TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 259, Number 2, June 1980

# AN APPLICATION OF HOMOLOGICAL ALGEBRA TO THE HOMOTOPY CLASSIFICATION OF TWO DIMENSIONAL CW-COMPLEXES

BY

MICHEAL N. DYER

ABSTRACT. Let $\pi$ be $Z_m \times Z_n$. In this paper the homotopy types of finite connected two dimensional CW-complexes with fundamental group $\pi$ are shown to depend only on the Euler characteristic. The basic method is to study the structure of the group $\operatorname{Ext}^1_{Z\pi}(I\pi^2, Z)$ as a principal $\operatorname{End}(I\pi^2)$-module.

**1.** In this paper $\pi$ *will denote the noncyclic group* $Z_m \times Z_n$, which is the product of two finite cyclic groups $Z_m$ and $Z_n$. Thus the $\gcd(m, n) \neq 1$. For convenience, we will always assume that $m$ divides $n$. This is no restriction.

Let $X_{\mathcal{P}}$ denote the two dimensional CW-complex modeled on the presentation $\mathcal{P} = \{x, y: x^m, y^n, [x, y]\}$ of $\pi$ and let $\pi_2 = \pi_2 X_{\mathcal{P}}$. $X_{\mathcal{P}}$ is called the standard model and $\pi_2$ the standard module. The study of this $\pi$-module $\pi_2$ forms the basis of this paper.

For any $\theta \in \operatorname{Aut} \pi$ and any $\pi$-module $M$, the module ${}_\theta M$ has action given by $g * m = \theta(g)m$ for any $m \in M$, $g \in \pi$. Two modules $M$, $N$ are said to be *$\theta$-isomorphic* iff there is an isomorphism $\alpha: M \to {}_\theta N$. The module $\pi_2$ splits as a short exact sequence $Z \rightarrowtail \pi_2 \twoheadrightarrow (I\pi)^2$ where $Z$ is the trivial $\pi$-module and $I\pi$ is the augmentation ideal in $Z\pi$. By studying the group $\operatorname{Ext}(I\pi^2, Z)$ we prove the following crucial theorem.

THEOREM A. *For any $\pi$-module $M$ such that $M \oplus Z\pi \cong \pi_2 \oplus Z\pi$, we have $M \cong_\theta \pi_2$ for some $\theta \in \operatorname{Aut} \pi$.*

Hence, $M$ is stably isomorphic to $\pi_2$ iff $M$ is $\theta$-isomorphic to $\pi_2$ for some $\theta \in \operatorname{Aut} \pi$.

The group $H^3(\pi; \pi_2)$ is isomorphic to the cyclic group $Z_{mn}$ [$\mathbf{D}_1$, §2]; to each integer $q$ prime to $mn$, there is a projective ideal $(q, N) \subset Z\pi$ generated by $q$ and $N = (\sum_{i=1}^m x^i)(\sum_{j=1}^n y^i)$. The function $\partial: Z^*_{mn} \to \tilde{K}_0 Z\pi$ given by $\partial(q + (mn)) = \{(q, N)\} \in \tilde{K}_0 Z\pi$ is a homomorphism. A $\theta$-isomorphism $\alpha: \pi_2 \to_\theta \pi_2$ has degree $k \in Z^*_{mn}$ iff $(\theta^*)^{-1}\alpha_*(1) = k$ in the diagram:

$$H^3(\pi; \pi_2) \xrightarrow{\alpha_*} H^3(\pi; {}_\theta\pi_2) \xleftarrow{\theta^*} H^3(\pi; \pi_2).$$

THEOREM B. *For any $k \in \ker \partial \subset Z^*_{mn}$ there is a $\theta \in \operatorname{Aut} \pi$ and a $\theta$-isomorphism $\alpha: \pi_2 \to_\theta \pi_2$ of degree $k$.*

Received by the editors April 17, 1979.
1980 *Mathematics Subject Classification*. Primary 57M20, 55U99.


0002-9947/80/0000-0260/$03.50

We prove Theorem A in §4 and Theorem B in §5.

The following Corollaries 1 and 2 follow from A and B just as in [$\mathbf{D_2}$, Theorem 5.5].

DEFINITION. A $(G, 2)$-*complex* is a finite, connected, 2-dimensional CW-complex having fundamental group isomorphic to $G$.

COROLLARY 1. *Any two $(Z_m \times Z_n, 2)$-complexes have the same homotopy type iff they have the same Euler characteristic.*

In the language of [$\mathbf{D_2}$], the homotopy trees $\mathrm{HT}(Z_m \times Z_n, 2)$ have essential height zero.

COROLLARY 2. *Let $X$ be a CW-complex with fundamental group isomorphic to $Z_m \times Z_n$ and suppose that $X$ is dominated by a $(G, 2)$-complex. Then $X$ has the homotopy type of a $(Z_m \times Z_n, 2)$-complex iff the Wall obstruction vanishes.*

In the homotopy classification of $G$-complexes for $G$ finite abelian, these results fill in a gap that existed between $G$ cyclic [$\mathbf{D_1}$] and $G$ having more than two torsion coefficients [**SD**]. A technique similar to this may be decisive in determining the isomorphism and $\theta$-isomorphism classes of the minimal $(G, 2)$-modules detected in [**SD**], for $G$ finite abelian.

