

RIEMANN SURFACES AND BOUNDED HOLOMORPHIC FUNCTIONS

BY

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ABSTRACT. The principal result of this article asserts the equivalence of the following four conditions on a hyperbolic Riemann surface X :

(a) the following set $\{z \mid |f(z)| < \sup |f| \text{ on } K \text{ for every bounded holomorphic section } f \text{ of } \xi\}$ is compact for every unitary vector bundle ξ and every compact set K ;

(b) every unitary line bundle has nontrivial bounded holomorphic sections and the condition in (a) holds for $\xi = i_{\mathbb{C}}$;

(c) every unitary line bundle has nontrivial bounded holomorphic sections and X is regular for potential theory;

(d) every unitary line bundle has nontrivial bounded holomorphic sections and X is its own B -envelope of holomorphy.

If X is a subset of \mathbb{C} , these are also equivalent to the following:

(e) for every unitary line bundle ξ the bounded holomorphic sections are dense in the holomorphic sections.

1. Introduction. This work provides results directed toward the problem of identifying a class \mathfrak{B} of complex manifolds which plays the same role for bounded holomorphic functions that Stein manifolds play for holomorphic functions. However, the results here are restricted to one-dimensional manifolds even though certain of them are meaningful to all dimensions.

As the work of H. Widom [12] has shown, the unitary vector bundles and their bounded holomorphic sections are fundamental in the study of the Hardy spaces on a Riemann surface and we should expect these objects to be of equal importance in other such situations. Studying the results of [12], we can conjecture that an appropriate class of manifolds \mathfrak{B} will be found by imposing conditions which assure enough bounded holomorphic sections for every unitary line bundle. A requirement which may lead to this end is that the hull, \hat{K} , of a set K relative to the set of bounded holomorphic sections of a unitary vector bundle shall be compact whenever K is compact and this is for every such bundle. The reasons for believing that this resolves the problem are developed in this paper.

We refer to [12] for the terminology and more details concerning the facts in this section. X denotes a hyperbolic Riemann surface; ξ denotes a unitary vector bundle over X . Each unitary line bundle can be associated with a representation of the first homology group G of X ; each such line bundle has an inverse $\bar{\xi}$ and a product is defined. Equivalence classes of such line bundles can be identified with G^* the dual group of G . With each unitary vector bundle ξ we have the space $H(\xi)$

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of holomorphic sections of ξ ; $H(i_d) = H$ is the space of holomorphic functions. Since the bundles are unitary every section has an absolute value which is a function, so we may define $B(\xi)$ to be the set of those members of $H(\xi)$ which have bounded absolute values; $B(i_d) = B$. The Hardy spaces are $H_p(\xi)$ where $0 < p < \infty$ and h is a member of $H_p(\xi)$ if h is a holomorphic section of ξ and $|h|^p$ has a harmonic majorant. One key fact to be used repeatedly is the following. A necessary and sufficient condition that a function on X be the absolute value of a member of some $B(\xi)$ for a unitary line bundle ξ is that it be identically zero or have the form $\exp(c - u - p)$ where c is a nonnegative number, u is a positive harmonic function and p is a discrete Green potential, i.e., $p(z) = \sum g(z, a_j)$ where each a_j may be repeated a finite number of times and $g(z, a)$ denotes the Green function for X with pole at a . This fact is a consequence of the Szegő-Solomentsev theorem of M. Heins [5].

We set $X_\alpha = \{z \mid g(z, a) > \alpha\}$ and let $\beta(\alpha, a)$ denote the first Betti number of X_α .

WIDOM'S THEOREM [12, p. 305]. *If $\int_0^\infty \beta(\alpha, a) d\alpha < \infty$, then every unitary vector bundle ξ has a nontrivial section in $B(\xi)$. If $\int_0^\infty \beta(\alpha, a) d\alpha = \infty$ then there is a line bundle ξ such that $H_1(\xi)$ is trivial.*

Trivial means that $B(\xi)$ consists of the zero section only. One of the consequences of this theorem is that if the integral converges for some a , then it converges for every a . Those hyperbolic domains for which that integral in Widom's theorem converges are referred to as Widom domains in [9].

