QUASI-LINEAR EVOLUTION EQUATIONS IN BANACH SPACES

BY

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Abstract. This paper is concerned with the quasi-linear evolution equation $u'(t) + A(t, u(t))u(t) = 0$ in $[0, T]$, $u(0) = x_0$ in a Banach space setting. The spirit of this inquiry follows that of T. Kato and his fundamental results concerning linear evolution equations. We assume that we have a family of semigroup generators that satisfies continuity and stability conditions. A family of approximate solutions to the quasi-linear problem is constructed that converges to a "limit solution." The limit solution must be the strong solution if one exists. It is enough that a related linear problem has a solution in order that the limit solution be the unique solution of the quasi-linear problem. We show that the limit solution depends on the initial value in a strong way. An application and the existence aspect are also addressed.

This paper is concerned with the quasi-linear evolution equation

$$u'(t) + A(t, u(t))u(t) = 0 \quad \text{in} [0, T], \quad u(0) = x_0$$

in a Banach space setting.

The spirit of this inquiry follows that of T. Kato. Kato wrote a fundamental paper on linear evolution equations in 1953 [9]; that is, investigation of

$$u'(t) + A(t)u(t) = 0 \quad \text{on} [0, T], \quad u(0) = x_0.$$  


After discussing the setting and method of attack, our theorem is stated and proved. We then give an application of the theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type. A proposition relevant to our theorem is also given.

Let $X$ and $Y$ be Banach spaces, with $Y$ densely and continuously embedded in $X$. Let $x_0 \in Y$, $T > 0$, $r > r_1 > 0$, $r_2 > 0$, $W = \overline{B}_X(x_0; r)$, $Z = B_X(x_0; r_1) \cap B_Y(x_0; r_2)$, and for each $t \in [0, T]$ and $w \in W$, let $-A(t, w)$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in $X$, with $Y \subset D(A(t, w))$.

We consider the quasi-linear evolution equation

$$v'(t) + A(t, v(t))v(t) = 0.$$  

(QL)
Given a function $u$ from $[0, T']$ into $W$, where $0 < T' < T$, we can also consider the linearized evolution equation

$$v'(t) + A(t, u(t))v(t) = 0.$$  \[(L; u)\]

By a solution (or strong solution) of (QL) or (L; $u$) on $[0, T']$, we mean a function $v$ on $[0, T']$ to $W$ which is absolutely continuous (A.e.) and differentiable (A.'e.) a.e., such that $v(t) \in Y$ a.e., ess sup\{$\|v(t)\|_Y\} < \infty$, and $v$ satisfies the appropriate equation, (QL) or (L; $u$), a.e. on $[0, T']$.

Our method is to produce, for each $x_1 \in Z$, a “limit solution” $u$ with initial value $x_1$ on an interval $[0, T']$, where $T' \in (0, T]$ is independent of $x_1$. For a partition $\Delta = \{t_0, t_1, \ldots, t_N\}$ of $[0, T']$, we use an iterative procedure to produce a Lipschitz continuous (A.) function $u_\Delta$ which satisfies

$$u_\Delta'(t) + A(t_\Delta(t_i))u_\Delta(t) = 0 \quad \text{for } t \in (t_i, t_{i+1})$$

and $i \in \{0, 1, \ldots, N - 1\}$, with $u_\Delta(0) = x_1$. This $u_\Delta$ is shown to be the time-ordered juxtaposition of the semigroups generated by the $-A(t, u_\Delta(t_i))$. These approximate solutions converge uniformly, as $|\Delta|$ goes to 0, to give the limit solution $u$. We show, in particular, that if $v = w$ is a solution of (QL) or (L; $u$) on $[0, T']$ with initial value $x_1$, then $w = u$. Thus, subject to an initial value, a solution of (QL) is unique if it exists, and whenever the linearized equation (L; $u$) has a solution, then so does the quasi-linear equation (QL). There are known conditions which are sufficient in order that (L; $u$) has a solution.


Theorem. Assume that

(i) $\{A(t, w)\}$ is stable in $X$ with constants of stability $M, \beta$; i.e.,

$$\|(A(t_k, w_k) + \lambda)^{-1}(A(t_{k-1}, w_{k-1}) + \lambda)^{-1} \ldots (A(t_1, w_1) + \lambda)^{-1}\|_X < M(\lambda - \beta)^{-k},$$

$\lambda > \beta$, for any finite family $\{(t_j, w_j)\}$, $0 < t_1 < \ldots < t_k < T$, $k = 1, 2, \ldots$.