**2. A study of** $\mathrm{Ext}((I\pi)^2, Z)$. By looking at the cellular chain complex of the universal cover $\tilde{X}_{\mathcal{P}}$ of the standard model $X_{\mathcal{P}}$, we may identify $\pi_2$ as the kernel of the following exact sequence:

$$\mathcal{C}_*(\tilde{X}_{\mathcal{P}}) : \pi_2 \rightarrowtail (Z\pi)^3 \xrightarrow[\partial_2]{\begin{bmatrix} N_x & 1-y & 0 \\ 0 & x-1 & N_y \end{bmatrix}} (Z\pi)^2 \xrightarrow[\partial_1]{(x-1,\, y-1)} Z\pi \overset{\epsilon}{\twoheadrightarrow} Z. \tag{2.1}$$

For an integer $r > 0$ and $z \in \pi$, let $\langle z, r\rangle = 1 + z + \cdots + z^{r-1}$. Then $N_x = \langle x, m\rangle$ and $N_y = \langle y, n\rangle$. The map $\varepsilon$: $Z\pi \to Z$ is the augmentation homomorphism. It is easy to see that $\pi_2$ has generators the columns of the matrix

$$\begin{bmatrix} x-1 & y-1 & 0 & 0 \\ 0 & N_x & -N_y & 0 \\ 0 & 0 & x-1 & y-1 \end{bmatrix}.$$

Label the columns $g_1$, $g_2$, $g_3$, and $g_4$ respectively.

Let $\eta_{13}$: $(Z\pi)^3 \to (Z\pi)^2$ denote the projection on the first and third coordinates. $\eta = \eta_{13}|\pi_2$ has image $I\pi^2$ and kernel $\pi_2^\pi = \{\alpha \in \pi_2 | g\alpha = \alpha \text{ for all } g \in \pi\} = Z = Z\pi(0, N, 0)$, where $N = \langle x, m\rangle\langle y, n\rangle$. Thus the extension class $[\mathcal{E}]$ of the extension

$$\mathcal{E} : Z \rightarrowtail \pi_2 \overset{\eta}{\twoheadrightarrow} I\pi^2$$

is a member of $E = \mathrm{Ext}^1_{Z\pi}(I\pi^2, Z)$. Sometimes, we will denote the class of the extension $\mathcal{F}$: $Z \rightarrowtail M \twoheadrightarrow I\pi^2$ by $[\mathcal{F}_M]$. Using the fact that $\mathrm{Ext}^1_{Z\pi}(I\pi, Z) \cong H^2(\pi; Z) \cong \mathrm{Ext}_Z(\pi, Z) \cong \pi$, we see that $\mathrm{Ext}(I\pi^2, Z) \cong \pi^2$. We will think of $E$ as

$2 \times 2$ matrices

$$E = \begin{bmatrix} Z_n & Z_n \\ Z_m & Z_m \end{bmatrix}.$$

$E$ may be considered as a right module over the ring $\mathrm{End}(I\pi)^2$ as follows: to each $\alpha \in \mathrm{End}(I\pi)^2$ and each extension class $[\mathfrak{F}] \in E$ we associate the extension class $[\mathfrak{F}\alpha]$ which is the pull-back of $M$ by $\alpha$. Thus

$$\begin{array}{lcccc} \mathfrak{F}\alpha: & Z \rightarrowtail & M\alpha & \twoheadrightarrow & (I\pi)^2 \\ & \| & \downarrow & & \downarrow\alpha \\ \mathfrak{F}: & Z \rightarrowtail & M & \twoheadrightarrow & (I\pi)^2 \end{array}$$

In fact, with this action, $E$ becomes a *principal* $\mathrm{End}(I\pi)^2$-module with generator $[\mathfrak{S}]$. To see this, we use the long exact sequence for $\mathrm{Ext}^i_{Z\pi}$ associated with $\mathfrak{S}$ **[HS**, p. 139]:

$$\mathrm{Hom}(I\pi^2, \pi_2) \to \mathrm{End}(I\pi)^2 \xrightarrow{\partial} \mathrm{Ext}(I\pi^2, Z) \to \mathrm{Ext}(I\pi^2, \pi_2) \to \dots.$$

The boundary operator $\partial$ is described by $\partial(\alpha) = [\mathfrak{S}\alpha]$. Sometimes, when the basic extension is clear, $\partial(\alpha)$ will be denoted by $[\alpha]$. But, by using the exact sequence 2.1, $\mathrm{Ext}^1_{Z\pi}((I\pi)^2, \pi_2) \cong \mathrm{Ext}^2_{Z\pi}(Z^2, \pi_2) \cong [H^2(\pi; \pi_2)]^2 = 0$ **[D₁**, Lemma 6.7]. Thus we have proved the following lemma.

2.2 Lemma. $\mathrm{Ext}((I\pi)^2, Z)$ *is a principal* $\mathrm{End}(I\pi)^2$*-module with generator* $[\mathfrak{S}]$. □

Definition. Let $\mathcal{G}(E) = \{[\mathfrak{F}] \in |[\mathfrak{F}]$ is a generator of $E$ as an $\mathrm{End}(I\pi)^2$-module$\}$.

2.3 Lemma. *Suppose that $M$ is stably isomorphic to the standard module $\pi_2$; i.e., $M \oplus Z\pi \cong \pi_2 \oplus Z\pi$. Then, if $M_\pi = M/M^\pi$, the extension $\mathfrak{F}_M$: $Z = M^\pi \rightarrowtail M \twoheadrightarrow M_\pi \cong I\pi^2$ generates $E$ as an* $\mathrm{End}((I\pi)^2)$*-module.*

Proof. If we can show that $M$ is an extension of $Z = M^\pi$ by $I\pi^2$, then $[\mathfrak{F}_M] \in \mathcal{G}(E)$ follows using the argument above (with the exact sequence $\mathfrak{F}_M$) together with the fact that

$$H^2(\pi, M) \cong H^2(\pi; M \oplus Z\pi) \cong H^2(\pi; \pi_2 \oplus Z\pi) \cong H^2(\pi; \pi_2) = 0$$

since $H^2(\pi; Z\pi) = 0$ for any finite group **[CE**, p. 233]. To prove the first statement, observe that

$$M \oplus Z\pi \cong \pi_2 \oplus Z\pi \Rightarrow M_\pi \oplus Z\pi/(N) \cong (\pi_2)_\pi \oplus Z\pi/(N).$$