Not only is there a group structure on the unitary line bundles but the sections corresponding to different bundles may be multiplied. If ξ and η are unitary line bundles and $f \in H(\xi)$, $h \in H(\eta)$, then $fh \in H(\xi\eta)$. As a consequence the set $B(\xi)$ are modules over B . When K is a subset of X and A is a set of sections of some unitary vector bundle, the A -hull of K is defined by

$$\hat{K}(A) = \{z \mid |f(z)| \leq p(K, f) \text{ for every } f \in A\},$$

here $p(K, f) = \sup|f|$ taken over the set K . The topology on the spaces $H(\xi)$ is the usual vector space topology defined by the seminorms $p(K, f)$ for K compact.

2. The class \mathfrak{B} and its properties. *A hyperbolic Riemann surface is of class \mathfrak{B} if for every unitary vector bundle ξ and for every compact set K the set $\hat{K}(B(\xi))$ is compact.* Every surface of class \mathfrak{B} is a Widom domain but the converse is false. The punctured disk is a Widom domain but not of class \mathfrak{B} for if K is a circle surrounding the origin, then $\hat{K}(B)$ is not compact.

It is easy to show that for any Widom domain the space B separates points and provides local coordinates. Some other properties of Widom domains and surfaces of class \mathfrak{B} follow.

PROPOSITION 1. *If X is of class \mathfrak{B} , then B is dense in H .*

PROOF. We make use here of Bishop's generalization of Mergelyan's theorem [2] and apply this result to the algebra B . Corollary 2 of [2, p. 48] applied to the compact set $\hat{K}(B)$ yields the result that every function holomorphic on the interior

of $\hat{K}(B)$ and continuous on $\hat{K}(B)$ is the uniform limit thereon of members of B . Now we can exhaust X by a sequence of compact sets K and apply the above reasoning to each of the sets $\hat{K}(B)$; since every $f \in H$ is continuous on $\hat{K}(B)$ and holomorphic on its interior, it follows that B is dense in H .

The surface X is regular if for each $\alpha > 0$ the set $\{z | g(z, a) > \alpha\}$ is relatively compact.

PROPOSITION 2. *If X is a Widom domain, then there is a Widom domain Y which is regular, contains X as a subset, and for which the complement of X is a discrete set.*

The proof of this result appears in [4, Theorem 3, p. 279].

The space B equipped with the sup norm topology is denoted by H^∞ .

PROPOSITION 3. *If X is a Widom domain, then X is homeomorphic to its image in the spectrum of H^∞ .*

PROOF. We suppose first that X is regular. We have the map i from X to the spectrum of H^∞ where $i(a)$ is the evaluation at $z = a$. The map is continuous and, because B separates points, i is injective. We must show that the inverse of i , restricted to the image of X , is continuous. A sequence of evaluations m_n converges to an evaluation m if and only if $\lim (m_n(f)) = m(f)$ for each $f \in H^\infty$. Suppose that $m_n(f) = f(a_n)$ and $m(f) = f(a)$; we want to show that the sequence $\{a_n\}$ converges in X to a . This is clear enough when the sequence converges to some point of X . Suppose $\{a_n\}$ has no convergent subsequences. Since X is regular it is possible to choose a subsequence of $\{a_n\}$ (denoted again by a_n) with the property that $\sum g(a, a_n) < \infty$. Then $\exp(-\sum g(z, a_n))$ is the absolute value of an f which belongs to $B(\xi)$ for some ξ . We choose an $h \in B(\bar{\xi})$ so $|h(a)| \neq 0$. Set $F = hf$ so $F \in B$, $F(a_j) = 0$ for each j while $F(a) \neq 0$. Therefore it is not possible that $\lim m_n = m$. If X is not regular it is contained in a regular Widom domain Y by Proposition 2 and its complement is discrete. From this it is clear that X and Y have the same spectrum and X is also homeomorphic to its image therein.