(ii) $Y \subset D(A(t, w))$ for each $(t, w)$, which implies that $A(t, w) \in B(Y, X)$, and the map $(t, w) \rightarrow A(t, w)$ is Lipschitz continuous with constant $C_1$; i.e.,

$$\|A(t_2, w_2) - A(t_1, w_1)\|_{Y,X} < C_1(\|t_2 - t_1\| + \|w_2 - w_1\|_X).$$

(iii) There is a family $\{S(t, w)\}$ of isomorphisms of $Y$ onto $X$ such that $S(t, w)A(t, w)S(t, w)^{-1} = A_1(t, w)$ is the negative of the infinitesimal generator of a strongly continuous semigroup in $X$ for each $(t, w)$, and $A_1(t, w)$ is stable in $X$, with constants of stability $M_1, \beta_1$. Furthermore, there is a constant $C_2$ such that

$$\|S(t, w)\|_{Y,X} < C_2, \|S(t, w)^{-1}\|_{X,Y} < C_2,$$

and the map $(t, w) \rightarrow S(t, w)$ is Lipschitz continuous with constant $C_3$ (see (ii) above).

Then, there exists a $T'$, with $0 < T' < T$, such that for each $x_1 \in Z$ and partition $\Delta = \{t_0, t_1, \ldots, t_n\}$ of $[0, T']$, we can find a function $u_\Delta$ which is Lipschitz continuous
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\( (X) \) on \([0, T']\) to \(W\), \(Y\)-bounded, and satisfies \( u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t) = 0 \) for \( t \in (t_i, t_{i+1}) \) and \( i \in \{0, 1, \ldots, n - 1\} \), with \( u_\Delta(0) = x_1 \). In fact, given \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( |\Delta| < \delta \) implies that \( \|u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t)\|_X < \varepsilon \) except at \( t_1, \ldots, t_n \). Further, the \( u_\Delta \) converge uniformly, as \( |\Delta| \) goes to 0, to a Lipschitz continuous \( (X) \) function \( u \) on \([0, T']\) to \(W\) which has initial value \( x_1 \) and is bounded, independent of \( x_1 \), in the relative completion of \( Y \) in \( X \) (the set of all points in \( X \) that are the limit in \( X\)-norm of sequences from \( Y \) that are bounded in \( Y\)-norm).

If \( x_2 \in Z \) and \( u \) is constructed as above but with initial value \( x_2 \), then \( \|u(t) - w(t)\|_X < C\|x_1 - x_2\|_X \) for \( t \in [0, T'] \), with \( C \) independent of \( x_1 \) and \( x_2 \).

Now, if \( v \) is a solution of (QL) or \((L; u)\) on \([0, T'']\), where \( 0 < T'' < T' \), with initial value \( x_1 \), then \( v = u \) on \([0, T'']\), and thus solutions to (QL) or \((L; u)\) are uniquely determined by their initial values.

**Corollary 1.** If \( Y \) is reflexive, then \((L; u)\) has a solution on \([0, T']\) with initial value \( x_1 \), and thus \( u \) is a solution of (QL) on \([0, T']\) with initial value \( x_1 \).

**Remarks.** (1) If \( D(A(t, w)) = Y \) for each \((t, w)\) and there is a \( \lambda > \beta \) such that

\[
\|\lambda I + A(t, w)\|^{-1}_{Y,X} < C_2 \quad \text{and} \quad \|\lambda I + A(t, w)\|_{Y,X} < C_2 \quad \text{for each} \quad (t, w),
\]

then (iii) is satisfied with \( S(t, w) = \lambda I + A(t, w) \).

(2) If \( Y \) is \( A(t, M>)-\text{admissible} \) \( (\exp(-sA(t, w)) \) takes \( y \) to \( y \) and forms a strongly continuous semigroup on \( Y \) \) for each \((t, w)\) and \( A(t, w) \) is stable in \( Y \), then (iii) is unnecessary.

We now begin to prove the Theorem. The proofs of the above remarks and Corollary will be given later.

Let \( T^0 = \min(T, r/\|A\|Me^{\beta T}(\|x_0\|_Y + r_2)) \), where \( \|A\| = \sup\{\|A(t, w)\|_{Y,X}: t \in [0, T], w \in W\} \) which is finite by (ii). Let \( K = C_2C_3M_1T^0 \) and

\[
T' = T^0/(1 + \|A\|C_2^2M_1e^{\kappa + \beta_1T}(\|x_0\|_Y + r_2)).
\]

**Lemma A.** If \( u \) is Lipschitz continuous \((X)\) on \([0, T']\) to \( W \) with Lipschitz constant

\[
\|A\|C_2^2M_1e^{\kappa + \beta_1T}(\|x_0\|_Y + r_2),
\]

then \( (A(t, u(t)): t \in [0, T']) \) is \( Y\)-stable with constants \( C_2^2M_1e^{\kappa} \) and \( \beta_1 \).