A careful, but elementary argument shows then that $M_\pi \oplus I\pi \cong (\pi_2)_\pi \oplus I\pi$. Because $I\pi$ (and hence $(I\pi)^2$) satisfies the Eichler condition **[SE**, p. 176] and $I\pi$ is a direct summand of $(\pi_2)_\pi \cong (I\pi)^2$, we have, using Jacobinski's cancellation theorem **[SE**, Theorem 19.8], that $M_\pi \cong I\pi^2$. □

Note. For any $(\pi, 2)$-complex $Y$, and any isomorphism $\alpha$: $\pi \to \pi_1 Y$ it follows that ${}_\alpha\pi_2(Y) \oplus Z\pi \cong \pi_2 \oplus Z\pi$. These modules are therefore of topological interest. We will show in Theorem 4.2 that the *converse* is also true; that is,

$[\mathfrak{F}] \in \mathcal{G}(E)$ implies that $M$ is stably isomorphic to $\pi_2$.

We identify the ring of endomorphisms of $\pi$ ($=$ End $\pi$) as a subset of $E$.

NOTATION. For each integer $a$, let $\bar{a}_k$ be the residue class of $a$ (mod $k$).

Let

$$\operatorname{End}\pi = \left\{ \alpha = \begin{bmatrix} \bar{a}_n & \bar{b}_n \\ \bar{c}_m & \bar{d}_m \end{bmatrix} \in E = \begin{bmatrix} Z_n & Z_n \\ Z_m & Z_m \end{bmatrix} \middle| n \text{ divides } bm \right\}.$$

Multiplication of two elements in $E$ (as $2 \times 2$-matrices) is well defined iff they are in End $\pi$. Aut $\pi \subset \mathcal{G}(E)$ is the subset of End $\pi \subset E$ consisting of invertible elements. Note that $\alpha(x^j y^i) = x^{cj+di} y^{aj+bi}$ can be computed from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} j \\ i \end{pmatrix} = \begin{pmatrix} aj + bi \\ cj + di \end{pmatrix}$$

(observe that we have interchanged $x$ and $y$).

In general $\mathcal{G}(E)$ is bigger than Aut $\pi$, as $\mathcal{G}(E)$ contains the image of GL(2, $Z$) in $E$. For example,

$$\left[\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 \\ \bar{2}_m & 1 \end{pmatrix}$$

is always in $\mathcal{G}(E)$, but never in Aut $\pi$.

2.4 LEMMA. *The boundary operator* $\partial$: $\operatorname{End}(I\pi)^2 \to E$ *is described by carrying each*

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \qquad (\alpha_{ij} \in Z\pi)$$

*to*

$$\begin{bmatrix} \overline{\varepsilon(\alpha_{11})}_n & \overline{\varepsilon(\alpha_{12})}_n \\ \overline{\varepsilon(\alpha_{21})}_m & \overline{\varepsilon(\alpha_{22})}_m \end{bmatrix}.$$

PROOF. We are thinking of $E$ as $\operatorname{End}(I\pi)^2/B$, where

$$B = \left\{ \alpha \in \operatorname{End}(I\pi)^2 \;\middle|\; \alpha \text{ coextends to } \pi_2\text{:} \begin{tikzcd} & (I\pi)^2 \arrow[dl, dashed, "\bar{\alpha}"'] \arrow[d, "\alpha"] \\ \pi_2 \arrow[r, two heads, "\eta"'] & (I\pi)^2 \end{tikzcd} \right\}.$$

$B$ is always a right ideal, but it is not a left ideal unless $m = n$. The identification of $E$ with $\pi^2$ is accomplished as follows: Identify each element $\alpha$ with a $2 \times 2$ matrix

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

where each $\alpha_{ij} \in Z\pi$. This can be done because End $I\pi \cong Z\pi/(N)$ for any finite group $\pi$. By direct computation one may show that any map $\beta = (\beta_{ij})$ coextends, provided each $\beta_{ij} \in I\pi$. One simply shows directly that, if $E^{ij}$ $(i, j = 1, 2)$ denotes the elementary $2 \times 2$ matrix with a one in the $ij$th slot and zeros elsewhere, then $(x - 1)E^{ij}$ and $(y - 1)E^{ij}$ coextend. The $\beta$ given above is a linear combination of $(x - 1)E^{ij}$ and $(y - 1)E^{ij}$ (because each $\beta_{ij} = \beta'_{ij}(x - 1) + \beta''_{ij}(y - 1)$), and hence coextends. For example, $\beta = \begin{pmatrix} 0 & 0 \\ x-1 & 0 \end{pmatrix}$ coextends by the map $\bar{\beta}$: $(I\pi)^2 \to \pi_2$ given by defining $\bar{\beta}(x-1, 0) = (x-1)(0, -N_y, x-1)$, $\bar{\beta}(y-1, 0) = (y-1)(0, 0, x-1)$, and $\bar{\beta}(0, y-1) = 0 = \bar{\beta}(0, x-1)$. Then $\eta \circ \bar{\beta} = \beta$ and we are done *provided* $\bar{\beta}$ is well defined. Using 2.1, we identify $\mathrm{Hom}_{Z\pi}(I\pi, M)$ with $\{\alpha: (Z\pi)^2 \to M: \alpha|\mathrm{im}\, \partial_2 = 0\}$. It is easy to check that the map $\bar{\alpha}$: $(Z\pi)^2 \to \pi_2$ which sends $(1, 0) \to (0, -(x-1)N_y, (x-1)^2)$ and $(0, 1) \to (0, 0, (y-1)(x-1))$ is zero when restricted to im $\partial_2$. Then $\bar{\beta} = (\bar{\alpha}, 0)$: $(I\pi)^2 \to \pi_2$.