Given a Riemann surface for which B separates points we want to consider the largest surface Y , $X \subseteq Y$, with the property that every bounded holomorphic function on X extends to a bounded holomorphic function on Y . This B -envelope of X exists and is unique up to conformal equivalence, (see [6, pp. 91–96]).

PROPOSITION 4. *If X is a Widom domain and is regular, then X is its own B -envelope of holomorphy.*

PROOF. Let Y denote the B -envelope of X and let c belong to the boundary of X in Y . Since X is regular there is a sequence $\{z_j\}$ in X which converges to c and converges so rapidly that $\sum g(a, z_j) = p(a)$ is convergent for some a . Then $\exp(-p(z))$ is the absolute value of a section f of some unitary line bundle ξ and $f \in B(\xi)$. We can choose $h \in B(\bar{\xi})$ so $|h(a)| \neq 0$. Then $F = hf \in B$ also extends to Y , i.e., extends to the point c and $F \neq 0$. But $F(z_j) = 0$ for each j and the sequence $\{z_j\}$ converges to c , so $F \neq 0$, a contradiction. Thus no such c exists and $X = Y$.

Let A denote the closure of B in H and suppose that X is a Widom domain. The results of Bishop [1, p. 508] imply that there is a Riemann surface Y and a uniformly closed subalgebra A' of holomorphic functions on Y such that

- (a) $X \subseteq Y$,
- (b) $A = A'|_X$,
- (c) for every compact $K \subseteq Y$, $\hat{K}(A')$ is the set K union with those components of its complement which are relatively compact and $\hat{K}(A')$ is compact.

PROPOSITION 5. *If X is a Widom domain and X is its own B -envelope of holomorphy, then $\hat{K}(B)$ is compact for every compact K .*

PROOF. It follows from (b) above that every member of B extends to Y . Since the B -envelope X_∞ of X is maximal with respect to this property, $Y \subseteq X_\infty$. Thus $X = Y$. Now (c) above means that $\hat{K}(A)$ is compact for every compact set K . But B is dense in A so $\hat{K}(B)$ and $\hat{K}(A)$ are identical.

3. The main theorems. Essentially every Widom domain X can be enlarged to a maximal Widom domain X_B and X_B is obtained by forming the B -envelope of X or equivalently by filling in the punctures of X .

THEOREM 1. *If X is a hyperbolic Riemann surface, then the following statements are equivalent.*

- (1) X is of class \mathfrak{B} ,
- (2) X is a Widom domain and for every compact set K , $\hat{K}(B)$ is compact,
- (3) X is a Widom domain and regular,
- (4) X is a Widom domain and X is its own B -envelope.

PROOF. *Statement one implies statement two.* This is trivial.

LEMMA 1. *Suppose that for every $a \in X$ and for every unitary vector bundle ξ there is an $f \in B(\xi)$ with $|f(a)| \neq 0$. Then for every set $K \subseteq X$, $\hat{K}(B(\xi)) \subseteq \hat{K}(B)$ for every ξ .*

PROOF. If $a \notin \hat{K}(B)$, then a standard argument shows that for every $M > 0$ and for every $\epsilon > 0$ there is an $h \in B$ such that $h(a) = M$ and $p(K, h) < \epsilon$. When $f \in B(\xi)$, $hf \in B(\xi)$. Suppose $a \notin \hat{K}(B)$ but $a \in \hat{K}(B(\xi))$. Then

$$M|f(a)| \leq p(K, hf) \leq p(K, h)p(K, f) \leq \epsilon p(K, f).$$

Hence for every $f \in B(\xi)$, $|f(a)| \leq (\epsilon/M)p(K, f)$. As ϵ and M are arbitrary $|f(a)| = 0$ which is the negation of the hypothesis.

In the case when ξ is a unitary line bundle one can show that $\hat{K}(B(\xi)) = \hat{K}(B)$ for every K and for every ξ .

