

QUASI-LINEAR EVOLUTION EQUATIONS IN BANACH SPACES¹

BY

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ABSTRACT. This paper is concerned with the quasi-linear evolution equation $u'(t) + A(t, u(t))u(t) = 0$ in $[0, T]$, $u(0) = x_0$ in a Banach space setting. The spirit of this inquiry follows that of T. Kato and his fundamental results concerning linear evolution equations. We assume that we have a family of semigroup generators that satisfies continuity and stability conditions. A family of approximate solutions to the quasi-linear problem is constructed that converges to a "limit solution." The limit solution must be the strong solution if one exists. It is enough that a related linear problem has a solution in order that the limit solution be the unique solution of the quasi-linear problem. We show that the limit solution depends on the initial value in a strong way. An application and the existence aspect are also addressed.

This paper is concerned with the quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = 0 \quad \text{in } [0, T], \quad u(0) = x_0$$

in a Banach space setting.

The spirit of this inquiry follows that of T. Kato. Kato wrote a fundamental paper on linear evolution equations in 1953 [9]; that is, investigation of

$$u'(t) + A(t)u(t) = 0 \quad \text{on } [0, T], \quad u(0) = x_0.$$

He strengthened and extended his analysis of the linear problem in 1970 [11]. Kato also wrote on the quasi-linear problem in 1975 [13]. We feel that our results give a natural approach to dealing with the quasi-linear problem.

After discussing the setting and method of attack, our theorem is stated and proved. We then give an application of the theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type. A proposition relevant to our theorem is also given.

Let X and Y be Banach spaces, with Y densely and continuously embedded in X . Let $x_0 \in Y$, $T > 0$, $r > r_1 > 0$, $r_2 > 0$, $W = \bar{B}_X(x_0; r)$, $Z = B_X(x_0; r_1) \cap B_Y(x_0; r_2)$, and for each $t \in [0, T]$ and $w \in W$, let $-A(t, w)$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in X , with $Y \subset D(A(t, w))$.

We consider the quasi-linear evolution equation

$$v'(t) + A(t, v(t))v(t) = 0. \quad (\text{QL})$$

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Given a function u from $[0, T']$ into W , where $0 < T' \leq T$, we can also consider the linearized evolution equation

$$v'(t) + A(t, u(t))v(t) = 0. \tag{L; u}$$

By a solution (or strong solution) of (QL) or (L; u) on $[0, T']$, we mean a function v on $[0, T']$ to W which is absolutely continuous (X) and differentiable (X) a.e., such that $v(t) \in Y$ a.e., $\text{ess sup}\{\|v(t)\|_Y\} < \infty$, and v satisfies the appropriate equation, (QL) or (L; u), a.e. on $[0, T']$.

Our method is to produce, for each $x_1 \in Z$, a "limit solution" u with initial value x_1 on an interval $[0, T']$, where $T' \in (0, T]$ is independent of x_1 . For a partition $\Delta = \{t_0, t_1, \dots, t_N\}$ of $[0, T']$, we use an iterative procedure to produce a Lipschitz continuous (X) function u_Δ which satisfies

$$u'_\Delta(t) + A(t_i, u_\Delta(t_i))u_\Delta(t) = 0 \quad \text{for } t \in (t_i, t_{i+1})$$

and $i \in \{0, 1, \dots, N - 1\}$, with $u_\Delta(0) = x_1$. This u_Δ is shown to be the time-ordered juxtaposition of the semigroups generated by the $-A(t_i, u_\Delta(t_i))$. These approximate solutions converge uniformly, as $|\Delta|$ goes to 0, to give the limit solution u . We show, in particular, that if $v = w$ is a solution of (QL) or (L; u) on $[0, T']$ with initial value x_1 , then $w = u$. Thus, subject to an initial value, a solution of (QL) is unique if it exists, and whenever the linearized equation (L; u) has a solution, then so does the quasi-linear equation (QL). There are known conditions which are sufficient in order that (L; u) has a solution.

Our hypotheses form a natural extension of Kato's assumptions for linear equations [11].

The term "limit solution" seems as appropriate as any to describe the function obtained by the iterative procedure; e.g., see Kobayashi [16] or Crandall and Evans [3].

