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p-SUBGROUPS OF COMPACT LIE GROUPS AND TORSION OF INFINITE HEIGHT IN $H^*(BG)$

BY

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ABSTRACT. The relation between elementary abelian p-subgroups of a connected compact Lie group G and the existence of p-torsion in $H^*(G)$ has been known for some time [B-S]. In this paper we prove that if G is any compact Lie group then $H^*(BG)$ contains p-torsion of infinite height iff G contains an elementary abelian p-group not contained in a maximal torus. The hard direction is proven using the double coset theorem for the transfer. A third equivalent condition is also given.

1. Let G be any compact Lie group and p any prime. Let BG be the classifying space of G. Let $H^*(BG)$ be singular cohomology with integral coefficients. An elementary abelian p-group is isomorphic to a product of a finite number of cyclic groups of order p. The rank of an elementary abelian p-group is the number of factors of p-cyclic groups. We shall reserve the letter L for elementary abelian p-groups. The p-rank of G, $R(p, G)$, is the largest rank of any elementary abelian p-subgroup of G. This number is always greater than or equal to the rank of G, $r(G)$, since a maximal torus of dimension l contains a subgroup L of p-rank l.

Borel and Serre [**B-S**] showed that if G is connected and $R(p, G) > r(G)$ then $H^*(G)$ contains p-torsion. Borel [**B**] later verified (using the classification of Lie groups) that if G is simple and simply connected and $H^*(G)$ has p-torsion then some subgroup L of G is not contained in any maximal torus. (A good example of a connected, compact Lie group to keep in mind is $\mathrm{SO}(n)$, $n \geqslant 3$. Viewed as matrices the subgroup of diagonal elements is an elementary abelian 2-group which is not contained in any maximal torus.)

The main theorem of this paper is a result similar to that of Borel and Serre which is true for all compact Lie groups. The classification of compact Lie groups is not used in the proof.

We say $y \in H^*(BG)$ is p-torsion of infinite height if $y^n \neq 0$ for all n and $p^i y = 0$ for some $i \geqslant 1$.

THEOREM 1.1. *For any compact Lie group G, the following are equivalent.*

(A) *$H^*(BG)$ has p-torsion of infinite height.*

(B) *G contains an elementary abelian p-group L not contained in any maximal torus.*

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We also give a third equivalent statement at the end of the paper (2.1). In addition we exhibit a nonconnected compact Lie group which has all of its subgroups L contained in maximal tori but whose classifying space nevertheless has 2-torsion (3.1). This example shows that one cannot remove the condition that the p-torsion is of infinite height from the theorem. This counterexample also shows that the cohomology of the classifying space of the Weyl group is not always contained in the cohomology of the classifying space of the normalizer of a maximal torus.

The implication (A) $\Rightarrow$ (B) follows easily from a theorem of Quillen [**Q**, 7.1]. The double coset theorem for the transfer [**F**] is used to prove (B) $\Rightarrow$ (A). Two key lemmas are proved along the way. One concerns the existence of elements in a finitely generated free ring over $\mathbf{Z}/p\mathbf{Z}$ which have the property that their sum under the action of a group of automorphisms of the ring is nontrivial (1.6). The second concerns the image of the restriction map of $H^*(BC)$ to $H^*(BL)$ where C is the centralizer of L in G (1.4).

(1.2) First we prove (A) $\Rightarrow$ (B). Let $y \in H^*(BG)$ be p-torsion of infinite height. Since y is of infinite height y reduced mod p, denoted $\bar{y}$, is also of infinite height. If not some power of y would be divisible by p. A theorem of Quillen [**Q**, 7.1] implies that every element of infinite height in $H^*(BG, \mathbf{Z}/p\mathbf{Z})$ is detected by some elementary abelian p-subgroup of G. Hence there is an $L \subset G$ such that $\rho^*(L, G)(\bar{y})$ is an element of infinite height in $H^*(BL, \mathbf{Z}/p\mathbf{Z})$. L is not contained in any maximal torus T since otherwise the integral class y would go to 0 in $H^*(BT)$ and hence in $H^*(BL)$.