Statement two implies statement one. If X is a Widom domain, the hypotheses of Lemma 1 hold. If $\hat{K}(B)$ is compact for every compact set K , then $\hat{K}(B(\xi)) \subseteq \hat{K}(B)$. As $\hat{K}(B(\xi))$ is closed it is also compact, so X is of class \mathfrak{B} .

Statement three implies statement two. Proposition 4 implies that X is its own B -envelope of holomorphy so Proposition 5 implies that $\hat{K}(B)$ is compact for every compact K .

Statement one implies statement three. Suppose X is of class \mathfrak{B} . The conclusion of Proposition 2 implies there is Widom domain Y which is regular and which contains X as the complement of a discrete set. If a belongs to Y but not to X , then a small loop K surrounding the point a is compact but $\hat{K}(B)$ must have a as a limit point, so it is not compact. Thus no such a exists and $X = Y$ and X is regular.

Statement four implies statement two. This is just Proposition 5. The concluding implication is *statement three implies statement four*. This is merely a restatement of a previous argument. If X is a regular Widom domain and $X \subseteq Y$, there exists a sequence $\{z_j\}$ in X converging to $c \in Y$ and an $F \in B$ which is not identically zero but for which $F(z_j) = 0$. No such F can extend to be holomorphic on Y . This completes the proof of Theorem 1.

In case more is known about the surface X another condition equivalent to those of Theorem 1 is available.

THEOREM 2. *If X is a hyperbolic subdomain of \mathbf{C} , then X is of class \mathfrak{B} if and only if $B(\xi)$ is dense in $H(\xi)$ for every unitary line bundle ξ .*

If $B(\xi)$ is dense in $H(\xi)$ for each ξ , then it follows that X is a Widom domain since $H(\xi)$ has nontrivial members for every ξ and this is valid whether or not X is a subset of \mathbf{C} . But then every $B(\xi)$ has nontrivial members, hence so does $H_1(\xi)$, so by Widom's theorem X is a Widom domain. To prove that when X is of class \mathfrak{B} , $B(\xi)$ is dense in $H(\xi)$ for each ξ we require two lemmas.

LEMMA 2. *If B is dense in H and if for every compact set K and for every unitary line bundle ξ there is an $h \in B(\xi)$ with no zeros on K , then $B(\xi)$ is dense in $H(\xi)$ for every unitary line bundle ξ .*

PROOF. Let K and ξ be given. We can assume with no loss of generality that $K = \hat{K}(B)$ since $\hat{K}(B)$ is itself compact. Let $F \in H(\xi)$ and suppose that $h \in B(\xi)$ has no zeros on K . Since X is a Stein manifold and since $K = \hat{K}(B) = \hat{K}(H)$ the holomorphic function F/h can be approximated on K by members of H [7, p. 239]. As B is dense in H , F/h can be approximated on K by members of B . Hence for each $\epsilon > 0$ there is an $f \in B$ such that $|F/h - f| < \epsilon$ on K . Then $|F - hf| < \epsilon p(K, h)$ holds on K . As $hf \in B(\xi)$ the conclusion of the lemma follows.

LEMMA 3. *Suppose that $X \subset \mathbf{C}$ is a Widom domain. For every compact set $K \subseteq X$ and for every unitary line bundle ξ on X there is an $h \in B(\xi)$ which has no zeros on K .*

PROOF. This is taken directly from p. 75 of [13]. Let $f \in B(\xi)$, $|f| \leq 1$, f not identically zero. Let a_1, \dots, a_n denote the zeros of f on K . Let $a \in X$, $a \notin K$ and put $k(z) = (z - a)^n / (z - a_1) \cdots (z - a_n)$. Then $kf \in B(\xi)$ and has no zeros on K .

To complete the proof of Theorem 2 let X be a subdomain of \mathbf{C} which is of class \mathfrak{B} . By the result of Proposition 1, B is dense in H . Now Lemmas 2 and 3 together imply that $B(\xi)$ is dense in $H(\xi)$ for every ξ .