**Proof of Lemma A.** We use Kato's Proposition 4.4 [11] with \( S(t) = S(t, u(t)) \). Then we estimate the variation of \( S \) by

\[
V_S < C_3(1 + \|A\|C_2^2M_1e^{\kappa + \beta_1T}(\|x_0\|_Y + r_2))T' < C_3T^0,
\]

whence \( (A(t, u(t))): t \in [0, T'] \) is \( Y\)-stable with constants \( C_2^2M_1e^{\kappa} \) and \( \beta_1 \).

This completes the proof of Lemma A.

By an evolution operator \( \{W(t, s): 0 < s < t < T'\} \) generated by \( \{\triangle(t): t \in [0, T']\} \subset \{A(t, w): t \in [0, T'], w \in W\} \) and a partition \( \Delta = \{t_0, \ldots, t_n\} \) of \([0, T']\), we mean the family of operators obtained by forming a time-ordered juxtaposition of the semigroups generated at the points of the partition; e.g., for \( t \in [t_i, t_{i+1}), s \in [t_j, t_{j+1}], s < t, \)

\[
W(t, s) = \exp(-(t - t_i)\triangle(t_i))\exp(-(t - t_{i-1})\triangle(t_{i-1})) \ldots \exp(-(t_{j+1} - s)\triangle(t_j)).
\]
It follows from (i) and Kato’s Proposition 3.3 [11] that \( \| W(t, s) \|_X < M e^{-\beta(t-s)} \). If \( \{ \tilde{\alpha}(t) \} \) is \( Y \)-stable with constants \( \tilde{M}, \tilde{\beta} \), then \( W(t, s) Y \subset Y \) and \( \| W(t, s) \|_Y < \tilde{M} e^{-\tilde{\beta}(t-s)} \) as a result of (iii) and Kato’s Propositions 2.4 and 3.3 [11]. Let \( \tilde{i} = i \), if \( t \in [t_i, t_{i+1}) \), \( i \neq N \), and \( \tilde{i}_N = t_N \). If \( f(t) = W(t, 0)x_1 \) on \([0, T')\), then \( f \) satisfies \( f'(t) + \tilde{\alpha}(i)f(t) = 0 \) for \( t \in \Delta \), with \( f(0) = x_1 \). The construction of an evolution operator from a family of semigroup generators and a partition, the notation \( \tilde{i} \), and the other results above will be used from this point on without further discussion.

**Lemma B.** Suppose \( \{ \tilde{\alpha}(t) : t \in [0, T'] \} \) is \( Y \)-stable with constants \( \tilde{M} \) and \( \tilde{\beta} \), and that \( \{ W(t, s) \} \) is generated by \( \{ \tilde{\alpha}(t) \} \) and a partition \( \Delta \) of \([0, T']\). Then, \( f(t) = W(t, 0)x_1 \) is Lipschitz continuous (\( X \)) with Lipschitz constant \( \| \tilde{M} e^{\tilde{\beta}T} \|_Y + r_2 \).

The result is also true if \( \{ W(t, s) \} \) is the evolution operator of Kato’s Theorem 4.1 [11].

**Proof of Lemma B.** For the partition case, since \( f'(t) = -\tilde{\alpha}(\tilde{i})f(t) \) except for \( t \in \Delta \), we get for \( s < t \)

\[
\| f(t) - f(s) \|_X = \left\| \int_s^t A(\tilde{\xi})f(\xi)d\tilde{\xi} \right\|_X < \| A \| \| f \|_Y |t-s| < \| A \| \tilde{M} e^{\tilde{\beta}T} \|_Y |t-s| < \| A \| \tilde{M} e^{\tilde{\beta}T} \|_Y + r_2 |t-s|.
\]

Now, the \( f \) on \([0, T')\) obtained from Kato’s evolution operator is the uniform (\( X \)) limit of the \( f \) corresponding to the partitions \( \Delta \) as \( |\Delta| \to 0 \). This establishes the result in the second case and completes the proof of Lemma B.

Together, Lemma A and Lemma B suggest an iteration scheme. We fix \( x_1 \) and \( \Delta \), then obtain sequences \( \{ u_n \}, \{ A_n(t) \}, \) and \( \{ U_n(t, s) \} \), with \( A_n(t) = A(t, u_n(t)), \{ U_n(t, s) \} \) the evolution operator generated by \( \{ A_n(t) \} \) and \( \Delta \), and \( u_{n+1}(t) = U_{n+1}(t, 0)x_1 \). Once Lemma A is satisfied, we have \( \{ A_n(t) : t \in [0, T'] \} \) is \( Y \)-stable with constants \( C_2M_1e^K \) and \( \beta_1 \); then, Lemma B applied to \( \{ \tilde{\alpha}(t) \} = \{ A_n(t) \}, \tilde{M} = C_2M_1e^K \) and \( \tilde{\beta} = \beta_1 \), implies that \( u_{n+1} \) is Lipschitz continuous (\( X \)) on \([0, T']\) with Lipschitz constant \( \| A \| C_2M_1e^{K+\beta_1T} \|_Y + r_2 \). Assuming \( u_{n+1}[0, T'] \subset W \), the stage is set to apply Lemma A to \( \{ A_{n+1}(t) : t \in [0, T'] \} \) and continue the process.