We use the double coset theorem to prove (B) $\Rightarrow$ (A). We shall refer to [**F**] for notation. Let L be a maximal elementary abelian p-group in G. Let $C = \{g \in G \mid glg^{-1} = l \ \forall\, l \in L\}$ be the centralizer of L. Let $N = \{g \in G \mid glg^{-1} \in L \ \forall\, l \in L\}$ be the normalizer of L. Then C is normal in N and N/C is a finite group. The general idea of the proof is to show that there is a $y \in H^*(BC)$ such that $\rho^*(L, G) \circ T(C, G)(y)$ is an element of infinite height in $H^*(BL)$. Then we essentially show that $T(C, G)(y)$ is in fact a p-torsion element.

PROPOSITION 1.3. *Let $y \in \tilde{H}^*(BC)$. Then*

$$\rho^*(L, G) \circ T(C, G)(y) = \left(\sum Cg\right) \circ \rho^*(L, C)(y)$$

where the sum is over the elements of N/C, where $gC \in N/C$. Cg is a conjugation automorphism of $H^*(BL)$ [**F**, II.3].

PROOF. By the double coset theorem [**F**, II.11],

$$\rho^*(L, G) \circ T(C, G) = \sum \chi^{\#}(M) T(C^g \cap L, L) \circ \rho^*(C^g \cap L, C^g) \circ Cg$$

where the sum is over orbit-type manifold components $\{M\}$ of the double coset space $L|G|C$.

Since L is an elementary abelian p-group so is $C^g \cap L = L'$. If $x \in \tilde{H}^*(BL')$ then $T(L', L)(x) = 0$ unless $L = L'$ since $\rho^*(L', L)$ is onto in H^* and $T(L', L) \circ \rho^*(L', L) = 0$ on $\tilde{H}^*(BL')$. This latter fact follows since all elements of $\tilde{H}^*(BL)$ have order p and p divides $\chi(L/L')$ if $L' \neq L$ [**F**, I.1].

Hence the only terms which are not zero are those where $g \in N$. (Since L is maximal it is in fact the unique maximal elementary abelian p-group in C. Hence $C^g \cap L = L^g \cap L$.) Hence we need only sum over the orbit-type manifold components of $L|N|C$. Since L is normal in N and is contained in C, $L|N|C$ is just the finite space N/C. $\chi^{\#}(\mathrm{pt}) = 1$. The proposition follows by noting that $T(L, L) = \mathrm{id}$ and $\rho^*(L, C^g) \circ Cg = Cg \circ \rho^*(L^{g^{-1}}, C) = Cg \circ \rho^*(L, C)$ [**F**, II.2, 3].

We now analyze im $\rho^*(L, C)$.

Let the polynomial subalgebra of $H^*(BL)$ be denoted by $\mathbf{Z}/p\mathbf{Z}[x_1, \ldots, x_l]$ where x_i corresponds to the generator of $H^*(B\mathbf{Z}/p\mathbf{Z})$ where $\mathbf{Z}/p\mathbf{Z}$ is the ith factor of L. x_i has dimension two. The p-rank of L is l.

LEMMA 1.4. *There exists an $S > 0$ such that*

$$\operatorname{im} \rho^*(L, C) \supset \mathbf{Z}/p\mathbf{Z}\left[x_1^{p^S}, x_2^{p^S}, \ldots, x_l^{p^S}\right].$$

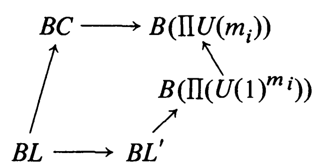
PROOF. The proof is a variation of an idea due to Swan [**S**]. We can embed C in the unitary group $U(n)$ for some n so that L is contained in the diagonal torus $U(1)^n$. This follows by the following argument. C can clearly be embedded in some $U(n)$ since it is a compact Lie group. Under this embedding L is contained in a maximal torus since all elementary abelian p-groups of $U(n)$ are contained in maximal tori [**B-S**, Example 1]. Since all maximal tori are conjugate we can assume L is contained in $U(1)^n$.