THEOREM. *Assume that*

(i) $\{A(t, w)\}$ is stable in X with constants of stability M, β ; i.e.,

$$\begin{aligned} &\|(A(t_k, w_k) + \lambda)^{-1}(A(t_{k-1}, w_{k-1}) + \lambda)^{-1} \dots (A(t_1, w_1) + \lambda)^{-1}\|_X \\ &\qquad < M(\lambda - \beta)^{-k}, \end{aligned}$$

$\lambda > \beta$, for any finite family $\{(t_j, w_j)\}$, $0 \leq t_1 \leq \dots \leq t_k \leq T$, $k = 1, 2, \dots$

(ii) $Y \subset D(A(t, w))$ for each (t, w) , which implies that $A(t, w) \in B(Y, X)$, and the map $(t, w) \rightarrow A(t, w)$ is Lipschitz continuous with constant C_1 ; i.e.,

$$\|A(t_2, w_2) - A(t_1, w_1)\|_{Y, X} \leq C_1(|t_2 - t_1| + \|w_2 - w_1\|_X).$$

(iii) There is a family $\{S(t, w)\}$ of isomorphisms of Y onto X such that $S(t, w)A(t, w)S(t, w)^{-1} = A_1(t, w)$ is the negative of the infinitesimal generator of a strongly continuous semigroup in X for each (t, w) , and $\{A_1(t, w)\}$ is stable in X , with constants of stability M_1, β_1 . Furthermore, there is a constant C_2 such that $\|S(t, w)\|_{Y, X} \leq C_2$, $\|S(t, w)^{-1}\|_{X, Y} \leq C_2$, and the map $(t, w) \rightarrow S(t, w)$ is Lipschitz continuous with constant C_3 (see (ii) above).

Then, there exists a T' , with $0 < T' \leq T$, such that for each $x_1 \in Z$ and partition $\Delta = \{t_0, t_1, \dots, t_n\}$ of $[0, T']$, we can find a function u_Δ which is Lipschitz continuous

(X) on $[0, T']$ to W , Y -bounded, and satisfies $u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t) = 0$ for $t \in (t_i, t_{i+1})$ and $i \in \{0, 1, \dots, n - 1\}$, with $u_\Delta(0) = x_1$. In fact, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\Delta| < \delta$ implies that $\|u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X < \varepsilon$ except at t_1, \dots, t_n . Further, the u_Δ converge uniformly, as $|\Delta|$ goes to 0, to a Lipschitz continuous (X) function u on $[0, T']$ to W which has initial value x_1 and is bounded, independent of x_1 , in the relative completion of Y in X (the set of all points in X that are the limit in X -norm of sequences from Y that are bounded in Y -norm).

If $x_2 \in Z$ and w is constructed as above but with initial value x_2 , then $\|u(t) - w(t)\|_X \leq C\|x_1 - x_2\|_X$ for $t \in [0, T']$, with C independent of x_1 and x_2 .

Now, if v is a solution of (QL) or (L; u) on $[0, T'']$, where $0 < T'' < T'$, with initial value x_1 , then $v = u$ on $[0, T'']$, and thus solutions to (QL) or (L; u) are uniquely determined by their initial values.

COROLLARY 1. *If Y is reflexive, then (L; u) has a solution on $[0, T']$ with initial value x_1 , and thus u is a solution of (QL) on $[0, T']$ with initial value x_1 .*

REMARKS. (1) If $D(A(t, w)) = Y$ for each (t, w) and there is a $\lambda > \beta$ such that $\|(\lambda I + A(t, w))^{-1}\|_{X,Y} \leq C_2$ and $\|\lambda I + A(t, w)\|_{Y,X} \leq C_2$ for each (t, w) , then (iii) is satisfied with $S(t, w) = \lambda I + A(t, w)$.

(2) If Y is $A(t, w)$ -admissible ($\{\exp(-sA(t, w))\}$ takes Y to Y and forms a strongly continuous semigroup on Y) for each (t, w) and $\{A(t, w)\}$ is stable in Y , then (iii) is unnecessary.

We now begin to prove the Theorem. The proofs of the above remarks and Corollary will be given later.

Let $T^0 = \min(T, r/\|A\|Me^{\beta T}(\|x_0\|_Y + r_2))$, where $\|A\| = \sup\{\|A(t, w)\|_{Y,X} : t \in [0, T], w \in W\}$ which is finite by (ii). Let $K = C_2C_3M_1T^0$ and

$$T' = T^0 / (1 + \|A\|C_2^2M_1e^{K+\beta_1T}(\|x_0\|_Y + r_2)).$$

LEMMA A. *If u is Lipschitz continuous (X) on $[0, T']$ to W with Lipschitz constant*

$$\|A\|C_2^2M_1e^{K+\beta_1T'}(\|x_0\|_Y + r_2),$$

then $\{A(t, u(t)) : t \in [0, T']\}$ is Y -stable with constants $C_2^2M_1e^K$ and β_1 .

PROOF OF LEMMA A. We use Kato's Proposition 4.4 [11] with $S(t) = S(t, u(t))$. Then we estimate the variation of S by

$$V_S \leq C_3(1 + \|A\|C_2^2M_1e^{K+\beta_1T'}(\|x_0\|_Y + r_2))T' \leq C_3T^0,$$

whence $\{A(t, u(t))\}$ is Y -stable with constants $C_2^2M_1e^{C_2M_1C_3T^0} = C_2^2M_1e^K$ and β_1 . This completes the proof of Lemma A.