Thus each $\alpha = (\alpha_{ij})$ in $\mathrm{End}(I\pi)^2$ is equivalent mod $B$ to the map $(\varepsilon(\alpha_{ij}))$ with integer entries. One may further show that the matrices $nE^{1j}$ and $mE^{2j}$ coextend ($j = 1, 2$). For example, $\langle y, n\rangle E^{11}$ coextends via a map $(I\pi)^2 \to \pi_2$ defined by carrying $(x-1, 0) \mapsto ((x-1)\langle y, n\rangle, 0, 0)$ and $(y-1, 0)$, $(0, x-1)$, $(0, y-1)$ all to zero. Thus we see that the map $\partial$: $\mathrm{End}(I\pi)^2 \twoheadrightarrow \mathrm{Ext}((I\pi)^2, Z)$ can be described by

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \mapsto \begin{bmatrix} \overline{\varepsilon(\alpha_{11})_n} & \overline{\varepsilon(\alpha_{12})_n} \\ \overline{\varepsilon(\alpha_{21})_m} & \overline{\varepsilon(\alpha_{22})_m} \end{bmatrix}. \quad \square$$

We will isolate several subsets of $\mathcal{G}(E)$, which we then proceed to study. Let

$$S\mathcal{G}(E) = \{[\mathcal{F}_M] \in E | M \oplus Z\pi \cong \pi_2 \oplus Z\pi\},$$
$${}_\theta\mathrm{Iso} = \{[\mathcal{F}_M] \in E | M \cong_\theta \pi_2 \text{ for some } \theta \in \mathrm{Aut}\, \pi\}, \quad \text{and}$$
$$\mathrm{Iso} = \{[\mathcal{F}_M] \in E | M \cong \pi_2\}.$$

Clearly, by 2.3, *any* module $M$ stably isomorphic to $\pi_2$ is already an extension of $Z$ by $I\pi^2$.

The following inclusions hold:

$$\mathcal{G}(E) \supset S\mathcal{G}(E) \supset {}_\theta\mathrm{Iso} \supset \mathrm{Iso}.$$

The first inclusion follows from Lemma 2.3. The last inclusion is clear; the second follows as any module $M$ which is $\theta$-isomorphic to $\pi_2$ may be embedded in the sequence: $0 \to M \to_\theta C_2 \to_\theta C_1 \to_\theta Z\pi \to Z$; thus $M$ is stably isomorphic to $\pi_2$ by Schanuel's lemma.

One can show that $\mathcal{G}(E) = \{[\mathcal{F}_M] \in E | M$ has the same genus as $\pi_2\}$. However, it is not true that any module $M$ of the same genus as $\pi_2$ is an extension of $Z$ by $I\pi^2$.

For future reference, we record the following easily proved characterization of Iso.

2.5 PROPOSITION. *Let* $[\mathcal{F}_M] \in \mathcal{G}(E)$. *Then* $M \cong \pi_2$ *iff* $[\mathcal{F}] = [\mathcal{E}\alpha] \in E$ *for some* $\alpha \in \mathrm{GL}(2, Z\pi/(N)) \subset \mathrm{End}(I\pi)^2$. $\square$

**3. Comparison of ${}_\theta\pi_2$ and $\pi_2\theta$.** In this section we compare the pullback $\pi_2\theta$ and the module ${}_\theta\pi_2$ for any $\theta \in \mathrm{Aut}\, \pi$. We continue the assumption that $m|n$.

Let ${}_\theta\mathcal{E}$ denote the extension $Z \rightarrowtail {}_\theta\pi_2 \twoheadrightarrow {}_\theta I\pi^2 \approx I\pi^2$, with the isomorphism given by $\begin{bmatrix} Z\theta^{-1} & 0 \\ 0 & Z\theta^{-1} \end{bmatrix}$. Recall that any $\theta \in \mathrm{Aut}\, \pi$ can be written as a product $\theta = E_1E_2 \ldots E_kD$ where each $E_i = \begin{bmatrix} 1 & a_i \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ b_i & 1 \end{bmatrix}$ ($a_i, b_i \in Z$ and $n$ divides $ma_i$, $i = 1, \ldots, k$) is an elementary automorphism and $D = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$ ($p, q \in Z$) is a diagonal automorphism, provided $p$ is prime to $n$ and $q$ is prime to $m$ [S, Proposition 6].

DEFINITION. If $\theta = [\begin{smallmatrix}1 & -r\\ 0 & 1\end{smallmatrix}]$ (respectively, $[\begin{smallmatrix}1 & 0\\ -s & 1\end{smallmatrix}]$, $[\begin{smallmatrix}p & 0\\ 0 & q\end{smallmatrix}]$) is a member of Aut $\pi$ (i.e., $mr$ is divisible by $n$) then let

$$\theta^0 = \begin{bmatrix} 1 & 0 \\ rm/n & 1 \end{bmatrix} \quad \left(\text{respectively} \begin{bmatrix} 1 & sn/m \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}\right).$$

Then, for any $\theta \in \operatorname{Aut} \pi$, $\theta^0$ is the element of Aut $\pi$ defined by writing $\theta = E_1 \ldots E_k[\begin{smallmatrix}p & 0\\ 0 & q\end{smallmatrix}]$, where $E_1, \ldots, E_k$ are elementary automorphisms, and setting $\theta^0 = \theta_1^0 \cdots \theta_k^0[\begin{smallmatrix}p & 0\\ 0 & q\end{smallmatrix}]$. Notice that $(\theta^0)^0 = \theta$.

