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FLOWS ON FIBRE BUNDLES

BY

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ABSTRACT. Conditions are given under which a fibrewise flow on a fibre bundle must have a nonempty catastrophe space.

1. The problem. When we formulate the catastrophe theory of R. Thom globally we have a fibre bundle E over a connected finite CW-complex B. The fibre M of E is a closed C^r manifold, and the structure group of E is a subgroup of the group Diff M of C^r diffeomorphisms $M \to M$ with the C^r topology ($r \geqslant 1$). We say that E is a C^r *bundle* for short. Then B is the space of *observables* and M is the manifold of *internal variables*. Let Vect M be the space of C^{r-1} vector fields on M with the C^{r-1} topology. We define an action '$\cdot$' of Diff M on Vect M by means of the identity $f \cdot V = (df)Vf^{-1}$ where $f \in$ Diff M, $V \in$ Vect M. In catastrophe theory the bundle with fibre Vect M associated with E has a cross-section. We think of this cross-section as a family V_b ($b \in B$) of fibrewise C^{r-1} vector fields on E.

We next define an attractor of a vector field. The definition in [5, §4.1] seems to be imprecise and we use the following definition instead.

DEFINITION. An *attractor* of $V \in$ Vect M is a closed invariant subspace A of M such that

(i) there is an invariant neighbourhood U of A for which $\bigcap_{t>0} \phi_{Vt} U = A$,

(ii) some trajectory of V is dense on A (here ϕ_V is the flow on M corresponding to V).

Perhaps an attractor ought to satisfy additional conditions but these would not affect our main results.

We suppose that we are given a *convention*, namely an assignment to each $b \in B$ of an attractor A_b of V_b. We think of a convention as a physical law, and of A_b as the physical state of B. Then $b \in B$ is said to be *regular* when it has a neighbourhood W for which there is a fibre-preserving homeomorphism h: $E|W \to W \times E_b$ onto the trivial bundle satisfying

(i) $h|E_b$ is the identity,

(ii) $h|A_c = A_b$ and $h\phi_{V_c} = \phi_{V_b}h$ for all $c \in W$.

The points that are not regular make up the *catastrophe subspace* K of B.

The purpose of this paper is to study the following problem. Suppose that we are given E and a closed connected subspace F of M. Then we wish to decide whether K can be empty with $A_0 = F$, that is to say whether there is a family V_b ($b \in B$) of fibrewise C^{r-1} vector fields on E and a convention such that K is empty and $A_0 = F$. Here 0 is the basepoint of B.

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If K can be empty with A_0 a point then the assignment $b \mapsto A_b$ defines a cross-section of E. Conversely let s be a cross-section of E. Then K can be empty with A_0 a point. To prove this we argue as follows. The tangent bundles TE_b of the manifolds E_b ($b \in B$) make up a vector bundle TFE over E. Choose a Riemannian metric on TFE and let T be $TFE|s(B)$. Then T is a Riemannian vector bundle over B and the T_b are the tangent spaces at $s(b)$ to the manifolds E_b ($b \in B$). Using the compactness of B we choose $\delta > 0$ so that the fibrewise exponential map ε: $T \to E$ maps the open disc bundle B_δ of radius δ homeomorphically into E. Then $N = \varepsilon(B_\delta)$ is a neighbourhood of $s(B)$ in E. We identify T with its own fibrewise tangent bundle and for $e \in E_b$ we define $V_b(e)$ to be either 0 or

$$(-d\varepsilon_b(t))\exp(-\sec\theta)$$

accordingly as $e \notin N$ or $e = \varepsilon(t)$ for $t \in B_\delta$. Here $\theta = \pi\|t\|/2\delta$. Taking $A_b = \{s(b)\}$ we see that K is empty.

In general when K is empty the A_b make up a fibre bundle A over B. In some cases A is a C^r bundle and there is a *fibrewise C^r embedding* f: $A \to E$. By this we mean that f is a fibre-preserving map such that

(i) f_b: $A_b \to E_b$ is a C^r embedding.

(ii) f_b varies continuously in the C^r topology with $b \in B$.

In §§2 and 3 we prove conditions necessary for the existence of fibrewise C^r embeddings, and in §4 we apply these results to our original problem. Our main results assert that under certain conditions the catastrophe space K must be nonempty.

2. A related problem. Let E, A be C^r bundles with fibres closed C^r manifolds M, F over a connected finite CW-complex B ($r \geqslant 1$).

LEMMA 1. *Let f: $A \to E$ be a fibrewise C^r embedding. Then the complement $E - f(A)$ is a C^r bundle over B.*

This requires only a local proof, and so we suppose that $E = B \times M$ and that there is a C^r trivialization g: $B \times F \to A$. Then it suffices to extend fg for each $b \in B$ over some neighbourhood U of b to a C^r trivialization h: $U \times M \to E|U = U \times M$. But this can be done because of the result, due to R. S. Palais, that the evaluation map on spaces of C^r embeddings is locally trivial. We refer to **[2]** for a short proof of Palais' theorem.