We now work with a fixed partition \( \Delta \) of \([0, T']\) and fixed \( x_1 \in Z \).

Let \( A_0(t) = A(t, x_1) \) for \( t \in [0, T'] \) and let \( \{ U_i(t, s) \} \) be the evolution operator generated by \( \{ A_0(t) \} \) and \( \Delta \). Define \( u_1(t) = U_i(t, 0)x_1 \). Then, \( u'_1(t) + A_0(t)u_1(t) = 0 \) except at \( t_1, t_2, \ldots, t_N \). Also,

\[
\| u_1(t) - x_1 \|_X = \| U_1(t, t)x_1 - U_i(t, 0)x_1 \|_X = \left\| \int_0^t U_1(t, s)A_0(s)x_1 ds \right\|_X < M e^{\beta T} \| A \| (\| x_0 \|_Y + r_2)t < r
\]

by the choice of \( T^0 \) and \( T' \). So, \( u_1(t) \in W \) for each \( t \in [0, T'] \). This argument also works for all the following \( u_n, n = 2, 3, \ldots \).
To start the procedure, we apply Lemma A to \( u = x \) and then Lemma B with \( \ell(t) = \{ A_0(t) \} \), \( \tilde{M} = C_2^2 M \), and \( \tilde{\beta} = \beta \), proving that \( u_1 \) is Lipschitz continuous \( (X) \) on \([0, T']\) with the Lipschitz constant \( \|A\|C_2^2 M \). For the next iteration, let \( A_1(t) = A(t, u(t)) \) for \( t \in [0, T'] \) and \( \{ U_2(t, s) \} \) be the evolution operator generated by \( \{ A_1(t) \} \) and \( \Delta \). Define \( u_2(t) = U_2(t, 0)x \). Then, \( u_2(t) + A_1(t)u_2(t) = 0 \) except at \( t_1, t_2, \ldots, t_N \). As with \( u_1, u_2 \in W \) for each \( t \in [0, T'] \).

As we commented before, we can continue in like manner. For convenience of notation, let \( M_2 = C_2^2 M \). Then, for \( n \geq 1 \), we have

\[
\| u_{n+1}(t) - u_n(t) \| \leq \left\| \int_0^t U_{n+1}(t, s)(A_n(s) - A_{n-1}(s))U_n(s, 0)ds \right\| X
\leq Me^{\beta T} C_2 M_2 e^{\beta T}(\|x_0\| Y + r_2) \cdot \int_0^t \|u_n(s) - u_{n-1}(s)\| X ds
\leq \left(MM_2 e^{\beta T} C_1(\|x_0\| Y + r_2)\right)^n \cdot \int_0^t \|u_1(s) - x_0\| X ds
\leq \left(MM_2 e^{\beta T} C_1(\|x_0\| Y + r_2)\right)^n \cdot \frac{r}{n!}
\]

which tends to 0 as \( n \to \infty \). Thus \( u(t) = u_\Delta(t), \ u_\Delta'(t) + A_\Delta(t)u_\Delta(t) = 0 \) except at \( t_1, \ldots, t_N \), \( u_\Delta(0) = x \), \( u_\Delta \) is Lipschitz continuous \( (X) \) with Lipschitz constant \( \|A\|C_2^2 M \). We now establish that \( \{ u_\Delta; u_\Delta(0) = x \} \) is a family of approximate solutions to (QL) on \([0, T']\) with initial value \( x \). Except for \( t_1, \ldots, t_N \), we have

\[
u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t) = u_\Delta'(t) + A(t, u_\Delta(t))u_\Delta(t)
+ (A(t, u_\Delta(t)) - A(t, u_\Delta(t)))u_\Delta(t)
= (A(t, u_\Delta(t)) - A(t, u_\Delta(t)))u_\Delta(t).
\]
So,
\[ \| u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t) \|_X \]
\[ \leq C_1 \left( |t - \tilde{t}| + \| u_\Delta(t) - u_\Delta(\tilde{t}) \|_X \right) M_2 e^{B \ell T} \left( \| x_0 \|_Y + r_2 \right) \]
\[ \leq C_1 M_2 e^{B \ell T} (\| x_0 \|_Y + r_2) \cdot \left( 1 + M_2 e^{B \ell T} \| A \| (\| x_0 \|_Y + r_2) \right) |t - \tilde{t}| \]
\[ = L |t - \tilde{t}|. \]

where \( L \) is independent of \( t \) in \([0, T']\) and \( \Delta \). Thus \( \| u'_\Delta(t) + A(t, u_\Delta(t))u_\Delta(t) \|_X \leq L|\Delta| \) except at \( t_1, t_2, \ldots, t_N \). This verifies that we have a family of approximate solutions.

