TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 259, Number 2, June 1980

LOCALLY FREE AFFINE GROUP ACTIONS

BY

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ABSTRACT. Differentiable actions by the nonabelian 2-dimensional Lie group on compact manifolds are considered. When the action is locally free and the orbits have codimension one it is shown that there are at most finitely many minimal sets each containing a countably infinite number of cylindrical orbits. Examples are given to show that various codimension, differentiability, and minimality restrictions are necessary.

Introduction. Let $\mathcal{A}$ denote the group of affine transformations of the line. As is well known, $\mathcal{A}$ is the only 2-dimensional Lie group other than the abelian group. For our purposes $\mathcal{A}$ will be the set of ordered pairs of real numbers with multiplication defined by

$$(u, v) \cdot (x, y) = (u + e^v x, v + y).$$

In the present paper we study locally free (left) actions by $\mathcal{A}$ on compact manifolds. This means that the isotropy group of the action at each point of the manifold is discrete. We assume also that the actions are differentiable of class $\mathcal{C}^k$ $(k \geqslant 1)$ which implies that the orbits are leaves of a 2-dimensional foliation of class $\mathcal{C}^k$. If $\Phi : \mathcal{A} \times M \to M$ is a locally free action with M compact, then the problem of classifying such actions raises the following questions.

(1) Which M admit such actions?

(2) If M admits a locally free $\mathcal{A}$-action, what dynamical properties must the action have?

Similar questions have been studied for actions of the abelian 2-dimensional group on compact 3-manifolds and, assuming extra smoothness ($\mathcal{C}^2$), very complete results have been obtained **[1]**. For the case of the affine group reasonable responses to question (2) are given below which extend modest observations made in **[6]**, **[8]**. There are well-known examples which shed some light on (1). Specifically, given an Anosov flow having one-dimensional strong stable (or strong unstable) manifolds the foliation by stable (or unstable) manifolds can be parametrized by an action of the affine group **[5]**. Thus, (1) is intimately related to the problem of determining which compact manifolds admit Anosov flows.

Let $\Phi : \mathcal{A} \times M \to M$ be a locally free action. For x in M let $\mathcal{O}(x)$ denote the orbit of x under Φ. There is a natural diffeomorphism between $\mathcal{O}(x)$ (in the leaf topology) and the homogeneous space $\mathcal{A}/\mathcal{I}(x)$ where $\mathcal{I}(x)$ denotes the isotropy

Received by the editors March 27, 1979 and, in revised form, August 7, 1979.

AMS (MOS) subject classifications (1970). Primary 57E20; Secondary 58F15.

Key words and phrases. Lie group actions, orbits, minimal sets, flows, invariant measures, contraction.

[1]Research partially supported by a grant from the National Science Foundation.

group of Φ at x and the action Φ restricted to $\mathcal{O}(x)$ corresponds to left multiplication. The action Φ is generated by the flows X_t and Y_t which are left translations by $(t, 0)$ and $(0, t)$, respectively,

$$X_t(x, y) = (x + t, y), \qquad Y_t(x, y) = (e^t x, y + t).$$

These flows commute with all right translations and, hence, determine well-defined flows of M.

1. Orbit structure. Let $\Phi : \mathcal{A} \times M \to M$ be a locally free $\mathcal{C}^k$ action with M compact and let X_t, Y_t be the flows on M described above.

1.1 PROPOSITION. *An orbit of Φ is diffeomorphic* (*in the leaf topology*) *to either a plane or a cylinder. An orbit of Φ is cylindrical if, and only if, it contains a periodic orbit for the flow Y_t, in which case the periodic orbit is unique.*

PROOF. Note that Y_t takes X-orbits to X-orbits and if σ is a segment in an X-orbit then length$(Y_t(\sigma)) = e^t$ length(σ).