3.1 THEOREM. *For any $\theta \in \operatorname{Aut} \pi$, $[{}_\theta\mathfrak{S}] = [\mathfrak{S}\theta^0]$ and $[\mathfrak{S}\theta] = [{}_{\theta^0}\mathfrak{S}]$.*

PROOF. Let $\theta \in \operatorname{Aut} \pi$ be written as a product of a diagonal automorphism $D = [\begin{smallmatrix}p & 0\\ 0 & q\end{smallmatrix}]$ and a product $E_1 \ldots E_k$ of elementary automorphisms. The theorem will follow from the fact that $[\mathfrak{S}(\alpha\beta)] = [(\mathfrak{S}\alpha)\beta]$ and ${}_{(\alpha\beta)}\pi_2 = {}_\beta({}_\alpha\pi_2)$ provided we can show that the theorem is true for $D$ and $E_i$. We will give only the proof for $\theta = [\begin{smallmatrix}1 & 0\\ -r & 1\end{smallmatrix}]$ (i.e. $\theta(x) = x$, $\theta(y) = x^{-r}y$), as the others are similar.

3.2 LEMMA. *Let $\pi = Z_m \times Z_n$, with $m|n$, and $\theta = [\begin{smallmatrix}1 & 0\\ -r & 1\end{smallmatrix}]$, $\theta^0 = [\begin{smallmatrix}1 & rn/m\\ 0 & 1\end{smallmatrix}]$. Then $[{}_\theta\mathfrak{S}] = [\mathfrak{S}\theta^0]$ in $\operatorname{Ext}(I\pi^2, Z)$.*

PROOF. Let $\beta = \sum_{i=1}^n y^{i-1}\langle x, ri\rangle$, $k = rn/m$, and $\bar\theta = \theta^{-1} = [\begin{smallmatrix}1 & 0\\ r & 1\end{smallmatrix}]$. Straightforward calculation using 2.1 shows that the matrix

$$\begin{bmatrix} 1 & 0 & k \\ 0 & x^r & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

defines a map from $C_2 = Z\pi e_1 \oplus Z\pi e_2 \oplus Z\pi e_2 \to {}_{\bar\theta}C_2 = {}_{\bar\theta}Z\pi e_1 \oplus {}_{\bar\theta}Z\pi e_2 \oplus {}_{\bar\theta}Z\pi e_3$ (i.e., $e_1 \to (1, 0, 0)$, $e_2 \to (0, x^r, 0)$, $e_3 \to (k, \beta, 1)$ and extend linearly) *which sends $\pi_2$ ($\subset C_2$) into ${}_{\bar\theta}\pi_2$ ($\subset {}_{\bar\theta}C_2$).* This same matrix defines a map from ${}_\theta C_2 \to {}_\theta({}_{\bar\theta}C_2) = C_2$ and hence a map $\theta_2$: ${}_\theta\pi_2 \to \pi_2$. On the generators for ${}_\theta\pi_2$, $\theta_2$ looks like

$$\begin{aligned}
&\bar\theta(x-1) * {}_\theta e_1 = (x-1)e_1 \\
&\qquad \mapsto \bar\theta(x-1)e_1, \bar\theta(y-1) * {}_\theta e_1 + \bar\theta(N_x) * {}_\theta e_2 = (y-1, N_x, 0) \\
&\qquad \mapsto \bar\theta(y-1)e_1 + x^r\bar\theta(N_x)e_2, \bar\theta(-N_y) * {}_\theta e_2 + \bar\theta(x-1) * {}_\theta e_3 \\
&\qquad = (0, -N_y, x-1) \\
&\qquad \mapsto k\bar\theta(x-1)e_1 + \big(-x^r\bar\theta(N_y) + \bar\theta(x-1)\beta\big)e_2 + \bar\theta(x-1)e_3
\end{aligned}$$

and finally,

$$\bar\theta(y-1) * {}_\theta e_3 = (y-1)e_3 \mapsto k\bar\theta(y-1)e_1 + \beta\bar\theta(y-1)e_2 + \bar\theta(y-1)e_3.$$

Inspection shows that $\theta_2$ induces a map $\theta_2'$: ${}_\theta(I\pi)^2 \to (I\pi)^2$ with matrix $[\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}]$. Here, 1: ${}_\theta I\pi \to I\pi$ is the map induced by $Z\bar\theta$: ${}_\theta Z\pi \to Z\pi$. Let $Z\theta$ denote the isomorphism $I\pi \to {}_\theta I\pi$ induced from $Z\theta$: $Z\pi \to {}_\theta Z\pi$. The composition

$$(I\pi)^2 \xrightarrow{\begin{pmatrix} Z\theta & 0 \\ 0 & Z\theta \end{pmatrix}} {}_\theta(I\pi)^2 \xrightarrow{\theta_2'} (I\pi)^2$$

yields a map with matrix $[\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}]$. □

**4. Characterizing** $\mathcal{G}(E)$. We will now characterize the set $\mathcal{G}(E)$ of generators of $E = \operatorname{Ext}(I\pi^2, Z)$. For each $\alpha = (\alpha_{ij}) \in \operatorname{End}(I\pi)^2$ ($\alpha_{ij} \in Z\pi$), let $\varepsilon(\alpha)$ denote the integer matrix with entries $\varepsilon(\alpha_{ij})$ ($i, j = 1, 2$).

4.1 PROPOSITION. *Let* $\alpha = (\alpha_{ij}) \in \operatorname{End}(I\pi)^2$ ($\alpha_{ij} \in Z\pi$). *The following are equivalent*:

(a) $[\alpha] \in \mathcal{G}(E)$.

(b) $\exists \alpha' \in \operatorname{End}(I\pi)^2$ *such that* $\alpha\alpha' - 1 \in B$.

(c) *The determinant of* $\varepsilon(\alpha)$ *is prime to* $m$ *and there are integers* $s$, $t$ *such that* $\varepsilon(\alpha_{11})s + \varepsilon(\alpha_{12})t \equiv 1 \pmod{n}$.