By an evolution operator $\{W(t, s) : 0 \leq s \leq t \leq T'\}$ generated by $\{\mathcal{Q}(t) : t \in [0, T']\} \subset \{A(t, w) : t \in [0, T'], w \in W\}$ and a partition $\Delta = \{t_0, \dots, t_N\}$ of $[0, T']$, we mean the family of operators obtained by forming a time-ordered juxtaposition of the semigroups generated at the points of the partition; e.g., for $t \in [t_i, t_{i+1}]$, $s \in [t_j, t_{j+1}]$, $s \leq t$,

$$W(t, s) = \exp(-(t - t_i)\mathcal{Q}(t_i))\exp(-(t_i - t_{i-1})\mathcal{Q}(t_{i-1})) \dots \exp(-(t_{j+1} - s)\mathcal{Q}(t_j)).$$

It follows from (i) and Kato's Proposition 3.3 [11] that $\|W(t, s)\|_X < Me^{\beta(t-s)}$. If $\{\mathcal{Q}(t)\}$ is Y -stable with constants $\tilde{M}, \tilde{\beta}$, then $W(t, s)Y \subset Y$ and $\|W(t, s)\|_Y < \tilde{M}e^{\tilde{\beta}(t-s)}$ as a result of (iii) and Kato's Propositions 2.4 and 3.3 [11]. Let $\bar{t} = t_i$ if $t \in [t_i, t_{i+1})$, $i \neq N$, and $\bar{t}_N = t_N$. If $f(t) = W(t, 0)x_1$ on $[0, T']$, then f satisfies $f'(t) + \mathcal{Q}(\bar{t})f(t) = 0$ for $t \notin \Delta$, with $f(0) = x_1$. The construction of an evolution operator from a family of semigroup generators and a partition, the notation \bar{t} , and the other results above will be used from this point on without further discussion.

LEMMA B. *Suppose $\{\mathcal{Q}(t): t \in [0, T']\}$ is Y -stable with constants \tilde{M} and $\tilde{\beta}$, and that $\{W(t, s)\}$ is generated by $\{\mathcal{Q}(t)\}$ and a partition Δ of $[0, T']$. Then, $f(t) = W(t, 0)x_1$ is Lipschitz continuous (X) with Lipschitz constant $\|A\|\tilde{M}e^{\tilde{\beta}T'}(\|x_0\|_Y + r_2)$.*

The result is also true if $\{W(t, s)\}$ is the evolution operator of Kato's Theorem 4.1 [11].

PROOF OF LEMMA B. For the partition case, since $f'(t) = -\mathcal{Q}(\bar{t})f(t)$ except for $t \in \Delta$, we get for $s < t$

$$\begin{aligned} \|f(t) - f(s)\|_X &= \left\| -\int_s^t A(\bar{\xi})f(\xi) d\xi \right\|_X < \|A\| \|f\|_Y |t - s| \\ &< \|A\| \tilde{M}e^{\tilde{\beta}T'} \|x_1\|_Y |t - s| < \|A\| \tilde{M}e^{\tilde{\beta}T'} (\|x_0\|_Y + r_2) |t - s|. \end{aligned}$$

Now, the f on $[0, T']$ obtained from Kato's evolution operator is the uniform (X) limit of the f corresponding to the partitions Δ as $|\Delta| \rightarrow 0$. This establishes the result in the second case and completes the proof of Lemma B.

Together, Lemma A and Lemma B suggest an iteration scheme. We fix x_1 and Δ , then obtain sequences $\{u_n\}$, $\{A_n(t)\}$, and $\{U_n(t, s)\}$, with $A_n(t) = A(t, u_n(t))$, $\{U_{n+1}(t, s)\}$ the evolution operator generated by $\{A_n(t)\}$ and Δ , and $u_{n+1}(t) = U_{n+1}(t, 0)x_1$. Once Lemma A is satisfied, we have $\{A_n(t): t \in [0, T']\}$ is Y -stable with constants $C_2^2 M_1 e^K$ and β_1 ; then, Lemma B applied to $\{\mathcal{Q}(t)\} = \{A_n(t)\}$, $\tilde{M} = C_2^2 M_1 e^K$ and $\tilde{\beta} = \beta_1$, implies that u_{n+1} is Lipschitz continuous (X) on $[0, T']$ with Lipschitz constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$. Assuming $u_{n+1}[0, T'] \subset W$, the stage is set to apply Lemma A to $\{A_{n+1}(t): t \in [0, T']\}$ and continue the process.

We now work with a fixed partition Δ of $[0, T']$ and fixed $x_1 \in Z$.