We first observe that X_t has no periodic orbits. Suppose $p \in M$ is such that $X_\tau(p) = p$ for some smallest $\tau > 0$. For every t, $Y_t(p)$ would be a periodic point for X_t of period τe^t. Letting $t \to -\infty$ we would have points of arbitrarily small X-period and compactness of M would imply that some sequence of X-periodic points converges to a *fixed point* of the X-flow. This would contradict the assumption that Φ is locally free. Thus, each X-orbit is a line. If σ is an X-orbit such that the X-orbits $Y_t(\sigma)$, $t \in \mathbf{R}$, are pairwise disjoint then the corresponding Φ-orbit is planar. On the other hand, suppose $Y_\tau(\sigma) = \sigma$ for some $\tau > 0$. σ is a complete metric space and $Y_{-\tau}$ is a contraction which possesses a unique fixed point whose Y-orbit is the unique Y-periodic orbit within the corresponding Φ-orbit. Since Y_τ preserves the orientation of σ the corresponding Φ-orbit is a cylinder and Proposition (1.1) is proved.

For the affine group actions whose orbit foliations are Anosov, it is well known that all but countably many of the orbits are planar. We now seek to determine the extent to which this holds for arbitrary locally free affine group actions. The first question to be answered is whether all the orbits can be planar, i.e., can $\mathcal{A}$ act *freely* on a compact manifold? The following result will be handy for constructing examples.

1.2 PROPOSITION. *Let $f : N \to N$ be a diffeomorphism of a compact manifold. Suppose there is a nowhere-vanishing vector field X on N and a positive real number $\lambda \neq 1$ such that $Df(X) = \lambda X$. Define M to be the quotient of $N \times \mathbf{R}$ by the identification $(x, s) \sim (f(x), s - 1)$. Then M admits a locally free affine group action whose cylindrical orbits correspond to the periodic points of f.*

PROOF. Let ϕ_t be the flow on N generated by the vector field X. Define flows X_t and Y_t on $N \times \mathbf{R}$ by

$$X_t(x, s) = (\phi_t(x), s), \qquad Y_t(x, s) = (\phi_{\lambda^t}(x), s \pm t)$$

where the sign in the definition of Y_t is $+$ if $\lambda > 1$ and $-$ if $\lambda < 1$. It is clear that X_t and Y_t determine well-defined flows on M and thus generate a locally free affine group action.

EXAMPLE. Let $h : T^2 \to T^2$ be a diffeomorphism of the 2-torus which is covered by a hyperbolic linear isomorphism of the plane which has a positive eigenvalue λ. There is a vector field X on T^2 such that $Dh(X) = \lambda X$. Let X_0 be the vector field on $T^2 \times S^1$ given by $X_0(x, \theta) = (X(x), 0)$ and let $f_\rho : T^2 \times S^1 \to T^2 \times S^1$ be the diffeomorphism defined by $f_\rho(x, \theta) = (h(x), \theta + \rho)$. Note that $Df(X_0) = \lambda X_0$ so (1.2) yields a manifold M^4 which admits a locally free affine group action. If ρ is rational, the action has uncountably many cylindrical orbits and when ρ is irrational there are none, i.e., the action is free.

2. Some results concerning codimension one actions. The observations made in the previous section suggest that we should focus attention on actions whose orbits have codimension one, so in this section we assume that M is a compact 3-manifold. A locally free affine group action on such a manifold will be referred to as a "codimension one" action.

2.1 THEOREM. *Suppose $\mathfrak{M}$ is a minimal set of a locally free codimension one affine group action. Then $\mathfrak{M}$ contains exactly countably (infinite) many cylindrical orbits and the rest are planar. In particular, such actions always have infinitely many cylindrical orbits.*

2.2 COROLLARY. *If $\Phi : \mathcal{Q} \times M \to M$ is a locally free codimension one action of class $\mathcal{C}^2$ then Φ has exactly countably many cylindrical orbits.*

PROOF. According to Duminy [**2**] the orbit foliation of Φ cannot have any exceptional minimal sets so M itself is a minimal set and Corollary 2.2 follows from Theorem 2.1.