To show that the \( \{ u_\Delta \} \) converge as \( |\Delta| \to 0 \), let \( \Delta_1 \) and \( \Delta_2 \) be two partitions of \([0, T']\) with \( |\Delta_1| \) and \( |\Delta_2| \) small enough that both \( \| f'(t) + A(t, f(t))f(t) \|_X < \varepsilon \) and \( \| g'(t) + A(t, g(t))g(t) \|_X < \varepsilon \) for \( t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2) \), where \( f(t) = u_{\Delta_1}(t), g(t) = u_{\Delta_2}(t), f(0) = x_1 = g(0), \) and \( \varepsilon > 0 \) is fixed. The preceding paragraph allows us to do this. Let \( \{ V(t, s) \} \) be the evolution operator obtained from Kato's Theorem 4.1 [11] for \( \{ A(t, f(t)): t \in [0, T'] \} \). For \( s, t \in [0, T'] \setminus (\Delta_1 \cup \Delta_2) \), \( s \leq t \), we get
\[ g'(s) - f'(s) = (g'(s) + A(s, g(s))g(s)) - (f'(s) + A(s, f(s))f(s)) \]
\[ + A(s, f(s)) (g(s) - f(s)) + (A(s, f(s)) - A(s, g(s)))g(s). \]

Moving the third expression on the right to the left side of the equation and applying \( V(t, s) \), we get
\[ V(t, s)(g'(s) - f'(s)) + V(t, s)A(s, f(s))(g(s) - f(s)) \]
\[ = V(t, s)(g'(s) + A(s, g(s))g(s)) \]
\[ - V(t, s)(f'(s) + A(s, f(s))f(s)) \]
\[ + V(t, s)(A(s, f(s)) - A(s, g(s)))g(s). \]

The left side is simply \( \partial V(t, s)(g(s) - f(s))/\partial s \). Integrating both sides in \( s \) from 0 to \( t \), evaluating the left side at the endpoints, and recognizing that \( V(t, t) = I \), we get
\[ g(t) - f(t) - V(t, 0)(x_1 - x_1) \]
\[ = \int_0^t V(t, s)(g'(s) + A(s, g(s))g(s)) \, ds \]
\[ - \int_0^t V(t, s)(f'(s) + A(s, f(s))f(s)) \, ds \]
\[ + \int_0^t V(t, s)(A(s, f(s)) - A(s, g(s)))g(s) \, ds. \]

So,
\[ \| g(t) - f(t) \|_X \leq T'Me^{B \ell T}e + T'Me^{B \ell T}e + Me^{B \ell T}C_1 M_2 e^{B \ell T}(\| x_0 \|_Y + r_2) \]
\[ \cdot \int_0^t \| f(s) - g(s) \|_X \, ds \]
\[ = L_1 \varepsilon + L_2 \int_0^t \| g(s) - f(s) \|_X \, ds. \]

This implies that
\[ \| u_{\Delta_1}(t) - u_{\Delta_2}(t) \|_X = \| g(t) - f(t) \|_X = O(\varepsilon) \]
independent of $t$ in $[0, T']$. Thus, $\{u_\Delta\}$ converges uniformly to a function $u$ on $[0, T']$ to $W$ as $|\Delta| \to 0$. We note that $u$ is Lipschitz continuous (X) with constant $\|u\|C_1^2M_1e^{K+\beta T}(\|x_0\|_Y + r_2)$, $u(0) = x_1$, and $u$ is bounded, independent of $x_1$, by $C_2^2M_1e^{K+\beta T}(\|x_0\|_Y + r_2)$ in the relative completion of $Y$ in $X$.