We embed L in $L' \subset U(1)^n$, another elementary abelian p-group, by adding new generators to L. Let ρ be a pth root of unity. We add the element with ρ in the places $(b_1, \ldots, b_j)$ in $U(1)^n$ and one elsewhere if the $(b_1, b_2, \ldots, b_j)$ places of each $x \in L$ are $(\rho^e, \ldots, \rho^e)$ for some e depending on x and $(b_1, \ldots, b_j)$ is maximal with respect to this property. In essence if the places $(b_1, \ldots, b_j)$ always agree and disagree with any other place for some element of L we let them act freely as a unit. By conjugating by some element of the symmetric group on n letters we can assume the b_i's are consecutive. Hence $U(1)^n$ is divided into segments of length m_i, where $\sum m_i = n$.

Note that L' is the elementary abelian p-group generated by diagonal representations of $\mathbf{Z}/p\mathbf{Z}$ in each block $U(m_i)$.

We claim that C is in fact contained in $\prod U(m_i)$. Let $c \in U(n)$ be an element which is not in $\prod U(m_i)$. Then there is a nonzero element in the matrix representation of c which lies outside of the blocks containing $\prod U(m_i)$. Say this element is in the dth row and kth column. Then by the maximality condition in the definition of L' there is an $x \in L$ which has different values in the dth and kth places in $U(1)^n$. c does not commute with this element and hence is not in C.

Consider the commutative diagram where all maps are induced by inclusions.



To prove the lemma it suffices to show that there is an $S > 0$ such that im $\rho^*(L, \prod U(m_i)) \supset \mathbf{Z}/p\mathbf{Z}[x_1^{p^S}, \ldots, x_l^{p^S}]$.

Let Σ_m be the symmetric group on m letters.

It is well known that im $\rho^*(U(1)^{m_i}, U(m_i))$ equals $H^*(BU(1)^{m_i})^{\Sigma_{m_i}}$, the invariants of the action of the Weyl group on the cohomology of the classifying space of a maximal torus of a unitary group.

Hence im $\rho^*(\prod U(1)^{m_i}, \prod U(m_i))$ equals $\prod H^*(BU(1)^{m_i})^{\Sigma_{m_i}}$.

We shall work on each factor separately. Let $m_i = m = p^s q$ for each i where q is prime to p. Let $t_j \in H^2(BU(1))$ be the generator of the jth factor of $U(1)^m$. Then $t = \sum_{j_1<j_2<\cdots<j_{p^s}} t_{j_1} t_{j_2} \cdots t_{j_{p^s}}$ is invariant under the action of Σ_m. Let y be the generator of $H^2(B\mathbf{Z}/p\mathbf{Z})$ where $\mathbf{Z}/p\mathbf{Z}$ corresponds to the diagonal representation of $\mathbf{Z}/p\mathbf{Z}$ in $U(1)^m$.

Then

$$\rho^*(\mathbf{Z}/p\mathbf{Z}, U(1)^m)(t) = \binom{m}{p^s} y^{p^s}$$

since $\rho^*(\mathbf{Z}/p\mathbf{Z}, U(1)^m)(t_j) = y$ for all j since $\mathbf{Z}/p\mathbf{Z}$ is diagonal. p does not divide the coefficient since

$$\binom{m}{p^s} = \frac{(p^s q)(p^s q - 1)\cdots(p^s q - p)\cdots(p^s q - p^s + 1)(m - p^s)!}{(p^s)(p^s - 1)\cdots(p^s - p)\cdots(p^s - p^s + 1)(m - p^s)!}$$

and p^u divides a factor in the numerator iff it divides the corresponding factor in the denominator.