2.3 THEOREM. *A codimension one locally free affine group action has only finitely many minimal sets.*

The proofs of Theorems 2.1 and 2.3 will be given later. Note that the codimension one hypothesis in Theorem 2.3 is essential since the examples given at the end of the previous section with ρ rational have uncountably many minimal sets. The question of whether a codimension one affine group action can have more than one minimal set appears to be open. We now proceed to describe examples which show that the minimality assumption in Theorem 2.1 and the extra smoothness assumption in Corollary 2.2 are essential.

As before, let $h : T^2 \to T^2$ be a hyperbolic automorphism of the 2-torus having a positive eigenvalue λ. Let μ denote the other eigenvalue and let X, Y be vector fields on T^2 such that $Dh(X) = \lambda X$ and $Dh(Y) = \mu Y$. We will modify h in a manner suggested by the DA construction of Smale [**10**]. Let p be a fixed point for h and let $g : T^2 \to T^2$ be a diffeomorphism, supported in a small neighborhood of p, which fixes p and preserves the foliation of T^2 by orbits of the Y-flow. Set $f = gh$. In terms of coordinates X and Y we can write

$$Df_x = \begin{pmatrix} \lambda & 0 \\ * & \nu(x) \end{pmatrix}$$

where ν is continuous. It is clear that g can be chosen in such a way that the following conditions are satisfied.

(i) f has uncountably many fixed points in a neighborhood of p.

(ii) $\sup_x|\nu(x)| < \lambda$ (when $\lambda > 1$) or $\inf_x|\nu(x)| > \lambda$ (when $\lambda < 1$).

Condition (ii) guarantees that f is normally hyperbolic to the foliation of T^2 by orbits of the Y-flow. According to Invariant Manifold Theory **[4]** there is an f-invariant $\mathcal{C}^1$ foliation $\mathfrak{F}$ by lines transverse to the Y-orbits. If η is the closed one form on T^2 such that $\eta(X) = 1$, $\eta(Y) = 0$ let X' be the vector field tangent to $\mathfrak{F}$ such that $\eta(X') = 1$. Thus, since $g^*\eta = \eta$ we have $Df(X') = \lambda X'$. So by (1.2), X' and f generate a locally free affine group action on a compact 3-manifold. When condition (i) is satisfied the induced action has uncountably many cylindrical orbits. Note that in this case the action has a unique *exceptional* minimal set whose intersection with T^2 (thought of as $T^2 \times \{0\}$) is an exceptional minimal set of $\mathfrak{F}$. By Denjoy's theorem $\mathfrak{F}$, and hence the affine group action, cannot be of class $\mathcal{C}^2$.

3. Invariant measure and holonomy. The first step is to show that a minimal set supports a finite measure which is invariant under the action by $\mathcal{Q}$ and use the measure to deduce information concerning the holonomy of the orbit foliation.

Suppose $\mathfrak{F}$ is a codimension one foliation of a manifold M and let $\mathcal{S} \subset M$ be a compact subset which is a union of leaves of $\mathfrak{F}$. Let J be an open interval containing 0 and let γ: $(J, 0) \to (\mathbf{R}, 0)$ be a local homeomorphism which represents the holonomy of a loop (based at the point corresponding to 0) in some leaf of $\mathcal{S}$. We will say that the element of holonomy is *contracting relative to* $\mathcal{S}$ if there is an $\varepsilon > 0$ such that whenever $0 < |t| < \varepsilon$ and t corresponds to a point in $\mathcal{S}$, $\gamma(t)$ lies between 0 and t. We say an element of holonomy is *expanding relative to* $\mathcal{S}$ if its inverse is contracting relative to $\mathcal{S}$.

