\|y\|^2 - \alpha}{\mu\|y\|^2 - \alpha}\right](M + \|u\|)$$

and so that

$$\|y\|^2 > \langle w, y\rangle \equiv \gamma\|y\|^2 > \mu\|y\|^2.$$

Set $\lambda = (\mu\|y\|^2 - \alpha)(\gamma\|y\|^2 - \alpha)^{-1}$. Since $\gamma > \mu$, we see that $\lambda < 1$. Since $\gamma < 1$, it follows that

$$\lambda > \big(\mu\|y\|^2 - \alpha\big)/\big(\|y\|^2 - \alpha\big) > 0. \tag{2.3}$$

Now set $v = \lambda w + (1 - \lambda)u$. Since $\lambda \in (0, 1]$, $v \in K$. We will show that $v \in E(y, \mu, K)$ and $\|v\| > M$. To this end, we compute $\langle v, y\rangle$.

$$\begin{aligned}\langle v,y\rangle &= \langle \lambda w + (1-\lambda)u, y\rangle = \lambda\langle w,y\rangle + (1-\lambda)\langle u,y\rangle \\ &= \lambda\gamma\|y\|^2 + (1-\lambda)\alpha = \lambda\gamma\|y\|^2 + \frac{(\gamma-\mu)\|y\|^2\alpha}{\gamma\|y\|^2-\alpha} \\ &= \frac{\gamma\mu\|y\|^2 - \gamma\alpha + (\gamma-\mu)\alpha}{\gamma\|y\|^2-\alpha}\|y\|^2 = \mu\|y\|^2,\end{aligned}$$

i.e., $\langle v, y\rangle = \mu\|y\|^2$.

Finally, using (2.3) and our choice of $\|w\|$,

$$\begin{aligned}\|v\| &= \|\lambda w + (1-\lambda)u\| \geqslant \lambda\|w\| - (1-\lambda)\|u\| \\ &\geqslant \frac{\mu\|y\|^2-\alpha}{\|y\|^2-\alpha}\|w\| - \|u\| \geqslant M + \|u\| - \|u\| = M.\end{aligned}$$

Thus $E(y, \mu, K)$ contains elements of arbitrarily large norm. Since $E(y, \mu, K)$ is clearly closed convex and linearly bounded, this completes the proof of Lemma 2.

LEMMA 3. *Suppose K is a closed convex set, $0 \in K$, and $\{y_n\}$ is a sequence in H such that $\sum_{j=1}^n y_j \in K$ for each n. If $\{\lambda_n\}$ is any nonincreasing sequence of nonnegative numbers bounded above by* 1, *then $\sum_{j=1}^n \lambda_j y_j \in K$ for each n.*

PROOF. The result holds trivially for $\{y_1\}$ and any scalar $\{\lambda_1\} \subseteq [0, 1]$. We now suppose the result holds for $\{y_1, \ldots, y_{m-1}\}$ and any nonincreasing sequence $\{\mu_1, \ldots, \mu_{m-1}\}$ bounded above by 1 and proceed by induction. Let $\{\lambda_1, \ldots, \lambda_m\}$ be any nonincreasing sequence bounded above by 1. We may suppose without loss of generality that $\lambda_1 > \lambda_m$ and compute

$$\sum_1^m \lambda_j y_j = \lambda_1 \sum_1^m \frac{\lambda_j}{\lambda_1} y_j = \lambda_1 \left\{ \frac{\lambda_1 - \lambda_m}{\lambda_1} \sum_{j=1}^{m-1} \left(\frac{\lambda_j - \lambda_m}{\lambda_1 - \lambda_m} y_j \right) + \frac{\lambda_m}{\lambda_1} \sum_{j=1}^m y_j \right\}.$$

Now since $\lambda_j \geqslant \lambda_{j+1}$, it follows that

$$(\lambda_j - \lambda_m)/(\lambda_1 - \lambda_m) \geqslant (\lambda_{j+1} - \lambda_m)/(\lambda_1 - \lambda_m),$$

and thus the inductive hypothesis implies that

$$x = \sum_{j=1}^{m-1} \left(\frac{\lambda_j - \lambda_m}{\lambda_1 - \lambda_m} y_j \right) \in K; \qquad x^1 = \sum_{j=1}^m y_j \in K$$

by supposition. Hence

$$\lambda_1\left[(1 - \lambda_m/\lambda_1)x + \lambda_m x^1/\lambda_1\right] \in K,$$

and the inductive step is verified.

REMARK. Lemma 3 holds in an arbitrary linear space.