Let S be the maximum of all the s. Let η_i be the generator of $H^2(B\mathbf{Z}/p\mathbf{Z})$ where $\mathbf{Z}/p\mathbf{Z}$ is the ith diagonal factor of L'. Since each factor of $\prod U(m_i)$ behaves independently we have shown

$$\operatorname{im} \rho^*\left(L', \prod U(m_i)\right) \supset \mathbf{Z}/p\mathbf{Z}\left[\eta_1^{p^S}, \eta_2^{p^S}, \ldots, \eta_r^{p^S}\right]$$

where the p-rank of L' is r.

We now show that for each x_i in $H^2(BL)$ which corresponds to a factor of L there is an element in $\mathbf{Z}/p\mathbf{Z}[\eta_1^{p^S}, \eta_2^{p^S}, \ldots, \eta_r^{p^S}]$ which hits precisely $x_i^{p^S}$ in $H^*(BL)$.

Since there is a map $\pi: BL' \to BL$ such that $\pi \circ \rho(L, L')$ is the identity we need only show that $\pi^*(x_i^{p^S}) \in \mathbf{Z}/p\mathbf{Z}[\eta_1^{p^S}, \ldots, \eta_r^{p^S}]$ for each i. $\pi^*(x_i) = \sum d_{j_i}\eta_{j_i}$, $d_{j_i} \in \mathbf{Z}/p\mathbf{Z}$. Hence

$$\pi^*\left(x_i^{p^S}\right) = \left(\sum d_{j_i}\eta_{j_i}\right)^{p^S} = \sum d_{j_i}\eta_{j_i}^{p^S}.$$

This establishes the lemma.

In summary we have shown there is an $S > 0$ such that

$$\operatorname{im} \rho^*(L, G) \circ T(C, G) \supset \sum Cg(x) \tag{1.5}$$

for all $x \in \mathbf{Z}/p\mathbf{Z}[x_1^{p^S}, \ldots, x_l^{p^S}]$ where the sum is over N/C. $gC \in N/C$.

Each Cg is an automorphism of the ring $\mathbf{Z}/p\mathbf{Z}[x_1, \ldots, x_l]$. It is not hard to see that Cg_1 and Cg_2 are different if g_1 and g_2 are in different cosets. Their action is determined by the conjugation action of g_1 and g_2 on L.

We need the following.

LEMMA 1.6. *Let D be a subgroup of $A = \mathrm{Aut}(\mathbf{Z}/p\mathbf{Z}[x_1, \ldots, x_l])$. Then given any $S > 0$ there exists an $x \in \mathbf{Z}/p\mathbf{Z}[x_1^{p^S}, \ldots, x_l^{p^S}]$ such that*

$$\sum_{d \in D} dx \neq 0.$$

PROOF. It is sufficient to find an x which works for A. Suppose otherwise, i.e. $\Sigma_{d \in D}\, dx = 0$. Let $a \in A$. Then $\Sigma_{d \in D}\, a\, dx = a \Sigma_{d \in D}\, dx = 0$. Since the sum over any coset of D is zero the sum over A must be zero which is a contradiction.

Hence we suppose $D = A$. A can be represented in $\mathrm{GL}(l, \mathbf{Z}/p\mathbf{Z})$ as a group of basis transformations. Let U equal the subset of A consisting of those elements which do not have any zeros in the diagonal, i.e. $u_{ii} \neq 0\ \forall\, i$ if $u \in U$. Let w equal the number of elements in U.

Let $\tilde{x} = x_1^{p^{a_1}} x_2^{p^{a_2}} \cdots x_l^{p^{a_l}}$ where $a_1 \geq 1$, $p^{a_i} > 2wp^{a_{i-1}}$ for $2 \leq i \leq l$. Also assume S divides a_i for all i.

Let $x = \Pi_{u \in U}\, u\tilde{x}$. We claim $\Sigma_{d \in A}\, dx \neq 0$. We shall see that this sum contains a nonzero multiple of $\tilde{x}^w$.