3.1 LEMMA. *Let $\mathfrak{M}$ be a minimal set of a locally free affine group action on a compact oriented 3-manifold. Every loop on a cylindrical orbit in $\mathfrak{M}$ which is not homotopic to zero in the cylinder determines an element of holonomy which is either contracting or expanding relative to $\mathfrak{M}$.*

PROOF. Since $\mathcal{Q}$ is amenable **[3]** there is a finite Borel measure μ on $\mathfrak{M}$ which is invariant under the action. Since $\mathfrak{M}$ is minimal the support of μ is all of $\mathfrak{M}$. Let $\mathcal{O} \subset \mathfrak{M}$ be a cylindrical orbit and let $p \in \mathcal{O}$ be a periodic point for the flow Y_t (described in the Introduction). A loop in $\mathcal{O}$ which is not homotopic to zero must be freely homotopic to a loop l: $[0, \tau] \to \mathcal{O}$ defined by $l(t) = Y_t(p)$ or by $l(t) = Y_{-t}(p)$. We consider the first case and show that the holonomy of l is contracting relative to $\mathfrak{M}$. A similar argument in the second case shows that l has expanding holonomy relative to $\mathfrak{M}$. Note that since the orbit foliation is transversely oriented (because the ambient manifold is) we may work on one "side" of $\mathcal{O}$ at a time. Since contraction relative to $\mathfrak{M}$ is vacuous on a proper side of $\mathcal{O}$ we consider a nonproper side. Let Q be an embedded compact arc transverse to the orbit foliation with one endpoint at p and assume p is a limit point of $Q \cap \mathfrak{M}$. For each q in Q and $a > 0$ define

$$U_q(a) = \{ Y_t X_s(q) | -a < s, t < a \}.$$

Fix a number $b > 0$ such that the set $U = \bigcup_{q \in Q} U_q(b)$ is a distinguished set for the orbit foliation. Let λ be left-invariant Haar measure on $\mathcal{Q}$ and for any Borel set $B \subset U$ and $q \in Q$ we let $B_q = B \cap U_q(b)$. We abuse notation by identifying B_q with the corresponding subset of $\mathcal{Q}$. By a result of Roklin **[9]** there is a Borel measure ν on Q such that, for any Borel set $B \subset U$, $\mu(B) = \int_Q \lambda(B_q)\, d\nu(q)$. Let $\delta > 0$ be sufficiently small and N a connected neighborhood of p in Q so that $Y_\tau(U_q(\delta)) \subset U$ for q in N. Note that since $Y_\tau(p) = p$, the set $Y_\tau(U_p(\delta))$ properly contains $U_p(\delta)$ and, hence,

$$\lambda(Y_\tau(U_p(\delta)))/\lambda(U_p(\delta)) > 1.$$

Now set $B = Y_\tau(\bigcup_{q \in N} U_q(\delta))$. Now let $r \neq p$, $r \in N \cap \mathfrak{M}$ be a point such that for every point q in the closed arc $[p, r]$ from p to r in Q we have $\lambda(B_q)/\lambda(U_q(\delta)) \geqslant c > 1$. We claim that the holonomy is contracting on $[p, r]$. If for some $q_0 \in (p, q]$ the image of $[p, q_0]$ under the holonomy map contains $[p, q_0]$ then for the set $A = \bigcup_{q \in [p, q_0]} U_q(\delta)$ we would have (from the formula defining ν), $\mu(Y_\tau(A)) \geqslant c\mu(A) > \mu(A)$ which contradicts the fact that μ is Y_t-invariant. This completes the proof of Lemma 3.1.