We need to know that $u$ "corresponds" to $\{A(t, u(t)) : t \in [0, T']\}$. Let $\{U(t, s)\}$ be the evolution operator obtained from Kato's Theorem 4.1 [11] for $\{A(t, u(t))\}$, and define $\bar{u}(t) = U(t, 0)x_1$. By Lemma A, $\{A(t, u(t))\}$ is $Y$-stable with constants $M_2$ and $\beta_1$. For any partition $\Delta$ of $[0, T']$ we have

$$\|\bar{u}(t) - u_\Delta(t)\|_X = \|U(t, 0)x_1 - U_\Delta(t, 0)x_1\|_X$$

$$\leq \left| \int_0^t U(t, s)(A(s, u(s)) - A_\Delta(s))U_\Delta(s, 0)x_1 ds \right|$$

$$\leq M_2e^{\beta T}C_1(|\Delta| + \sup\|u(s) - u_\Delta(s)\| : s \in [0, t])$$

$$\cdot M_2e^{\beta T}(\|x_0\|_Y + r_2).$$

Since $u_\Delta$ converges to $u$ uniformly on $[0, T']$ as $|\Delta| \to 0$, and $u_\Delta$ is Lipschitz continuous (X) with a Lipschitz constant that is independent of $\Delta$, we see that $\|\bar{u}(t) - u_\Delta(t)\|_X$ goes to $\|\bar{u}(t) - u(t)\|_X$ and to 0 as $|\Delta| \to 0$. Thus $u(t) = \bar{u}(t) = U(t, 0)x_1$.

Suppose $x_2 \in Z$ and that $w_\Delta$ and $w$ are obtained in the same manner as $u_\Delta$ and $u$, except that the initial value for $w_\Delta$ and $w$ is $x_2$. Analogous to the technique employed to obtain $u_\Delta$ and $u$, we get

$$\frac{d}{ds} U_\Delta(t, s)(u_\Delta(s) - w_\Delta(s)) = U_\Delta(t, s)(A(s, w_\Delta(s)) - A(s, u_\Delta(s)))w_\Delta(s)$$

for $s, t \in [0, T']$, $s < t$, $s \notin \Delta$. Integrating both sides in $s$ from 0 to $t$ yields

$$u_\Delta(t) - w_\Delta(t) - U_\Delta(t, 0)(x_1 - x_2)$$

$$= \int_0^t U_\Delta(t, s)(A(s, w_\Delta(s)) - A(s, u_\Delta(s)))w_\Delta(s) ds,$$

and so

$$\|u_\Delta(t) - w_\Delta(t)\|_X < M_2e^{\beta T}\|x_1 - x_2\|_X + M_2e^{\beta T}C_1M_2e^{\beta T}(\|x_0\|_Y + r_2)$$

$$\cdot \int_0^t \|u_\Delta(s) - w_\Delta(s)\|_X ds.$$

Thus, $\|u_\Delta(t) - w_\Delta(t)\|_X < C\|x_1 - x_2\|_X$, with $C$ independent of $t$ in $[0, T']$, $\Delta$, and $x_1$ and $x_2$. It follows that $\|u(t) - w(t)\|_X < C\|x_1 - x_2\|_X$ for $t \in [0, T']$, with $C$ also independent of the initial values.

We now turn to the uniqueness of solutions to (QL) or (L; $u$) on $[0, T'']$, where $0 < T'' < T'$, with initial value $x_1 \in Z$.

Suppose $v$ is such a solution to (L; $u$). Then, $v'(s) + A(s, u(s))v(s) = 0$ a.e., so

$$\frac{d}{ds} U(t, s)v(s) = U(t, s)v'(s) + U(t, s)A(s, u(s))v(s) = 0 \quad \text{a.e.}$$

Integrating in $s$ from 0 to $t$, we get

$$U(t, t)v(t) - U(t, 0)v(0) = v(t) - U(t, 0)x_1 = v(t) - u(t) = \text{constant}.$$
value $x_1$. In fact, this also makes $u$ a solution of (QL). We note that it is not necessary that $v([0, T]) \subset W$.

Now suppose that $v$ is a solution to (QL) on $[0, T]$ with initial value $x_1$. Then,
\[ v'(s) + A(s, v(s))v(s) = 0 \text{ a.e.}, \]
and so
\[ v'(s) + A(s, u(s))v(s) = (A(s, u(s)) - A(s, v(s)))v(s) \text{ a.e.} \]
Thus,
\[ \frac{\partial}{\partial s} U(t, s)v(s) = U(t, s)v'(s) + U(t, s)A(s, u(s))v(s) \text{ a.e.} \]
Integrating in $s$ from 0 to $t$, we get
\[ v(t) - u(t) = U(t, t)v(t) - U(t, 0)v(0) \]
\[ = \int_0^t U(t, s)(A(s, u(s)) - A(s, v(s)))v(s) \, ds. \]
This implies that
\[ \|v(t) - u(t)\|_X \leq M\epsilon^{\theta T}\|v\| \gamma C_1 \int_0^t \|u(s) - v(s)\|_X \, ds, \]
and thus $\|v(t) - u(t)\|_X = 0$ for all $t$ in $[0, T]$. This makes $u$ the unique solution of (QL) on $[0, T]$ with initial value $x_1$. This completes the proof of our Theorem. $\Box$