The sole impediment to proving Theorem 2 in general is the assertion of Lemma 3 when X is not a subdomain of \mathbf{C} . If X is a Widom domain, then for each point a

and for each ξ there is an h in $B(\xi)$, $|h(a)| \neq 0$; so there is a neighborhood of a on which h is never zero. But to extend from the local condition to the same assertion for any compact set seems difficult. Despite this Theorem 2 is probably true without the restriction to subsets of C .

The next theorem shows that surfaces of class \mathfrak{B} are maximal with respect to each of the classes $B(\xi)$.

THEOREM 3. *Suppose X is a Widom domain. Suppose W is a Widom domain with the following properties:*

(a) $X \subseteq W$,

(b) *for some unitary line bundle ξ on W every bounded holomorphic section of the restriction of ξ to X extends to a bounded holomorphic section of ξ on W .*

Then $W \subseteq X_B$.

PROOF. We claim first that every bounded holomorphic function on X extends to be holomorphic on W . For let b denote such a function on X . Let $a \in W$ and let ξ denote a unitary line bundle as in (b) above. Let $F \in B(\xi)$ with $|F(a)| \neq 0$. Then $bF = h$ is a bounded holomorphic section of the restriction of ξ to X so extends to W . Hence near the point a , $b = h/F$ is holomorphic. Since W is connected, b can be continued analytically to each point of W . Now we consider W_B the B -envelope of W and claim that X is dense in W_B . For if not, there is an open connected set $Y \subseteq W_B$ for which $Y \cap X = \emptyset$. Let $a \in Y$. As W_B is regular there is an $\alpha > 0$ such that the set $\{z | g(a, z) > \alpha\}$ ($g(a, z)$ is the Green function for W_B) is contained in Y . There is a section F_a corresponding to some η such that $|F_a| = \exp(-g(a, \cdot))$. We can choose an $h \in B(\eta)$ on W_B so $|h(a)| \neq 0$. On the complement of Y , which contains X , $k = h/F_a$ is a bounded holomorphic function. So k extends to be holomorphic on W . But k has a pole at $z = a$ so $a \notin W$ and this is true for every $a \in Y$. This is not possible as $W_B - W$ is discrete. So X is dense in W_B . Now every bounded holomorphic function on X extends to be a bounded holomorphic function on W . Therefore W is contained in the B -envelope of X .

4. Example and counterexamples. The punctured disk is an example of a surface which is a Widom domain but for which $B(\xi)$ is not dense in $H(\xi)$ for any unitary line bundle ξ . Given ξ there is a number t , $0 < t < 1$, such that $z^t \in B(\xi)$; in fact every member of $B(\xi)$ has the form $z^t b(z)$ for some $b \in B$ [12, p. 312]. Every member of $H(\xi)$ has the form $z^t h(z)$ for some $h \in H$. Now we can see that $B(\xi)$ is dense in $H(\xi)$ for some ξ if and only if B is dense in H .

There is a surface for which B is dense in H but which is not a Widom domain. It suffices to find a surface for which B is dense in H but which is not regular; such a surface cannot be a Widom domain since it would have to be of class \mathfrak{B} hence be regular. Also, such a surface is its own B -envelope. To find such a surface we use the Δ domains defined in [14]. From the punctured disk we remove a sequence of disks clustering at the origin, $|z - x_n| \leq r_n$, $n = 1, 2, \dots$, where $\sum r_n/x_n < \infty$. Wiener's criterion [11, p. 104] shows that the origin is not a regular point. For any Δ domain X , B is dense in H . This is easily seen as follows. Runge's theorem shows that for any compact set $K \subseteq X$, any $f \in H$ can be approximated uniformly on K

using rational functions whose poles lie in the excised disks $|z - x_n| < r_n$, at the origin and outside $|z| \leq 1$. On the other hand, every function of the form z^{-s} , $s = 1, 2, \dots$, can be approximated uniformly on K by functions of the form $(z - a)^{-s}$ where a is in an excised disk. Hence every $f \in H$ can be approximated uniformly on K by rational functions whose poles are in the excised disks or outside $|z| \leq 1$. The restriction of any such function to X is bounded.