4. Proof of Theorem 2.1.

4.1 Proposition. *If $\mathfrak{M}$ is a (compact) minimal set of a codimension one locally free affine group action, then $\mathfrak{M}$ contains at most countably many cylindrical orbits.*

Proof. If the manifold is not oriented we lift the action to a 2-fold covering which is oriented. This procedure cannot reduce the number of cylindrical orbits. It suffices to show that for each $T > 0$, $\mathfrak{M}$ contains at most finitely many periodic orbits of Y_t of period $\leqslant T$. Suppose $\{p_n\}$ is an infinite sequence of points on distinct orbits and τ_n are positive real numbers such that $Y_{\tau_n}(p_n) = p_n$ and $\tau_n \leqslant T$. Passing to a subsequence we may assume that $p_n \to p$ in $\mathfrak{M}$ and $\tau_n \to \tau$ in $[0, T]$. By continuity of the flow we have $Y_\tau(p) = p$. Lemma 3.1 implies that the holonomy of the orbit foliation determined by the loop $l(t) = Y_t(p)$, $t \in [0, \tau]$, is contracting relative to $\mathfrak{M}$. This means that there are neighborhoods U of p in $\mathfrak{M}$ and N of τ in $\mathbf{R}$ such that for $q \in U$ (q and p in different plaques) and $t \in N$, q and $Y_t(q)$ lie in different plaques of a distinguished neighborhood (for the orbit foliation of the affine group action). This contradicts the fact that for sufficiently large n, $p_n \in U$ and $\tau_n \in N$. Thus, the set of periodic orbits of period $\leqslant T$ is finite and Proposition 4.1 is proved.

4.2 Proposition. *If $\mathfrak{M}$ is a (compact) minimal set of a codimension one locally free affine group action, then $\mathfrak{M}$ contains infinitely many cylindrical orbits.*

Proof. Passing to a 2-fold covering space we assume that the ambient manifold is oriented. (This procedure at most doubles the number of cylindrical orbits.)

We observe first that $\mathfrak{M}$ contains at least one cylindrical orbit by (9.2) of **[8]**. (Note: (9.2) of **[8]** is not stated correctly. It can be corrected by replacing the word "every" in the last line by "some".) The argument for showing the existence of infinitely many cylindrical orbits involves showing the existence of periodic orbits of Y_t in any neighborhood of a "homoclinic" point **[10]**. In our case, however, the periodic point may not be hyperbolic so some special arguments are needed. Let $p \in \mathfrak{M}$ be such that $Y_\tau(p) = p$, $\tau > 0$. Since $\mathfrak{M}$ is a minimal set for the affine group action, the orbit $\mathcal{O}(p)$ is nonproper on at least one side (call it the positive side) relative to transverse orientation. Since the holonomy of the loop $l(t) = Y_t(p)$, $t \in [0, \tau]$, is contracting relative to $\mathfrak{M}$ the Y-periodic orbit through p has two locally defined invariant manifolds **[4]** which are transverse and whose intersection is $\{Y_t(p)\}$. The unique (global) unstable manifold coincides with $\mathcal{O}(p)$. The other manifold is either a stable manifold (if p is hyperbolic) or a center manifold which is locally unique on the positive side of $\mathcal{O}(p)$ **[4**, §5A]. Let $W_+(p)$ denote the invariant (stable or center) manifold emanating from the positive side of $\mathcal{O}(p)$ along $\{Y_t(p)\}$. Thus, $W_+(p)$ is bounded on one side by the periodic orbit $\{Y_t(p)\}$. Let Q be a compact arc in $W_+(p)$ transverse to Y_t-orbits with one endpoint at p. Let U be a distinguished set as defined in the proof of Theorem 3.1, $U = \bigcup_{q\in Q} U_q(b)$. Let $Q_0 \subset Q$ be a segment with endpoints in $\mathfrak{M}$, one of them at p, and $b_0 < b$ a positive number such that $Y_{-\tau}(U_0) \subset U$ where $U_0 = \bigcup_{q\in Q_0} U_q(b_0)$. Take Q_0 small enough so that the holonomy of the loop $l(t) = Y_t(p)$, $t \in [0, \tau]$, is contracting relative to $\mathfrak{M}$ in U_0. Since $\mathcal{O}(p)$ is dense in $\mathfrak{M}$ there exist infinitely many points $q \neq p$ in $Q_0 \cap \mathcal{O}(p)$. We will show that for any fixed $q_0 \in$ (interior Q_0) $\cap\, \mathcal{O}(p)$ and any neighborhood V of q_0 there is a periodic point r of the Y-flow such that $r \in V \cap \mathfrak{M}$. This will imply that $\mathfrak{M}$ contains infinitely many periodic orbits of Y_t and complete the proof of Proposition 4.2.