(d) *There is an integer matrix* $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ *such that*

$$\bar{\gamma}_n = \begin{bmatrix} \bar{a}_n & \bar{b}_n \\ \bar{c}_n & \bar{d}_n \end{bmatrix} \in \operatorname{GL}(2, Z_n)$$

*and* $[\alpha] = [\gamma] \in E$.

PROOF. (a) $\Rightarrow$ (b). If $[\mathcal{E}\alpha] \in \mathcal{G}(E)$, then $[\mathcal{E}]$ itself must be a pullback of $Z \rightarrowtail \pi_2\alpha \twoheadrightarrow I\pi^2$ by some $\alpha'$: $I\pi^2 \to I\pi^2$. Thus $[\mathcal{E}] = [\mathcal{E}\alpha\alpha']$ implies $\alpha\alpha' - 1 \in B$.

(b) $\Rightarrow$ (c). It is easy to see that the correspondence $[\alpha] \in E \mapsto \det \varepsilon(\alpha) \in Z$ is well defined modulo $m$. $\alpha\alpha' - 1 \in B$ implies $\det \varepsilon(\alpha) \cdot \det \varepsilon(\alpha') \equiv 1 \pmod{m}$. Thus $\det \varepsilon(\alpha)$ is prime to $m$. Furthermore, $\varepsilon(\alpha_{11})\varepsilon(\alpha'_{11}) + \varepsilon(\alpha_{12})\varepsilon(\alpha'_{21}) - 1 \equiv 0 \pmod{n}$ follows by looking at the 11-coordinate of $\partial(\alpha\alpha' - 1)$.

(c) $\Rightarrow$ (d). Because $m$ divides $n$, the natural map $\bar{a}_n \mapsto \bar{a}_m$ ($a \in Z$) induces a surjection $Z_n^* \to Z_m^*$. Let $d = \det \varepsilon(\alpha)$. $d$ is prime to $m$, so there is an integer $l$ such that $d + lm$ is prime to $n$. $s\varepsilon(\alpha_{11}) + t\varepsilon(\alpha_{12}) = 1 + kn$ then yields $d + l(1 + kn)m$ is prime to $n$. Now consider the integer matrix

$$\gamma = \begin{bmatrix} \varepsilon(\alpha_{11}) & \varepsilon(\alpha_{12}) \\ \varepsilon(\alpha_{21}) - ltm & \varepsilon(\alpha_{22}) + lsm \end{bmatrix}.$$

We claim that $\gamma$ represents an element of $\operatorname{GL}(2, Z_n)$; i.e., that $\det \gamma$ is prime to $n$. Consider

$$\det \gamma = d + lm(\varepsilon(\alpha_{11}) \cdot s + \varepsilon(\alpha_{12}) \cdot t) = d + lm(1 + kn),$$

which is prime to $n$. Clearly $[\gamma] = [\alpha]$, by Lemma 2.4.

(d) $\Rightarrow$ (a). Choose an integer matrix $\gamma$ so that $[\gamma] = [\alpha]$ in $E$ and $\bar{\gamma}_n \in \operatorname{GL}(2, Z_n)$. Let $\gamma'$ be an integer matrix which represents the inverse $(\bar{\gamma}_n)^{-1}$. Then $\overline{(\gamma\gamma' - 1)}_n = 0$ implies that $\partial(\gamma\gamma' - 1) = 0$. We claim that $\alpha\gamma' - 1 \in B$. Choose $b \in B$ so that $\alpha = \gamma + b$. Then $\alpha\gamma' - 1 = (\gamma + b)\gamma' - 1 = \gamma\gamma' - 1 + b\gamma'$. But $B$ is a *right* ideal, so $b\gamma' \in B$. Hence $\alpha\gamma' - 1 \equiv \gamma\gamma' - 1 \equiv 0 \pmod{B}$ and $\alpha \in \mathcal{G}(E)$. □

4.2 THEOREM. $\mathcal{G}(E) = {}_\theta\mathrm{Iso}$.

PROOF. By Proposition 4.1, given any $[\alpha] \in \mathcal{G}(E)$ there is a $2 \times 2$ integer matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ whose determinant is prime to $n$, and such that $[A] = [\alpha] \in E$. Consider $\bar{A}_n$ as the matrix mod $n$, that is to say, $A$ represents an element of $\operatorname{GL}(2, Z_n) = \operatorname{Aut}(Z_n \times Z_n)$. By Proposition 6 of [S], $A \equiv E_1E_2 \ldots E_kD \pmod{n}$ where $D = \left[\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix}\right]$ is a diagonal automorphism ($p$, $q$ are prime to $n$) and each $E_i$ is an elementary matrix of the form $\left[\begin{smallmatrix}1 & \gamma_i\\ 0 & 1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1 & 0\\ \gamma_i & 1\end{smallmatrix}\right]$, with $p, q, \gamma_i \in Z$. Let $M = E_1E_2 \cdots E_kD$ be the integer matrix which is the product of $E_1, E_2, \ldots, E_k$, and $D$. Then

$$\begin{aligned}[\mathcal{E}\alpha] &= [\mathcal{E}A] = [\mathcal{E}M] \quad (\text{because } \alpha \equiv A \equiv M \bmod B)\\ &= [(\ldots((\mathcal{E}E_1)E_2)\cdots E_k)D]\end{aligned}$$

(because pullbacks commute with composition).