If $Y$ is reflexive, then by Kato's Theorem 5.1 [11], we have the result that $v = u$ is a solution of $(L; u)$ on $[0, T]$ with initial value $x_1$, and thus $u$ is a solution of (QL). This gives us Corollary 1. $\Box$

The remarks following the statement of the Theorem and Corollary are straightforward. We also note that Remark (1) deals with a particular case of condition (iii) of the theorem. Remark (2) contains a condition which greatly simplifies the proof of the Theorem, but which would be extremely difficult to verify in the absence of conditions stronger than condition (iii); e.g., see [2].

We now turn our attention to an application of our Theorem using the Sobolevskii-Tanabe theory of linear evolution equations of parabolic type.

Corollary 2. Let $S$ be the sector of the complex plane $C$ consisting of 0 and \( \{\lambda \in C: -\theta < \arg \lambda < \theta \} \), where $\theta \in (\pi/2, \pi)$ is fixed. We assume that conditions (i) and (ii) of the Theorem hold with $Y = D(A(t, w))$ for each $t, w$, and that

(iii)’ the resolvent set of $-A(t, w)$ contains $S$ and
\[ \| (\lambda I + A(t, w))^{-1} \|_X \leq C_4/ (1 + |\lambda|) \]

for each $\lambda \in S$, $t \in [0, T]$, and $w \in W$, where $C_4$ is a constant independent of $\lambda, t, \text{ and } w$.

Then, the conclusion of the Theorem holds and $(L; u)$ has a continuously differentiable $(X)$ solution on $[0, T']$ with initial value $x_1$, and thus $u$ is a continuously differentiable $(X)$ solution of (QL) on $[0, T']$ with initial value $x_1$.

Proof. Under these conditions the hypotheses of the Theorem hold, where $S(t, w) = A(t, w)$ for each $t$ and $w$. This gives us the limit solution $u$. The plan of attack is to produce a solution to $(L; u)$ which is continuously differentiable $(X)$ on
[0, T'] and has initial value \( x_1 \). This is where the Sobolevskii-Tanabe theory enters. Let \( A(t) = A(t, u(t)) \) for each \( t \in [0, T] \), and we see for \( t_1, t_2, t_3 \in [0, T] \) that

\[
\| (A(t_1) - A(t_2))A(t_3)^{-1} \|_X < \| A(t_1) - A(t_2) \|_{Y,X} \| A(t_3)^{-1} \|_{X,Y} < C_4 \| A(t_1, u(t_1)) - A(t_2, u(t_2)) \|_{Y,X} < C_5|t_2 - t_1| \quad \text{by (ii)}
\]

and the Lipschitz continuity of \( u \), where \( C_5 \) is independent of the choice of \( t_1, t_2, t_3 \). It follows from the Sobolevskii-Tanabe theory [14], [15], [19], [20], [21], [22] that there is an evolution operator \( \{ V(t, s) : 0 < s < t < T' \} \) such that \( v(t) = V(t, 0)x_1 \) defines a continuously differentiable \((X)\) function that satisfies \( v'(t) + A(t)v(t) = 0 \), \( v(0) = x_1 \). The operator also satisfies \( \| A(t)V(t, 0)A(0)^{-1} \|_X < C_6 \) on \([0, T']\), with \( C_6 \) independent of \( t \) [19, p. 5], thus

\[
\| v(t) \|_Y = \| V(t, 0)x_1 \|_Y = \| A(t)^{-1}A(i)V(t, 0)A(0)^{-1}A(0)x_1 \|_Y < \| A(t)^{-1} \|_{X,Y} \| A(t)V(t, 0)A(0)^{-1}X \|_X \| A(0) \|_{Y,X} \| x_1 \|_Y < C_7\| x_1 \|_Y,
\]

where \( C_7 \) is independent of \( t \). So, except for the image of \( v \) lying in \( W \), we have that \( v \) is a solution of \((L; u)\). Since the proof of the uniqueness of a solution to \((L; u)\) does not depend on \( v([0, T']) \subset W \), we still have that \( v(t) = u(t) \) on \([0, T']\). Consequently, \( u \) is the solution of \((QL)\) on \([0, T']\) with initial value \( x_1 \). In fact, \( u \) is continuously differentiable \((X)\), without exception, on \([0, T']\). □

We note that in general an application of the Theorem involves finding conditions that guarantee the existence of a solution to \((L; u)\), which then implies that \( u \) is the solution of \((QL)\).