Two fibrewise C^r embeddings f_0, f_1: $A \to E$ are said to be *isotopic* when there is a fibrewise C^r embedding F: $A \times [0, 1] \to E \times [0, 1]$ over $B \times [0, 1]$ such that

$$F|A \times \{0\} = f_0, \qquad F|A \times \{1\} = f_1.$$

Applying Lemma 1 to F and using **[3**, 11.4**]** we have the following lemma.

LEMMA 2. *Let f_0, f_1: $A \to E$ be isotopic fibrewise C^r embeddings. Then the C^r bundles $E - f_0(A)$, $E - f_1(A)$ are equivalent.*

Let E, A be orthogonal sphere bundles of fibre dimensions $q > p \geqslant 1$ and let f: $A \to E$ be a fibrewise C^r embedding. We suppose that f is orthogonal when restricted to the fibre over the basepoint 0 so that, in particular, the $f_b: A_b \to E_b$ are unknotted when $q = p + 2$.

PROBLEM. (i) Is there a fibrewise orthogonal embedding of A in E?

(ii) If so then is f isotopic to a fibrewise orthogonal embedding?

An affirmative answer to (i) would mean that E was the fibre join of A with an orthogonal $q - p - 1$-sphere bundle. The Whitney duality theorem would then give conditions on the Stiefel-Whitney classes of E necessary for the existence of f.

REMARK. If A has a cross-section s (for example if A is an oriented circle bundle and $H^2(B; Z) = 0$) then the $df(TFA_{s(b)})$ define an orthogonal subbundle of E. We can identify this subbundle with A and so there is a fibrewise orthogonal embedding of A in E.

When A does not have a cross-section we can still obtain a condition necessary for the existence of f by pulling everything back over the principal bundle associated with A. We can then apply our remark, together with the Whitney duality theorem. However we shall do better than this.

Let W be the path component of the orthogonal embeddings in the space of C^r embeddings of S^p in S^q with the C^r topology. Note that any two orthogonal embeddings of S^p in S^q are isotopic, since $p < q$. We define an action '$\cdot$' of the direct product of the orthogonal groups $O(p + 1) \times O(q + 1)$ on W by means of the identity

$$((P, Q) \cdot g)(x) = Qg(P^{-1}x)$$

where $(P, Q) \in O(p + 1) \times O(q + 1), g \in W, x \in S^p$.

Let L be the fibre product of the principal bundles associated with A, E. Let D be the bundle associated with L and with fibre W. Then f corresponds to a cross-section of D. The Stiefel manifold $V_{q+1,p+1}$ is the $O(p + 1) \times O(q + 1)$-invariant subspace V of W consisting of the orthogonal embeddings. Let C be the bundle associated with L and with fibre V. Then the inclusion j of V in W extends to a fibre-preserving map from C to D.

Taking derivatives at the basepoint of S^p defines a retraction of W onto V and so $j_*: \pi_k V \to \pi_k W$ is the inclusion of a direct summand. By [**4**, Proposition 2] (see also [**7**]), j_* is surjective when $k \leqslant 2q - 4p - 3$. We use this to prove the following lemma.

LEMMA 3. (i) *If* $\dim B \leqslant 2q - 4p - 2$ *then there is a fibrewise orthogonal embedding of A in E.*

(ii) *If* $\dim B \leqslant 2q - 4p - 3$ *then f is isotopic to a fibrewise orthogonal embedding.*

For the proof let s be the cross-section of D corresponding to f and note that, since j_* is an isomorphism when $k \leqslant 2q - 4p - 3$, the vertical homotopy class of $s|B^{2q-4p-3}$ comes from a cross-section s_1 of $C|B^{2q-4p-3}$. This proves (ii). To prove (i) let $\theta_1 \in H^{2q-4p-2}(B; \pi_{2q-4p-3}V)$ be the obstruction to extending $s_1|B^{2q-4p-4}$ to a cross-section of $C|B^{2q-4p-2}$. Then $j_{**}\theta_1 \in H^{2q-4p-2}(B; \pi_{2q-4p-3}W)$ is the obstruction to extending $s|B^{2q-4p-4}$: $B^{2q-4p-4} \xrightarrow{s_1} C \to D$ to a cross-section of $D|B^{2q-4p-2}$. Since this obstruction is zero so also is θ_1, and $s_1|B^{2q-4p-4}$ extends to a cross-section of $C|B^{2q-4p-2}$. But cross-sections of C correspond bijectively and naturally to fibrewise orthogonal embeddings of A in E. This proves Lemma 3.

3. The general case. For the applications in §4 we take $p = 1$, and the most interesting values of q are probably 2, 3, 4. When $p = 1$ Lemma 3(i) holds trivially for these values of q, without the hypothesis that there is a fibrewise C^r embedding of A in E. We therefore need a more general result.

We continue to work in the context of §2 and include E in its fibre suspension ΣE in the usual way. Following f by j such inclusions ($j \geqslant 1$) we obtain a fibrewise C^r embedding f' of A in $E' = \Sigma^j E$.