COROLLARY 1. *Suppose K is a closed convex set, $0 \in K$, and $\{y_n\}$ is an orthonormal sequence in H such that $\sum_{j=1}^n y_j \in K$ for each n. If*

$$K_0 = \Big\{ w \in \overline{\mathrm{sp}}(\{y_n\}) \colon \{\langle w, y_n\rangle\} \text{ is a nonincreasing sequence of nonnegative numbers bounded above by } 1 \Big\},$$

then K_0 is a closed convex subset of K.

3. Proofs of the theorems.

PROOF OF THEOREM 2. Fix $\mu \in (0, 1)$. Choose $y_1 \in K$ such that $\|y_1\| \geq \mu^{-1}$, set $K_1 = E(y_1, \mu, K)$ and let u_1 be the (unique) element of K_1 of minimal norm. Since K_1 is admissible, we may choose $w_1 \in K_1$ so that $\|w_1\| > \mu^{-1}\|u_1\|$ and so that $\|w_1\| \geq (\mu + 1)\|y_1\|$. Then setting

$$y_2 = w_1 - \langle w_1, y_1\rangle y_1/\|y_1\|^2,$$

we see that $\langle y_2, y_1\rangle = 0$, $\|y_2\| \geq \|y_1\|$ and $w_1 = \mu y_1 + y_2$. Moreover, if we set

$$\lambda_2 = \left(\mu\|y_2\|^2 + \mu^2\|y_1\|^2\right)/\left(\|y_2\|^2 + \mu^2\|y_1\|^2\right),$$

then $K_2 = E(w_1, \lambda_2, K_1)$ is admissible and also, if $w \in K_2$, then $\langle w, y_1\rangle = \mu\|y_1\|^2$ and $\langle w, y_2\rangle = \mu\|y_2\|^2$. To see these assertions, first recall that Lemma 2 implies $E(w_1, \lambda_2, K_1)$ is admissible if $\|w_1\| > \lambda_2^{-1}\|u_1\|$. But

$$\lambda_2^{-1}\|u_1\| = \frac{\|u_1\|}{\mu}\left(\frac{\|y_2\|^2 + \mu^2\|y_1\|^2}{\|y_2\|^2 + \mu\|y_1\|^2}\right) < \|u_1\|/\mu < \|w_1\|$$

by construction. Since $w \in K_1$, $\langle w, y_1\rangle = \mu\|y_1\|^2$. Finally,

$$\begin{aligned}\langle w, y_2\rangle + \mu^2\|y_1\|^2 &= \langle w, y_2\rangle + \langle w, \mu y_1\rangle = \langle w, w_1\rangle \\ &= \lambda_2\|w_1\|^2 = \lambda_2\left(\|y_2\|^2 + \mu^2\|y_1\|^2\right) \\ &= \mu\|y_2\|^2 + \mu^2\|y_1\|^2,\end{aligned}$$

and hence $\langle w, y_2\rangle = \mu\|y_2\|^2$.

We now proceed by induction; suppose we have defined the sequence $\{y_1, \ldots, y_n\}$. For $j = 1, \ldots, n$ set

$$\lambda_j = \frac{\mu\|y_j\|^2 + \mu^2\sum_{m=1}^{j-1}\|y_m\|^2}{\|y_j\|^2 + \mu^2\sum_{m=1}^{j-1}\|y_m\|^2}$$

and for $j = 1, \ldots, n - 1$ set

$$w_j = y_{j+1} + \mu\sum_{m=1}^{j} y_m.$$

Suppose in addition our sequence satisfies the following conditions.

(i) $\|y_1\| \leq \|y_2\| \leq \cdots \leq \|y_n\|$.
(ii) $\langle y_i, y_j\rangle = 0$ $(j \neq i)$.
(iii) $K_j = E(w_{j-1}, \lambda_j, K_{j-1})$ is admissible and has element of minimal norm u_j.
(iv) $w_j \in K_j$ and $\|w_j\| > \mu^{-1}\|u_j\|$, and
(v) if $w \in K_i$ $(i = 1, \ldots, n)$ and if $j = 1, \ldots, i$, then $\langle w, y_j\rangle = \mu\|y_j\|^2$.

The paragraphs at the beginning of this proof provide the basis for our induction. We will show that we can define y_{n+1} so that, with λ_{n+1}, w_n and K_{n+1} defined as above, properties (i)–(v) hold.