Let $A_0(t) = A(t, x_1)$ for $t \in [0, T']$ and let $\{U_1(t, s)\}$ be the evolution operator generated by $\{A_0(t)\}$ and Δ . Define $u_1(t) = U_1(t, 0)x_1$. Then, $u_1'(t) + A_0(\bar{t})u_1(t) = 0$ except at t_1, t_2, \dots, t_N . Also,

$$\begin{aligned} \|u_1(t) - x_1\|_X &= \|U_1(t, t)x_1 - U_1(t, 0)x_1\|_X \\ &= \left\| \int_0^t U_1(t, s)A_0(\bar{s})x_1 ds \right\|_X \\ &< Me^{\beta T'} \|A\| (\|x_0\|_Y + r_2) t < r \end{aligned}$$

by the choice of T^0 and T' . So, $u_1(t) \in W$ for each $t \in [0, T']$. This argument also works for all the following u_n , $n = 2, 3, \dots$

To start the procedure, we apply Lemma A to $u \equiv x_1$ and then Lemma B with $\{\mathcal{Q}(t)\} = \{A_0(t)\}$, $\tilde{M} = C_2^2 M_1 e^K$ and $\tilde{\beta} = \beta_1$, proving that u_1 is Lipschitz continuous (X) on $[0, T']$ with the Lipschitz constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$.

For the next iteration, let $A_1(t) = A(t, u_1(t))$ for $t \in [0, T']$ and let $\{U_2(t, s)\}$ be the evolution operator generated by $\{A_1(t)\}$ and Δ . Define $u_2(t) = U_2(t, 0)x_1$. Then, $u_2'(t) + A_1(\bar{t})u_2(t) = 0$ except at t_1, t_2, \dots, t_N . As with u_1 , $u_2(t) \in W$ for each $t \in [0, T']$.

As we commented before, we can continue in like manner. For convenience of notation, let $M_2 = C_2^2 M_1 e^K$. Then, for $n > 1$, we have

$$\begin{aligned} \|u_{n+1}(t) - u_n(t)\|_X &= \|U_{n+1}(t, 0)x_1 - U_n(t, 0)x_1\|_X \\ &= \left\| -\int_0^t U_{n+1}(t, s)(A_n(\bar{s}) - A_{n-1}(\bar{s}))U_n(s, 0)x_1 ds \right\|_X \\ &\leq M e^{\beta T'} C_1 M_2 e^{\beta_1 T'} (\|x_0\|_Y + r_2) \cdot \int_0^t \|u_n(\bar{s}) - u_{n-1}(\bar{s})\|_X ds \\ &\leq (M M_2 e^{(\beta+\beta_1)T'} C_1 (\|x_0\|_Y + r_2))^n \cdot \int_0^t \dots \int_0^t \|u_1(\bar{s}) - x_0\|_X ds \\ &\leq \frac{(M M_2 e^{(\beta+\beta_1)T'} C_1 (\|x_0\|_Y + r_2) T')^n}{n!} r. \end{aligned}$$

It follows that there exists a continuous function u_Δ on $[0, T']$ to W such that $u_n \rightarrow u_\Delta$ uniformly on $[0, T']$ as $n \rightarrow \infty$. The rate of convergence is independent of Δ and x_1 .

Now, let $A_\Delta(t) = A(t, u_\Delta(t))$ for $t \in [0, T']$ and let $\{U_\Delta(t, s)\}$ be the evolution operator generated by $\{A_\Delta(t)\}$ and Δ . Define $\hat{u}(t) = U_\Delta(t, 0)x_1$, then $\hat{u}'(t) + A_\Delta(\bar{t})\hat{u}(t) = 0$ except at t_1, t_2, \dots, t_N , and

$$\begin{aligned} \|\hat{u}(t) - u_n(t)\|_X &= \|U_\Delta(t, 0)x_1 - U_n(t, 0)x_1\|_X \\ &= \left\| -\int_0^t U_\Delta(t, s)(A_\Delta(\bar{s}) - A_{n-1}(\bar{s}))U_n(s, 0)x_1 ds \right\|_X \\ &\leq M e^{\beta T'} C_1 M_2 e^{\beta_1 T'} (\|x_0\|_Y + r_2) \cdot \int_0^t \|\hat{u}(\bar{s}) - u_{n-1}(\bar{s})\|_X ds \\ &\leq ((M M_2 e^{(\beta+\beta_1)T'} C_1 (\|x_0\|_Y + r_2) T')^n / n!) \|\hat{u} - x_0\|_X \end{aligned}$$

which tends to 0 as $n \rightarrow \infty$. Thus $\hat{u}(t) = u_\Delta(t)$, $u_\Delta'(t) + A_\Delta(\bar{t})u_\Delta(t) = 0$ except at t_1, \dots, t_N , $u_\Delta(0) = x_1$, u_Δ is Lipschitz continuous (X) with Lipschitz constant $\|A\| C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$, and $\|u_\Delta(t)\|_Y = \|U_\Delta(t, 0)x_1\|_Y < C_2^2 M_1 e^{K+\beta_1 T'} (\|x_0\|_Y + r_2)$, independent of t and x_1 .