In order to prove the above assertion we first show that there is an arc $[q_0, q_1] \subset Q_0$ (Q_0 ordered in the obvious way) which can be deformed to an arc arbitrarily near Q_0 by an isotopy preserving the orbits of the Y-flow. Let $S \subset U$ be the local section of the Y-flow given by (b as in definition of U)

$$S = \bigcup_{q\in Q} \{X(q)| - b < s < b\}$$

and define projections Π_Q: $U \to Q$, Π_S: $U \to S$ by $\Pi_Q(Y_t X_s(q)) = q$ and $\Pi_S(Y_t X_s(q)) = X_s(q)$. Note that any arc in U transverse to the orbit foliation is isotopic along Y-orbits to its image under Π_S.

Since $q_0 \in \mathcal{O}(p)$, there is a $T > 0$ and a $q_1 > q_0$ in $\mathfrak{M} \cap Q_0$ such that $Y_{-T}([q_0, q_1]) \subset U_0$ and $\Pi_Q(Y_{-T}([q_0, q_1])) = [p, p_1]$ for some $p_1 \neq p$ in $\mathfrak{M} \cap Q_0$. Let A_1 be the arc $\Pi_S(Y_{-T}([q_0, q_1]))$. Let $\mathcal{P}$: $S \cap U_0 \to S$ be the Poincaré map $\Pi_S Y_{-\tau}$. If $\Pi_Q(A_1)$ does not contain Q_0 let $A_2 = \mathcal{P}(A_1)$. Inductively, if $\Pi_Q(A_k)$ does not contain Q_0 let $A_{k+1} = \mathcal{P}(A_k)$. We claim that for some n, $A_n \supset Q_0$. Note that $\Pi_Q(A_k) = [p, p_k]$ for some $p_k \in \mathfrak{M} \cap Q$ where $p_k > p_{k-1}$. If $p^* = \sup_k\{p_k\}$ were in Q_0 then we would have $\mathcal{P}(p^*) = p^*$ which is impossible since $p^* \in \mathfrak{M} \cap Q_0$ and $\mathcal{P}$ is expanding on $\mathfrak{M} \cap Q_0$. So $A_n \supset Q_0$ for some n and by redefining q_1 we may assume that $\Pi_Q(A_n) = Q_0$. Let $\mathfrak{f}$ denote the restriction of $\Pi_S Y_{-T}$ to

$[q_0, q_1]$. We have $A_n = \mathcal{P}^{n-1}\mathcal{G}([q_0, q_1])$. Now the further iterates $\mathcal{P}^m(A_n)$ (restricted by intersection with U_0 at each stage) converge in the sense of graph transform to Q_0 [**4**]. Thus replacing q_1 at each stage when necessary by a point (still denoted q_1) closer to q_0 we have arcs

$$A_{m+n} = \mathcal{P}^{m+n-1}\mathcal{G}([q_0, q_1])$$

such that $\Pi_Q(A_{m+n}) = Q_0$ and the continuous functions f_m: $[q_0, q_1] \to \mathbf{R}$, defined by $f(q) = t$ when $X_t(q)$ is in A_{m+n}, approach zero uniformly.