By the inductive hypothesis, the set K_n is unbounded, and hence we may choose $w_n \in K_n$ with norm so large that $\|w_n\| > \mu^{-1}\|u_n\|$ and so large that if

$$y_{n+1} = w_n - \sum_{j=1}^{n}\frac{\langle w_n, y_j\rangle}{\|y_j\|^2}y_j$$

then $\|y_{n+1}\| > \|y_n\|$ (and thus (i) is fulfilled). By (v), $\langle w_n, y_j\rangle = \mu\|y_j\|^2$ ($j = 1, \ldots, n$), and thus our choice of w_n and y_{n+1} guarantees that $w_n = y_{n+1} + \mu\sum_{j=1}^n y_j$, as desired.

By (ii), $\langle y_{n+1}, y_j\rangle = 0$ for $j = 1, \ldots, n$, and thus (ii) holds for $\{y_1, \ldots, y_{n+1}\}$.

By Lemma 2, the set $K_{n+1} = E(w_n, \lambda_{n+1}, K_n)$ will be admissible if $\|w_n\| > \lambda_{n+1}^{-1}\|u_n\|$. But

$$\lambda_{n+1}^{-1}\|u_n\| = \frac{\|u_n\|}{\mu}\left(\frac{\|y_{n+1}\|^2 + \mu^2\sum_{m=1}^n \|y_m\|^2}{\|y_{n+1}\|^2 + \mu\sum_{m=1}^n \|y_m\|^2}\right) < \|u_n\|/\mu < \|w_n\|$$

by construction. Thus (iii) is fulfilled; our construction guarantees that (iv) holds.

Finally observe that if $w \in K_{n+1} \subseteq K_n$, then (v) implies that $\langle w, y_j\rangle = \mu\|y_j\|^2$ ($j = 1, \ldots, n$); consequently, to verify (v) it suffices to show that $\langle w, y_{n+1}\rangle = \mu\|y_{n+1}\|^2$. Using the fact that $\langle w, y_j\rangle = \mu\|y_j\|^2$, we see that

$$\begin{aligned}\langle w, y_{n+1}\rangle + \mu^2\sum_{j=1}^n \|y_j\|^2 &= \langle w, y_{n+1}\rangle + \mu\sum_{j=1}^n \langle w, y_j\rangle \\ &= \left\langle w, y_{n+1} + \mu\sum_{j=1}^n y_j\right\rangle = \langle w, w_n\rangle \\ &= \lambda_{n+1}\|w_n\|^2 = \mu\|y_{n+1}\|^2 + \mu^2\sum_{j=1}^n \|y_j\|^2\end{aligned}$$

the last equality following from the definition of λ_{n+1} and the orthogonality of $\{y_1, \ldots, y_{n+1}\}$. This shows that (v) is fulfilled, and the inductive step is complete.

We have now shown that there is an orthogonal sequence $\{y_j\} \subseteq H$ such that, for each n, $y_n + \mu\sum_{j=1}^{n-1} y_j \in K$ and such that $\|y_1\| < \|y_2\| < \cdots < \|y_n\| < \cdots$. An easy induction, applying Lemma 3 to the sequence of vectors $\{\mu y_1, \ldots, \mu y_{n-1}, y_n\}$ and the sequence of scalars $\{1, 1, \ldots, 1, \mu\}$, shows that $\sum_{j=1}^n \mu y_j \in K$. Another application of Lemma 3 shows that if $e_j = \|\mu y_j\|^{-1}\mu y_j$, then $\sum_{j=1}^n e_j \in K$ for each n. This completes the proof of Theorem 2.

Proof of Theorem 1. Sufficiency is well known; we show only necessity. Without loss of generality we may assume that K is linearly bounded and that $0 \in K$. Take $\{e_j\}$ to be the orthonormal sequence guaranteed by Theorem 2, and H^1 to be the closed subspace spanned by the $\{e_j\}$. Set $K_0 = \{w \in H^1\colon \{\langle w, e_j\rangle\}$ is a nonincreasing sequence of nonnegative numbers bounded above by $1\}$. The corollary to Lemma 3 implies that K_0 is a closed convex subset of K. For $w = \sum_{j=1}^\infty \langle w, e_j\rangle e_j \in K_0$, define $F(w)$ by $F(w) = e_1 + \sum_{j=1}^\infty \langle w, e_j\rangle e_{j+1}$. Then F: $K_0 \to K_0$, is nonexpansive (indeed, is an isometry) and is fixed-point free. For $x \in K$, take $P(x)$ to be the point in K_0 nearest to x, and set $T = FP$. Since, as is well known, P is nonexpansive, T will be a nonexpansive fixed-point free mapping of K into itself.

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