First we note that if $d \in A$ then

$$d\left(x_i^{p^{a_i}}\right) = \sum \alpha_j x_j^{p^{a_i}}$$

where the sum is over j and $\alpha_j \in \mathbf{Z}/p\mathbf{Z}$. This follows since $x_i^{p^{a_i}}$ is a pth power of a basis element. Now $dx = \Pi_{u \in U}\, du\, \tilde{x}$ and x_l appears with multiplicity $p^{a_l}w$ in $\tilde{x}^w$. Since $p^{a_l} > \Sigma_{i<l}\, p^{a_i}w$ the only way $\tilde{x}^w$ can occur in $\Pi\, du\, \tilde{x}$ is if a factor of exactly $x_l^{p^{a_l}}$ comes from each $du\, \tilde{x}$. One cannot miss a factor of $x_l^{p^{a_l}}$ from any $du\, \tilde{x}$ since one could not make up the difference from all the other possible factors. Similarly since $p^{a_i} > \Sigma_{j<i}\, p^{a_j}w\ \forall\, i$ we have that a factor of $x_i^{p^{a_i}}$ must come from each $du\, \tilde{x}$. Hence the only way dx can contain a nonzero multiple of $\tilde{x}^w$ is if $du\, \tilde{x}$ contains a nonzero multiple of $\tilde{x}$ for each $u \in U$.

We claim that if d is not diagonal there exists a $u \in U$ such that $du\, \tilde{x}$ fails to contain a nonzero multiple of $\tilde{x}$. First suppose d is not an element of U. Then d times the identity is not in U and hence the factor $d\tilde{x}$ does not contain a nonzero multiple of $\tilde{x}$. On the other hand if $d \in U$ but is not diagonal there is an i and a $j \neq i$ such that $d_{ij} = r \neq 0$ and $d_{ii} = q \neq 0$. r generates $\mathbf{Z}/p\mathbf{Z}$ additively. Let $m \in \mathbf{Z}/p\mathbf{Z}$ be such that $mr + q = 0$. Let $u \in U$ have ones down the diagonal, m in the ji place and zeros elsewhere. Then du is not contained in U. Hence $du\, \tilde{x}$ fails to contain a nonzero multiple of $\tilde{x}$.

Suppose d is a diagonal element. Then d permutes the elements of U. Hence dx contains the same multiple of $\tilde{x}^w$ as does x. Since there are exactly $(p-1)^l$ diagonal matrices and this number is prime to p some nonzero multiple of $\tilde{x}^w$ remains in the sum. This establishes the lemma.

Note. This lemma is certainly not as precise as possible. I suspect that the image of the map $\Sigma_{d \in D}\, d$ in $\mathbf{Z}/p\mathbf{Z}[x_1, \ldots, x_l]$ contains l algebraically independent elements.

We now finish the proof of (B) $\Rightarrow$ (A).

From (1.5) and 1.6 it follows that there is a $y \in H^*(BC)$ such that $\rho^*(L, G) \circ T(C, G)(y)$ is an element of infinite height in $H^*(BL)$. It remains to show that $T(C, G)(y)$ is p-torsion. Note that up to this point we have not used the assumption that L is not contained in any maximal torus.

Let G_0 be the identity component of G. We have two cases.

Case 1. $L \cap G_0$ is not contained in a maximal torus. Then we claim that the rank of C is less than the rank of G. Let T_1 be a maximal torus of C. Let $g \in G_0 \cap L$ be an element not in T_1. Since g commutes with each element of T_1 there is a maximal torus of G_0 which contains both g and T_1 [**Br**, 0.6.7].

Let T be a maximal torus of G. By [**F**, VI.3] it follows that $\rho^*(T, G) \circ T(C, G)(y) = 0$. This suffices to prove that $T(C, G)(y)$ is torsion since $\rho^*(T, G)$ is injective on torsion free elements. This follows since $T(T, G) \circ \rho^*(T, G)$ is multiplication by $\chi(G/T) \neq 0$. Furthermore since $\rho^*(L, G) \circ T(C, G)(y) \neq 0$ it is clear that the order of $T(C, G)(y)$ is $p^a q$ where q is prime to p and $a \geqslant 1$. Since $T(C, G)$ is a homomorphism we can assume $q = 1$, by multiplying y by q if $q \neq 1$.