Given a neighborhood V of q_0 choose q_1 in $\mathfrak{M} \cap Q_0$, $\varepsilon > 0$, and $m > 0$ so that in addition to the above we have for q in $[q_0, q_1]$

$$X_t(q) \in V \quad \text{whenever } |t| \leqslant \varepsilon\big(e^{\tau-b}/\,(e^{\tau-b}-1)\big) \tag{i}$$

and

$$\sup_q\ |f_m(q)| < \varepsilon. \tag{ii}$$

The map $\Pi_Q\mathcal{P}^{m+n-1}\mathcal{G}$: $[q_0, q_1] \to Q_0$ is a homeomorphism and, therefore, has a fixed point q^*. We may assume q^* is in $\mathfrak{M}$ because if q^* is in $Q_0 - \mathfrak{M}$ then the endpoints of the maximal open interval in $Q_0 - \mathfrak{M}$ containing q^* must also be fixed by our orientability assumption. Thus, the set $\{X_s(q^*)|s \in \mathbf{R}\}$ is sent into itself by Y_{t_0} for some $t_0 > \tau - b$ which implies that $X_{s_0}(q^*)$ is Y-periodic for some $s_0 \leqslant \varepsilon(e^{t_0}/(e^{t_0}-1))$. Hence, $X_{s_0}(q^*)$ is a Y-periodic point contained in V by condition (i). The proof of Proposition 4.2 is complete.

Theorem 2.1 follows immediately from Propositions 4.1 and 4.2.

5. Proof of Theorem 2.3. We suppose that $\{\mathfrak{M}_k\}$ is a sequence of minimal sets for an affine group action. Let

$$\mathfrak{M}^* = \{x \in M | x_k \to x, x_k \in \mathfrak{M}_k\}.$$

$\mathfrak{M}^*$ is closed and invariant under the action by $\mathcal{Q}$. Let $\mathfrak{M} \subset \mathfrak{M}^*$ be a minimal set for the action. We claim that for sufficiently large k, $\mathfrak{M}_k = \mathfrak{M}$. Let $p \in \mathfrak{M}$, $\tau > 0$ be such that $Y_\tau(p) = p$ and let U_0 be a distinguished set containing p of the form considered in the previous section. In particular, we assume that the holonomy of the loop $Y_t(p)$, $t \in [0, \tau]$, is contracting relative to $\mathfrak{M}$ in U_0. Since $\mathfrak{M} \subset \mathfrak{M}^*$ there is a sequence $\{x_k\}$ converging to p with $x_k \in \mathfrak{M}_k$. For k sufficiently large the orbit of x_k under the affine group action will intersect U_0, so we can assume that $x_k \in U_0$. This implies that the point $\Pi_Q(x_k)$ is in $W_+(p)$ and, hence, since the holonomy of the periodic orbit is contracting relative to $\mathfrak{M}$ in U_0, the closure of the set $\{Y_t(\Pi_Q(x_k))|t > 0\}$ contains p. Since $\mathfrak{M}_k$ is closed and invariant $\mathfrak{M}_k \supset \mathfrak{M}$. Since $\mathfrak{M}_k$ is minimal $\mathfrak{M}_k = \mathfrak{M}$ for k sufficiently large and the proof of Theorem 2.3 is complete.

6. Generalizations. It is evident from an examination of the proofs that the results in §2 can be generalized somewhat. Consider Lie groups which are semidirect products of the form $N \cdot \mathbf{R}$ where N is a simply-connected n-dimensional nilpotent group with Lie algebra $\mathfrak{N}$. Let Y be a vector field which spans $\mathbf{R}$ in $\mathfrak{N} \cdot \mathbf{R}$. If the real parts of the eigenvalues of $\mathrm{ad}_Y|\mathfrak{N}$ are all nonzero and have the same sign (this forces $\mathfrak{N}$ to be nilpotent) then the obvious analogues of Theorem 2.1, Corollary

2.2, Theorem 2.3 hold for locally free actions by $N \cdot \mathbf{R}$ on compact manifolds of dimension $n + 2$. In this case the word "cylindrical" refers to orbits homeomorphic to $\mathbf{R}^n \times S^1$.

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