We now establish that $\{u_\Delta: u_\Delta(0) = x_1$ and Δ is a partition of $[0, T']\}$ is a family of approximate solutions to (QL) on $[0, T']$ with initial value x_1 . Except for t_1, \dots, t_N , we have

$$\begin{aligned} u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t) &= u_\Delta'(t) + A(\bar{t}, u_\Delta(\bar{t}))u_\Delta(t) \\ &\quad + (A(t, u_\Delta(t)) - A(\bar{t}, u_\Delta(\bar{t})))u_\Delta(t) \\ &= (A(t, u_\Delta(t)) - A(\bar{t}, u_\Delta(\bar{t})))u_\Delta(t). \end{aligned}$$

So,

$$\begin{aligned} & \|u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X \\ & \leq C_1(|t - \bar{t}| + \|u_\Delta(t) - u_\Delta(\bar{t})\|_X)M_2e^{\beta_1 T'} \cdot (\|x_0\|_Y + r_2) \\ & \leq C_1M_2e^{\beta_1 T'}(\|x_0\|_Y + r_2) \cdot (1 + M_2e^{\beta_1 T'}\|A\|(\|x_0\|_Y + r_2))|t - \bar{t}| \\ & = L|t - \bar{t}|. \end{aligned}$$

where L is independent of t in $[0, T']$ and Δ . Thus $\|u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X \leq L|\Delta|$ except at t_1, t_2, \dots, t_N . This verifies that we have a family of approximate solutions.

To show that the $\{u_\Delta\}$ converge as $|\Delta| \rightarrow 0$, let Δ_1 and Δ_2 be two partitions of $[0, T']$ with $|\Delta_1|$ and $|\Delta_2|$ small enough that both $\|f'(t) + A(t, f(t))f(t)\|_X < \epsilon$ and $\|g'(t) + A(t, g(t))g(t)\|_X < \epsilon$ for $t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2)$, where $f(t) = u_{\Delta_1}(t)$, $g(t) = u_{\Delta_2}(t)$, $f(0) = x_1 = g(0)$, and $\epsilon > 0$ is fixed. The preceding paragraph allows us to do this. Let $\{V(t, s)\}$ be the evolution operator obtained from Kato's Theorem 4.1 [11] for $\{A(t, f(t)): t \in [0, T']\}$. For $s, t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2)$, $s < t$, we get

$$\begin{aligned} g'(s) - f'(s) &= (g'(s) + A(s, g(s))g(s)) - (f'(s) + A(s, f(s))f(s)) \\ &\quad - A(s, f(s))(g(s) - f(s)) + (A(s, f(s)) - A(s, g(s)))g(s). \end{aligned}$$

Moving the third expression on the right to the left side of the equation and applying $V(t, s)$, we get

$$\begin{aligned} & V(t, s)(g'(s) - f'(s)) + V(t, s)A(s, f(s))(g(s) - f(s)) \\ & = V(t, s)(g'(s) + A(s, g(s))g(s)) \\ & \quad - V(t, s)(f'(s) + A(s, f(s))f(s)) \\ & \quad + V(t, s)(A(s, f(s)) - A(s, g(s)))g(s). \end{aligned}$$

The left side is simply $\partial V(t, s)(g(s) - f(s))/\partial s$. Integrating both sides in s from 0 to t , evaluating the left side at the endpoints, and recognizing that $V(t, t) = I$, we get

$$\begin{aligned} & g(t) - f(t) - V(t, 0)(x_1 - x_1) \\ & = \int_0^t V(t, s)(g'(s) + A(s, g(s))g(s)) \, ds \\ & \quad - \int_0^t V(t, s)(f'(s) + A(s, f(s))f(s)) \, ds \\ & \quad + \int_0^t V(t, s)(A(s, f(s)) - A(s, g(s)))g(s) \, ds. \end{aligned}$$

So,

$$\begin{aligned} \|g(t) - f(t)\|_X & \leq T'Me^{\beta T'}\epsilon + T'Me^{\beta T'}\epsilon + Me^{\beta T'}C_1M_2e^{\beta_1 T'}(\|x_0\|_Y + r_2) \\ & \quad \cdot \int_0^t \|f(s) - g(s)\|_X \, ds \\ & = L_1\epsilon + L_2\int_0^t \|g(s) - f(s)\|_X \, ds. \end{aligned}$$

This implies that

$$\|u_{\Delta_1}(t) - u_{\Delta_2}(t)\|_X = \|g(t) - f(t)\|_X = O(\epsilon)$$

independent of t in $[0, T']$. Thus, $\{u_\Delta\}$ converges uniformly to a function u on $[0, T']$ to W as $|\Delta| \rightarrow 0$. We note that u is Lipschitz continuous (X) with constant $\|A\|C_2^2M_1e^{K+\beta_1T'}(\|x_0\|_Y + r_2)$, $u(0) = x_1$, and u is bounded, independent of x_1 , by $C_2^2M_1e^{K+\beta_1T'}(\|x_0\|_Y + r_2)$ in the relative completion of Y in X .