The interior of any compact bordered surface is of class \mathfrak{B} . The simplest way to see this is to know first that any such surface is a Widom domain [12, p. 307]. Since it is also regular it is also of class \mathfrak{B} . It is also the case that for such surfaces $B(\xi)$ is dense in $H(\xi)$ for every unitary vector bundle ξ .

Several authors have considered classes of surfaces which include the Widom domains. In [3] and [4] Hasumi considers surfaces of type (B) and in [8] Neville considers admissible surfaces. In each case the author assumes the existence of an outer section in $B(\xi)$ for each unitary line bundle ξ . This means that there is for each ξ an $f \in B(\xi)$ such that $|f| = \exp(-u)$ where u is a quasibounded harmonic function. The existence of outer sections in each $B(\xi)$ implies that the surface is of class \mathfrak{B} as the next proposition shows. From this result it follows that surfaces of type (B) and admissible surfaces are of class \mathfrak{B} .

PROPOSITION 6. *If the hyperbolic surface X has the property that for every unitary line bundle ξ , $B(\xi)$ has an outer member, then X is of class \mathfrak{B} .*

PROOF. Since every $B(\xi)$ is nontrivial, X is a Widom domain. Let Y denote the B -envelope of X . Every quasibounded harmonic function on X extends to a harmonic function on Y . Let $a \in Y$, $a \notin X$ and consider the section f , $|f| = \exp(-tg_a)$ where g_a is the Green function for Y with pole at a and where $0 < t < 1$. There is a unitary line bundle ξ on X for which $f \in B(\xi)$. We choose an $h \in B(\xi)$ so that h is outer and $|h| \leq 1$. Then $F = fh$ is a bounded holomorphic function on X , hence extends to Y . Also, $|F| \leq 1$ on Y and $F(a) = 0$. Therefore, $|F| \leq \exp(-g_a)$ so $|h| \leq \exp(-(1-t)g_a)$ and consequently $u \geq (1-t)g_a$ on Y . This is impossible as u is harmonic at $z = a$. Thus no such an a exists and $X = Y$.

In [8, p. 67] it is observed that the interior of any compact bordered surface is an admissible surface. Since admissible surfaces are of class \mathfrak{B} this is another way of seeing that such surfaces are of class \mathfrak{B} .

In [13] Widom studied the function $m_\infty(\xi, a) = \sup |f(a)|$ where $f \in B(\xi)$ and $|f| < 1$. If we put the weak topology on the dual group G^* , then we may speak of the continuity of m_∞ as a function of ξ as is done in [13]. In [10] it is shown that for any Widom domain for which m_∞ is continuous in ξ , $B(\xi)$ is dense in $H(\xi)$ for every unitary line bundle ξ . Thus any surface for which m_∞ is continuous is of class \mathfrak{B} . A construction for obtaining surfaces for which m_∞ is continuous is given in [13].

5. Concluding remarks. Any surface which is "good" for bounded holomorphic functions should possess the property that $B(\xi)$ is dense in $H(\xi)$ for each unitary vector bundle ξ . Hence the first question which should be answered is—does Theorem 2 hold in the general case? Besides this there are other properties which one might expect.

(1) *The set H^∞ is dense in the Hardy spaces H_p and in the Smirnov class N^+ . When m_∞ is continuous these properties hold [13].*

(2) *Every positive bounded divisor is the divisor of a member of B . This refers to the following. A necessary condition that a positive divisor (z_j) , where each z_j may be repeated a finite number of times, be the divisor of a bounded holomorphic function is that $\sum g(a, z_j) < \infty$ for some a . Is this condition also sufficient when X is of class \mathfrak{B} ?*

(3) *Every unitary line bundle is determined by a bounded divisor. By this we mean that if ξ is a unitary line bundle, there is some divisor (z_j) such that $\sum g(a, z_j) < \infty$ for some a and there is an $f \in B(\xi)$ such that $|f(z)| = \exp(-\sum g(z, z_j))$. This is analogous to the fact that every vector bundle is determined by a divisor.*

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