But, by Proposition 2.5

$$\pi_2 \cong \pi_2E_1 \cong \pi_2E_1E_2 \cong \cdots \cong \pi_2E_1\cdots E_k$$

because each $E_i \in \mathrm{Aut}(I\pi)^2$. $D$ is a member of Aut $\pi$ implies that

$$\pi_2 \cong \pi_2E_1\cdots E_k \cong {}_D\pi_2M \cong {}_D\pi_2\alpha$$

by invoking Theorem 3.1. □

We have seen in §2 that the map $\bar{\partial} = \partial|\mathrm{GL}(2, Z\pi/(N))\colon \mathrm{GL}(2, Z\pi/(N)) \to \mathcal{G}(E)$ is onto the subset Iso. We would like to determine im $\bar{\partial}$. As above, each elementary automorphism $\alpha \doteq \left[\begin{smallmatrix}1 & \gamma\\ 0 & 1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}1 & 0\\ \gamma & 1\end{smallmatrix}\right]$ yields $\pi_2\alpha \cong \pi_2$. The diagonal automorphisms $\beta = \left[\begin{smallmatrix}p & 0\\ 0 & q\end{smallmatrix}\right]$ $(p, q \in Z)$ of $\pi$ which are clearly in im $\bar{\partial}$ have $\bar{p}_{mn}, \bar{q}_{mn} \in \ker\{\partial\colon Z_{mn}^* \to \tilde{K}_0Z\pi\}$; for, choosing units mod $N$ $u, v \in Z\pi/(N)^*$ such that $\varepsilon(u) = p$, $\varepsilon(v) = q$ [4, 2.1], we have $\alpha = \left[\begin{smallmatrix}u & 0\\ 0 & v\end{smallmatrix}\right] \in \mathrm{GL}(2, Z\pi/(N))$ and $\bar{\partial}\alpha = \beta$. We make the following convenient hypothesis [**SD**, §4].

Reduction hypothesis. The natural projection $Z_{mn}^* \to Z_n^*$ remains surjective when restricted to ker $\partial$.

The reduction hypothesis is satisfied whenever $n = m = p$, an odd prime [**U**]. Whenever the reduction hypothesis is satisfied, the above argument implies that $\bar{\partial}$ is surjective.

4.3 Corollary. *If the reduction hypothesis is true, then each* $[\alpha] \in \mathcal{G}(E)$ *has* $\pi_2\alpha \cong \pi_2$; *i.e.*, $\mathcal{G}(E) = \mathrm{Iso}$. □

**5. Shuffling $k$-invariants.** Recall that $\pi = Z_m \times Z_n$, with $m|n$ and generators $x, y$ of order $m$ and $n$, respectively. For each $\pi$-module $M$, $H^3(\pi; M)$ is isomorphic to $\mathrm{Hom}_{Z\pi}(\pi_2, M)/\mathcal{B}$, where $\mathcal{B} = \{\beta\colon \pi_2 \to M \mid \beta$ extends to a map $\bar{\beta}\colon C_2 \to M\}$. For each $\alpha \in \mathrm{Hom}_{Z\pi}(\pi_2, M)$, let $\{\alpha\} = \alpha + \mathcal{B}$ be the class of $\alpha$ in $H^3(\pi; M)$. If $\theta$ is a member of Aut $\pi$, then $\theta^*\colon H^*(\pi; \pi_2) \to H^*(\pi; {}_\theta\pi_2)$ may be computed by choosing a chain map from $\mathcal{C}_*(\tilde{X}_\mathcal{P}) \to {}_\theta\mathcal{C}_*(\tilde{X}_\mathcal{P})$ covering the identity.

$$\begin{array}{ccccccccc} & & C_2 & & & & & & \\ & & \| & & & & & & \\ \pi_2 & \to & (Z\pi)^3 & \xrightarrow{\partial_2} & Z\pi^2 & \xrightarrow{\partial_1} & Z\pi & \to & Z\\ \downarrow\theta_2 & & \downarrow\psi & & \downarrow & & \downarrow Z\theta & & \|\\ {}_\theta\pi_2 & \to & {}_\theta(Z\pi)^3 & \xrightarrow{\partial_2} & {}_\theta Z\pi^2 & \xrightarrow{\partial_1} & {}_\theta Z\pi & \to & Z \end{array}$$

It follows that for any $\alpha \in \mathrm{End}\,\pi_2$, $\theta^*\{\alpha\} = \{\theta_2\alpha\} \in H^3(\pi; {}_\theta\pi_2)$.

If $\theta = [\begin{smallmatrix}1 & 0\\ 0 & p\end{smallmatrix}]$, where $p$ is prime to $m$ (i.e., $\theta(x) = x^p$, $\theta(y) = y$), then

$$\psi = \begin{bmatrix} \langle x,p\rangle & 0 & 0 \\ 0 & \langle x,p\rangle & 0 \\ 0 & 0 & 1 \end{bmatrix}: (Z\pi)^3 \to {}_\theta(Z\pi)^3 \text{ induces } \theta_2\colon \pi_2 \to {}_\theta\pi_2.$$

The following lemma will prove useful.

5.1 LEMMA. *Let $v \in Z\pi$ be a unit* (mod $N$) *having augmentation $\varepsilon(v) = p$. Then there is a unit* (mod $N$) *$u \in Z\pi$ having $\varepsilon(u) = p$ and such that $u = \langle x,p\rangle + (y-1)\alpha$, for some $\alpha \in Z\pi$.*

PROOF. For each $v \in Z\pi$, let $v_y \in Z(Z_m(x))$ denote the image of $v$ under the map $\varepsilon_y\colon Z\pi \to Z(Z_m)$ obtained by setting $y = 1$. $v$ is a unit mod $N$ implies $v_y$ is a unit mod $N_x$, which in turn implies that $v_y = w\langle x,p\rangle$, where $w$ is a *unit* in $Z(Z_m)$. $w$ is also a unit in $Z\pi$ so $u = w^{-1}v$ has $u_y = \langle x,p\rangle$. Hence $u - \langle x,p\rangle$ is in the kernel of the map $\varepsilon_y$. It is easy to see that $u - \langle x,p\rangle = (y-1)\alpha$ for some $\alpha \in Z\pi$. □