We need to know that u "corresponds" to $\{A(t, u(t)): t \in [0, T']\}$. Let $\{U(t, s)\}$ be the evolution operator obtained from Kato's Theorem 4.1 [11] for $\{A(t, u(t))\}$, and define $\bar{u}(t) = U(t, 0)x_1$. By Lemma A, $\{A(t, u(t))\}$ is Y -stable with constants M_2 and β_1 . For any partition Δ of $[0, T']$ we have

$$\begin{aligned} \|\bar{u}(t) - u_\Delta(t)\|_X &= \|U(t, 0)x_1 - U_\Delta(t, 0)x_1\|_X \\ &= \left\| -\int_0^t U(t, s)(A(s, u(s)) - A_\Delta(\bar{s}))U_\Delta(s, 0)x_1 ds \right\|_X \\ &\leq Me^{\beta T'}C_1(|\Delta| + \sup\{\|u(s) - u_\Delta(\bar{s})\|: s \in [0, t]\}) \\ &\quad \cdot M_2e^{\beta_1T'}(\|x_0\|_Y + r_2)T'. \end{aligned}$$

Since u_Δ converges to u uniformly on $[0, T']$ as $|\Delta| \rightarrow 0$, and u_Δ is Lipschitz continuous (X) with a Lipschitz constant that is independent of Δ , we see that $\|\bar{u}(t) - u_\Delta(t)\|_X$ goes to $\|\bar{u}(t) - u(t)\|_X$ and to 0 as $|\Delta| \rightarrow 0$. Thus $u(t) = \bar{u}(t) = U(t, 0)x_1$.

Suppose $x_2 \in Z$ and that w_Δ and w are obtained in the same manner as u_Δ and u , except that the initial value for w_Δ and w is x_2 . Analogous to the technique employed to obtain u , we get

$$\frac{\partial}{\partial s} U_\Delta(t, s)(u_\Delta(s) - w_\Delta(s)) = U_\Delta(t, s)(A(\bar{s}, w_\Delta(\bar{s})) - A(\bar{s}, u_\Delta(\bar{s})))w_\Delta(s)$$

for $s, t \in [0, T']$, $s \leq t$, $s \notin \Delta$. Integrating both sides in s from 0 to t yields

$$\begin{aligned} u_\Delta(t) - w_\Delta(t) - U_\Delta(t, 0)(x_1 - x_2) \\ = \int_0^t U_\Delta(t, s)(A(\bar{s}, w_\Delta(\bar{s})) - A(\bar{s}, u_\Delta(\bar{s})))w_\Delta(s) ds, \end{aligned}$$

and so

$$\begin{aligned} \|u_\Delta(t) - w_\Delta(t)\|_X &\leq Me^{\beta T'}\|x_1 - x_2\|_X + Me^{\beta T'} \cdot C_1M_2e^{\beta_1T'}(\|x_0\|_Y + r_2) \\ &\quad \cdot \int_0^t \|u_\Delta(\bar{s}) - w_\Delta(\bar{s})\|_X ds. \end{aligned}$$

Thus, $\|u_\Delta(t) - w_\Delta(t)\|_X \leq C\|x_1 - x_2\|_X$, with C independent of t in $[0, T']$, Δ , and x_1 and x_2 . It follows that $\|u(t) - w(t)\|_X \leq C\|x_1 - x_2\|_X$ for $t \in [0, T']$, with C also independent of the initial values.

We now turn to the uniqueness of solutions to (QL) or (L; u) on $[0, T'']$, where $0 < T'' \leq T'$, with initial value $x_1 \in Z$.

Suppose v is such a solution to (L; u). Then, $v'(s) + A(s, u(s))v(s) = 0$ a.e., so

$$\frac{\partial}{\partial s} U(t, s)v(s) = U(t, s)v'(s) + U(t, s)A(s, u(s))v(s) = 0 \quad \text{a.e.}$$

Integrating in s from 0 to t , we get $U(t, t)v(t) - U(t, 0)v(0) = v(t) - U(t, 0)x_1 = v(t) - u(t) = \text{constant}$. Since $v(0) - u(0) = x_1 - x_1 = 0$, we have $v(t) = u(t)$ for all t in $[0, T'']$. This makes u the unique solution of (L; u) on $[0, T'']$ with initial

value x_1 . In fact, this also makes u a solution of (QL). We note that it is not necessary that $v([0, T'']) \subset W$.