It may be difficult at times to recognize that the conditions for our Theorem hold. The following Proposition gives criteria that obtain the Banach space \( Y \) and verify most of condition (iii) of the Theorem. If, in particular, we are able to use \( \lambda I + A(t, w) \), where \( \lambda > \beta \) is fixed, for \( S(t, w) \) in the Proposition, then condition (ii) of the Theorem holds as well as all of condition (iii).

**Proposition.** Let \( Y \) be a dense linear subspace of \( X \). Suppose for each \( t \in [0, T] \) and \( w \in W \) that \( S(t, w) \) is an isomorphism (algebraically) from \( Y \) onto \( X \), \( S(t, w) \) is a closed operator in \( X \), \( S(t, w)^{-1} \in B(X) \) with \( \| S(t, w)^{-1} \|_X < L_1 \), and the bounded linear operator \( S(t, w)S(t_0, w_0)^{-1} \) satisfies

\[
\| S(t_2, w_2)S(t_0, w_0)^{-1} - S(t_1, w_1)S(t_0, w_0)^{-1} \|_X < L_2(|t_2 - t_1| + \| w_2 - w_1 \|_X),
\]

where \( L_1, L_2, t_0 \) from \([0, T]\), and \( w_0 \in W \) are fixed. Suppose further that \( Y \) has the graph norm induced by \( S(t_0, w_0) \); i.e., for \( y \in Y \), \( \| y \|_Y = \| y \|_X + \| S(t_0, w_0)y \|_X \). Then,

(i) \( Y \) is a Banach space under this norm, and \( Y \) is continuously embedded in \( X \).

(ii) \( S(t, w)^{-1} \in B(X, Y) \) for each \( t \) and \( w \), and \( \| S(t, w)^{-1} \|_{X,Y} < 1 + L_1 + L_2(T + 2r) \), where \( r \) is the radius of the ball \( W \).

(iii) \( S(t, w) \in B(Y, X) \) for each \( t \) and \( w \), and \( \| S(t, w) \|_{Y,X} < 1 + L_2(T + 2r) = L_3 \).

(iv) \( \| S(t_2, w_2) - S(t_1, w_1) \|_{Y,X} < L_2L_3(|t_2 - t_1| + \| w_2 - w_1 \|_X) \).
Proof. Since $S(t_0, w_0)$ is a closed linear operator with domain $Y$, it is clear that $Y$ is a Banach space under the indicated norm. It is also immediate that $Y$ is continuously embedded in $X$.

Let $x \in X$, then

$$
\|S(t, w)^{-1}x\|_Y = \|S(t, w)^{-1}x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x\|_X \\
\leq L_1\|x\|_X + \|S(t_0, w_0)S(t, w)^{-1}x\|_X + \|x\|_X \\
\leq (L_1 + 1)\|x\|_X + L_2(|t - t_0| + \|w - w_0\|_X)\|x\|_X \\
\leq (L_1 + 1)\|x\|_X + L_2(T + 2r)\|x\|_X \\
= (1 + L_1 + L_2(T + 2r))\|x\|_X.
$$

So, $S(t, w)^{-1} \in B(X, Y)$ and $\|S(t, w)^{-1}\|_{X,Y} < 1 + L_1 + L_2(T + 2r)$.

Let $y \in Y$, then

$$
\|S(t, w)y\|_X = \|S(t, w)S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \\
\leq (1 + L_2(|t - t_0| + \|w - w_0\|_X)) \cdot \|S(t_0, w_0)y\|_X \\
\leq (1 + L_2(T + 2r))\|y\|_Y - \|y\|_X \\
\leq (1 + L_2(T + 2r))\|y\|_Y.
$$

So, $S(t, w) \in B(Y, X)$ and $\|S(t, w)\|_{Y,X} < 1 + L_2(T + 2r)$.

To show (iv), let $y \in Y$, then

$$
\|S(t_2, w_2)y - S(t_1, w_1)y\|_X = \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}S(t_0, w_0)y\|_X \\
\leq \|(S(t_2, w_2) - S(t_1, w_1))S(t_0, w_0)^{-1}\|_X \cdot \|S(t_0, w_0)y\|_Y \\
\leq L_2(|t_2 - t_1| + \|w_2 - w_1\|_X) \cdot \|S(t_0, w_0)\|_{Y,X}\|y\|_Y \\
\leq L_2L_3(|t_2 - t_1| + \|w_2 - w_1\|_X)\|y\|_Y.
$$

We also note that condition (i) for our Theorem holds when each $A(t, w)$ satisfies $\|\exp(-sA(t, w))\|_X < e^{Bt}$.

References

3. M. G. Crandall and L. C. Evans, On the relation of the operator $\partial / \partial s + \partial / \partial t$ to evolution governed by accretive operators, Israel J. Math. 21 (1975), 261–278.


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