Case 2. $L \cap G_0$ is contained in a maximal torus. Then there is a $g \in L$ which is not contained in G_0. Let g generate $\mathbf{Z}/p\mathbf{Z}$ in the following diagram where $\Gamma = G/G_0$.

$$B\mathbf{Z}/p\mathbf{Z} \to BL \to BG \to B\Gamma.$$

The first two maps are induced by inclusions and the last by projection. Since the composition of these maps is induced by the inclusion of $\mathbf{Z}/p\mathbf{Z}$ in Γ the induced map in cohomology is nontrivial by Swan's theorem [**S**]. Since Γ is finite this implies $H^*(BG)$ contains a p-torsion element of infinite height which pulls back to an element of infinite height in $H^*(BL)$.

This completes the proof of the main theorem.

Note that as in Case 1 the p-torsion element in $H^*(BG)$ pulls back to zero in $H^*(BT)$ where T is any maximal torus. This follows since the composition $T \to G \to \Gamma$ is trivial.

2. As a corollary of the proof we have

THEOREM 2.1. *The following are equivalent for G any compact Lie group*

(B) *G contains an elementary abelian p-group not contained in any maximal torus.*

(C) *$H^*(BG, \mathbf{Z}/p\mathbf{Z})$ contains an element of infinite height which pulls back to zero in the cohomology of the classifying space of any maximal torus.*

(C) $\Rightarrow$ (B) is a slight modification of the argument (A) $\Rightarrow$ (B) (1.2). (B) $\Rightarrow$ (C) can be proven in almost the same manner as (B) $\Rightarrow$ (A) was, just using $\mathbf{Z}/p\mathbf{Z}$ coefficients. As noted in both Case 1 and Case 2 of the proof the element of infinite height pulls back to zero on any maximal torus.

Theorems 1.1 and 2.1 imply that (A), (B), and (C) are equivalent for any compact Lie group G. Thus the existence of p-torsion of infinite height in $H^*(BG)$ is detected by a condition in mod p cohomology and also by the existence of a particular kind of subgroup of G.

3.

EXAMPLE 3.1. We now give an example of a compact Lie group G such that $H^*(BG)$ has 2-torsion and such that the only elementary abelian 2-group in G is contained in a maximal torus. The group is not connected. It has two components and looks like a continuous version of the union of all the generalized quaternion groups.

The identity component is the circle S^1. The second component contains an element v of order four. The square of v is -1 in S^1. If $s \in S^1$ then vsv^{-1} is defined to equal s^{-1}. This is enough to define the group. Every element which is in the nonidentity component vS^1 is of order four. Hence the only element of order two is $-1 \in S^1$.

Let $\mathbf{Z}/4\mathbf{Z}$ be the group generated by v. Consider the maps

$$B\mathbf{Z}/4\mathbf{Z} \to BG \to B\mathbf{Z}/2\mathbf{Z}$$

where the first map is induced by the inclusion and the second by dividing out by the identity component which in this case is also a maximal torus. The composition is induced by sending v to the generator of $\mathbf{Z}/2\mathbf{Z}$. This induces a nontrivial map in cohomology. Hence $H^*(BG)$ contains 2-torsion. However the 2-torsion is not of infinite height. This can be seen either by direct calculation of $H^*(BG)$ or by using Quillen's theorem as used in (A) $\Rightarrow$ (B) of the main theorem (1.2). This example shows that the theorem would be false if one eliminated the hypothesis that the p-torsion be of infinite height.

Note. G normalizes its maximal torus. Hence this example also shows that it is not true in general that the cohomology of the classifying space of the Weyl group injects into the cohomology of the classifying space of the normalizer of a maximal torus.

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