Now suppose that v is a solution to (QL) on $[0, T'']$ with initial value x_1 . Then, $v'(s) + A(s, v(s))v(s) = 0$ a.e., and so

$$v'(s) + A(s, u(s))v(s) = (A(s, u(s)) - A(s, v(s)))v(s) \quad \text{a.e.}$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial s} U(t, s)v(s) &= U(t, s)v'(s) + U(t, s)A(s, u(s))v(s) \quad \text{a.e.} \\ &= U(t, s)(A(s, u(s)) - A(s, v(s)))v(s) \quad \text{a.e.} \end{aligned}$$

Integrating in s from 0 to t , we get

$$\begin{aligned} v(t) - u(t) &= U(t, t)v(t) - U(t, 0)v(0) \\ &= \int_0^t U(t, s)(A(s, u(s)) - A(s, v(s)))v(s) \, ds. \end{aligned}$$

This implies that

$$\|v(t) - u(t)\|_X \leq Me^{\beta T''} \|v\|_Y C_1 \int_0^t \|u(s) - v(s)\|_X \, ds,$$

and thus $\|v(t) - u(t)\|_X = 0$ for all t in $[0, T'']$. This makes u the unique solution of (QL) on $[0, T'']$ with initial value x_1 . This completes the proof of our Theorem. \square

If Y is reflexive, then by Kato's Theorem 5.1 [11], we have the result that $v = u$ is a solution of $(L; u)$ on $[0, T']$ with initial value x_1 , and thus u is a solution of (QL). This gives us Corollary 1. \square

The remarks following the statement of the Theorem and Corollary are straightforward. We also note that Remark (1) deals with a particular case of condition (iii) of the theorem. Remark (2) contains a condition which greatly simplifies the proof of the Theorem, but which would be extremely difficult to verify in the absence of conditions stronger than condition (iii); e.g., see [2].

We now turn our attention to an application of our Theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type.

COROLLARY 2. *Let S be the sector of the complex plane C consisting of 0 and $\{\lambda \in C: -\theta \leq \arg \lambda \leq \theta\}$, where $\theta \in (\pi/2, \pi)$ is fixed. We assume that conditions (i) and (ii) of the Theorem hold with $Y = D(A(t, w))$ for each t, w , and that (iii)' the resolvent set of $-A(t, w)$ contains S and*

$$\|(\lambda I + A(t, w))^{-1}\|_X \leq C_4 / (1 + |\lambda|)$$

for each $\lambda \in S, t \in [0, T]$, and $w \in W$, where C_4 is a constant independent of λ, t , and w .

Then, the conclusion of the Theorem holds and $(L; u)$ has a continuously differentiable (X) solution on $[0, T']$ with initial value x_1 , and thus u is a continuously differentiable (X) solution of (QL) on $[0, T']$ with initial value x_1 .

PROOF. Under these conditions the hypotheses of the Theorem hold, where $S(t, w) = A(t, w)$ for each t and w . This gives us the limit solution u . The plan of attack is to produce a solution to $(L; u)$ which is continuously differentiable (X) on

$[0, T']$ and has initial value x_1 . This is where the Sobolevskii-Tanabe theory enters. Let $A(t) = A(t, u(t))$ for each $t \in [0, T']$, and we see for $t_1, t_2, t_3 \in [0, T']$ that

$$\begin{aligned} \|(A(t_1) - A(t_2))A(t_3)^{-1}\|_X &< \|A(t_1) - A(t_2)\|_{Y,X} \|A(t_3)^{-1}\|_{X,Y} \\ &< C_4 \|A(t_1, u(t_1)) - A(t_2, u(t_2))\|_{Y,X} \\ &< C_5 |t_2 - t_1| \quad \text{by (ii)} \end{aligned}$$

and the Lipschitz continuity of u , where C_5 is independent of the choice of t_1, t_2, t_3 . It follows from the Sobolevskii-Tanabe theory [14], [15], [19], [20], [21], [22] that there is an evolution operator $\{V(t, s); 0 \leq s < t < T'\}$ such that $v(t) = V(t, 0)x_1$ defines a continuously differentiable (X) function that satisfies $v'(t) + A(t)v(t) = 0, v(0) = x_1$. The operator also satisfies $\|A(t)V(t, 0)A(0)^{-1}\|_X < C_6$ on $[0, T']$, with C_6 independent of t [19, p. 5], thus

$$\begin{aligned} \|v(t)\|_Y &= \|V(t, 0)x_1\|_Y = \|A(t)^{-1}A(t)V(t, 0)A(0)^{-1}A(0)x_1\|_Y \\ &< \|A(t)^{-1}\|_{X,Y} \|A(t)V(t, 0)A(0)^{-1}\|_X \cdot \|A(0)\|_{Y,X} \|x_1\|_Y \\ &< C_7 \|x_1\|_Y, \end{aligned}$$

where C_7 is independent of t . So, except for the image of v lying in W , we have that v is a solution of $(L; u)$. Since the proof of the uniqueness of a solution to $(L; u)$ does not depend on $v([0, T']) \subset W$, we still have that $v(t) = u(t)$ on $[0, T']$. Consequently, u is the solution of (QL) on $[0, T']$ with initial value x_1 . In fact, u is continuously differentiable (X), without exception, on $[0, T']$. \square

We note that in general an application of the Theorem involves finding conditions that guarantee the existence of a solution to $(L; u)$, which then implies that u is the solution of (QL) .