Recall the homomorphism $\partial\colon Z^*_{mn}$ (= units in $H^3(\pi;\pi_2)) \to \tilde{K}_0 Z\pi$ given by $\partial(\bar{p}_{mn}) = \{(p,N)\} \in \tilde{K}_0 Z\pi$. It is known that $\bar{p}_{mn} \in \ker\partial$ iff there is a $v \in Z\pi$ which is a unit mod $N$ whose augmentation $\varepsilon(v) = p$, [$\mathbf{D}_2$, 2.1]. A $\theta$-homomorphism $\alpha\colon \pi_2 \to_\theta \pi_2$ has degree $k \in H^3(\pi;\pi_2)$ iff $k\{\theta_2\} = \{\alpha\}$. *For each $k \in \ker\partial$, we will construct a $\theta$-isomorphism $\pi_2 \to_\theta \pi_2$* of degree $k$. This will prove Theorem B. We commence the proof.

Given $\bar{k}_{mn} \in \ker\partial \subset Z^*_{mn}$. Choose $u \in Z\pi$ such that $\varepsilon(u) = k$ and $u$ is a unit mod $N$. $k$ is prime to $mn$ implies that $k$ is prime to $m^2n$. Choose an integer $p$ such that $pk + sm^2n = 1$. Let $\theta$ be the automorphism which carries $x \to x^p$, $y \to y$. $\bar{p}_{mn} = \bar{k}^{-1}_{mn}$ in $Z^*_{mn}$ implies $\bar{p}_{mn} \in \ker\partial$. Thus there is a $v \in Z\pi$ such that $\varepsilon(v) = p$, $v$ is a unit mod $N$, and $v = \langle x,p\rangle + (y-1)\alpha$ for some $\alpha \in Z\pi$.

Let $e_i$ ($i = 1,2,3$) denote the natural basis for $(Z\pi)^3$. Define homomorphisms $p_{ij}\colon (Z\pi)^3 \to_\theta (Z\pi)^3$ sending $e_i \to g_j$ ($g_j$ is a generator of $\pi_2$, hence, of ${}_\theta\pi_2$, see §2) and $e_k \to 0$ ($k \neq i$) ($i = 1,2,3$; $j = 1,2,3,4$). Note that $\bar{p}_{ij} = p_{ij}|\pi_2$ defines a degree 0 map from $\pi_2 \to_\theta \pi_2$.

Consider the map $M = \psi + \alpha p_{12}\colon (Z\pi)^3 \to_\theta (Z\pi)^3$. $M$ has matrix

$$\begin{bmatrix} v = \langle x,p\rangle + (y-1)\alpha & 0 & 0 \\ \alpha N_x & \langle x,p\rangle & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\overline{M} = M|_{\pi_2}\colon \pi_2 \to_\theta \pi_2$ carries $g_1 \mapsto (x^p - 1)(v,0,0)$, $g_2 \mapsto (v(y-1), \alpha N_x(y-1) + pN_x, 0)$, $g_3 \mapsto (0, -N_y\langle x,p\rangle, x^p - 1)$, and $g_4 \mapsto (0,0,y-1)$. The map $\overline{\overline{M}}\colon (I\pi)^2 \to_\theta (I\pi)^2$ induced by $\overline{M}$ then has matrix $[\begin{smallmatrix}v & 0\\ 0 & 1\end{smallmatrix}]$ and is an isomorphism because $v$ is a unit (mod $N$).

We will now alter $\overline{M}$ to give an isomorphism $\pi_2 \to_\theta \pi_2$ of degree $\bar{k}_{mn}$. Look at $Q = uM + sNp_{22}\colon (Z\pi)^3 \to_\theta (Z\pi)^3$. $Q$ then has matrix

$$\begin{bmatrix} uv & 0 & 0 \\ \alpha u N_x & u\langle x,p\rangle + smN & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This clearly restricts to a map $\bar{Q}\colon \pi_2 \to_\theta \pi_2$ having degree $\bar{k}_{mn}$ because $\{\bar{Q}\} = \{u\theta_2 + u\alpha\bar{p}_{12} + sN\bar{p}_{22}\} = \{u\theta_2\} = \{\varepsilon(u)\theta_2\} = k\{\theta_2\}$. Clearly $\bar{Q}$ induces the same map as $u\bar{M}$ on $I\pi^2 \to_\theta I\pi^2$ ($sN\bar{p}_{22}$ restricted to $(I\pi)^2$ is zero) and is an isomorphism. We need only show that $Q|_{\pi_2^*}$ is the identity: $Q(0, N, 0) = (u\langle x, p\rangle + smN)(0, N, 0) = (kp + sm^2n)(0, N, 0) = (0, N, 0)$. □

## References

[CE] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

[$D_1$] M. Dyer, *Homotopy classification of* $(\pi, m)$*-complexes*, J. Pure Appl. Algebra **7** (1976), 429–482.

[$D_2$] ______, *On the essential height of homotopy trees with finite fundamental group*, Compositio Math. **36** (1978), 209–224.

[HS] P. Hilton and U. Stammbach, *A course in homological algebra*, Springer-Verlag, New York, 1970.

[S] A. Sieradski, *Combinatorial isomorphisms and combinatorial homotopy equivalences*, J. Pure Appl. Algebra **7** (1976), 59–95.

[SD] A. Sieradski and M. Dyer, *Distinguishing arithmetic for certain stably isomorphic modules*, J. Pure Appl. Algebra **15** (1979), 199–217.

[SE] R. G. Swan and E. G. Evans, *K-theory of finite groups and orders*, Lecture Notes in Math., vol. 149, Springer-Verag, Berlin and New York, 1970.

[U] S. Ullom, *Non-trivial lower bounds for class groups of integral group rings*, Illinois J. Math. **20** (1976), 314–329.

Department of Mathematics, University of Oregon, Eugene, Oregon 97403