It may be difficult at times to recognize that the conditions for our Theorem hold. The following Proposition gives criteria that obtain the Banach space Y and verify most of condition (iii) of the Theorem. If, in particular, we are able to use $\lambda I + A(t, w)$, where $\lambda > \beta$ is fixed, for $S(t, w)$ in the Proposition, then condition (ii) of the Theorem holds as well as all of condition (iii).

PROPOSITION. *Let Y be a dense linear subspace of X . Suppose for each $t \in [0, T]$ and $w \in W$ that $S(t, w)$ is an isomorphism (algebraically) from Y onto X , $S(t, w)$ is a closed operator in X , $S(t, w)^{-1} \in B(X)$ with $\|S(t, w)^{-1}\|_X \leq L_1$, and the bounded linear operator $S(t, w)S(t_0, w_0)^{-1}$ satisfies*

$$\|S(t_2, w_2)S(t_0, w_0)^{-1} - S(t_1, w_1)S(t_0, w_0)^{-1}\|_X \leq L_2(|t_2 - t_1| + \|w_2 - w_1\|_X),$$

where L_1, L_2, t_0 from $[0, T]$, and $w_0 \in W$ are fixed. Suppose further that Y has the graph norm induced by $S(t_0, w_0)$; i. e., for $y \in Y, \|y\|_Y = \|y\|_X + \|S(t_0, w_0)y\|_X$. Then,

- (i) Y is a Banach space under this norm, and Y is continuously embedded in X .
- (ii) $S(t, w)^{-1} \in B(X, Y)$ for each t and w , and $\|S(t, w)^{-1}\|_{X,Y} \leq 1 + L_1 + L_2(T + 2r)$, where r is the radius of the ball W .
- (iii) $S(t, w) \in B(Y, X)$ for each t and w , and $\|S(t, w)\|_{Y,X} \leq 1 + L_2(T + 2r) \equiv L_3$.
- (iv) $\|S(t_2, w_2) - S(t_1, w_1)\|_{Y,X} \leq L_2L_3(|t_2 - t_1| + \|w_2 - w_1\|_X)$.

PROOF. Since $S(t_0, w_0)$ is a closed linear operator with domain Y , it is clear that Y is a Banach space under the indicated norm. It is also immediate that Y is continuously embedded in X .

Let $x \in X$, then

$$\begin{aligned} \|S(t, w)^{-1}x\|_Y &= \|S(t, w)^{-1}x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x\|_X \\ &\leq L_1\|x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x - x\|_X + \|x\|_X \\ &\leq (L_1 + 1)\|x\|_X + L_2(|t - t_0| + \|w - w_0\|_X)\|x\|_X \\ &\leq (L_1 + 1)\|x\|_X + L_2(T + 2r)\|x\|_X \\ &= (1 + L_1 + L_2(T + 2r))\|x\|_X. \end{aligned}$$

So, $S(t, w)^{-1} \in B(X, Y)$ and $\|S(t, w)^{-1}\|_{X, Y} \leq 1 + L_1 + L_2(T + 2r)$.

Let $y \in Y$, then

$$\begin{aligned} \|S(t, w)y\|_X &= \|S(t, w)S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \\ &\leq (1 + L_2(|t - t_0| + \|w - w_0\|_X)) \cdot \|S(t_0, w_0)y\|_X \\ &\leq (1 + L_2(T + 2r))(\|y\|_Y - \|y\|_X) \\ &\leq (1 + L_2(T + 2r))\|y\|_Y. \end{aligned}$$

So, $S(t, w) \in B(Y, X)$ and $\|S(t, w)\|_{Y, X} \leq 1 + L_2(T + 2r)$.

To show (iv), let $y \in Y$, then

$$\begin{aligned} \|S(t_2, w_2)y - S(t_1, w_1)y\|_X &= \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \\ &\leq \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}\|_X \cdot \|S(t_0, w_0)y\|_X \\ &\leq L_2(|t_2 - t_1| + \|w_2 - w_1\|_X) \cdot \|S(t_0, w_0)\|_{Y, X}\|y\|_Y \\ &\leq L_2L_3(|t_2 - t_1| + \|w_2 - w_1\|_X)\|y\|_Y. \quad \square \end{aligned}$$

We also note that condition (i) for our Theorem holds when each $A(t, w)$ satisfies $\|\exp(-sA(t, w))\|_X \leq e^{\beta t}$.

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