LEMMA 1. *The inclusion of $\Sigma^j(E - f(A))$ in $E' - f'(A)$ is a fibre homotopy equivalence.*

By §2, Lemma 1 both $\Sigma^j(E - f(A))$ and $E' - f'(A)$ are fibre bundles. Therefore, by a result due to Dold **[1]**, it suffices to prove that the inclusion is a homotopy equivalence over the basepoint. But the inclusion $S(X - Y) \to SX - Y$ is a homotopy equivalence whenever (X, Y) is a polyhedral pair with X, Y compact. Iterating this j times with $X = M$, $Y = F$ we have Lemma 1.

LEMMA 2. *Let S^p be embedded orthogonally in S^q and let S^{q-p-1} be the $q - p - 1$-sphere orthogonal to S^p. Then the inclusion of S^{q-p-1} in $S^q - S^p$ is a homotopy equivalence.*

One proof is by induction on q, beginning at $q = p + 1$ and arguing as in the proof of Lemma 1.

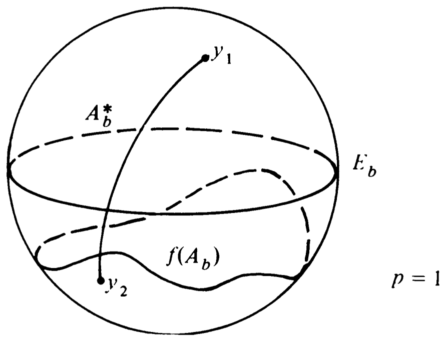
LEMMA 3. *Suppose that $j \geqslant [(\dim B + 1)/2] + 2p - q + 2$. Then $E' = \Sigma^j E$ is the fibre join of A with a bundle E'' which is fibre homotopy equivalent to the j-fold fibre suspension of a homotopy $q - p - 1$-sphere bundle.*

To prove the lemma we apply §2, Lemma 3(ii) to the fibrewise C^r embedding f' of A in E'. Thus f' is isotopic to a fibrewise orthogonal embedding f''. Let E'' be the orthogonal complement to $f''(A)$ in E'. Then E' is equivalent to the fibre join of A with E''. But by Lemma 2 the inclusion of E'' in $E' - f''(A)$ is a fibre homotopy equivalence. But $E' - f''(A)$ is equivalent to $E' - f'(A)$ by §2, Lemma 2, and $E' - f'(A)$ is fibre homotopy equivalent to $\Sigma^j(E - f(A))$ by Lemma 1. But $E' - f(A)$ is a homotopy $q - p - 1$-sphere bundle by Lemma 2. This completes the proof.

When $q = p + 1$ we also have the following result.

LEMMA 4. *Let $q = p + 1$. Then E is the fibre join of a 0-sphere bundle with an orthogonal p-sphere bundle A^* which is fibre homotopy equivalent to A.*

To prove this note, by §2, Lemma 1, the complement $E - f(A)$ is a C^r bundle over B whose fibre is the disjoint union of two copies of an open q-disc. Since the q-disc is contractible we may continuously assign to each $b \in B$ a subset $\{y_1, y_2\}$ of $E_b - f(A_b)$ so that y_1, y_2 lie in different path components. Let A_b^* be the p-sphere orthogonal to a geodesic joining y_1, y_2. Let E_b^* be the 0-sphere orthogonal to A_b^* in E_b. Then the A_b^*, E_b^* make up an orthogonal p-sphere bundle A^* and a 0-sphere bundle E^*. But E is the fibre join of A^* with E^*. To complete the proof we note that both A^* and A are fibre homotopy equivalent to the bundle $E - \bigcup_{b \in B}\{y_1, y_2\}$.



4. Applications. We resume the discussion of §1. For $q \geq 1$ let E be an orthogonal q-sphere bundle over a connected finite CW-complex B. Let $A_0 = F$ be a circle embedded orthogonally in $M = S^q$. If K is empty then the attractors A_b make up a circle bundle A' over B. From the definition of an attractor each $V_b|A_b$ has at most one zero. Therefore, and since all points are regular, A' is oriented. If $q = 1$ this means that E is orientable. Evidently the converse also holds. If $q = 1$ and E is orientable then K can be empty. From now on we suppose that $q \geq 2$.

If K is empty and $V_0|A_0$ has a zero then, by regularity, so has each other $V_b|A_b$ and the zeros define a cross-section of A'. Therefore A' is trivial, and so E has two orthogonal cross-sections. It follows that the trivial circle bundle A' embeds fibrewise orthogonally in E.

If K is empty and $V_0|A_0$ is never zero then, by regularity, so is each other $V_b|A_b$. Therefore the flow ϕ_{V_b} defines a C^r embedding of S^1 in E_b whose image is A_b and whose derivative maps the clockwise unit tangent field on S^1 to $V_b|A_b$. Such embeddings are unique up to rotations of S^1. Therefore A' is the image by a fibrewise C^r embedding of an oriented orthogonal circle bundle A. Summarizing, we have the first part of the following lemma.

LEMMA 1. (i) *If K can be empty then there is a fibrewise C^r embedding of an oriented orthogonal bundle A in E.*

(ii) *The converse also holds. We prove this in an appendix.*

Comparing Lemma 1(i) with §3, Lemma 4 we have the following result.

THEOREM 1. *Let $q = 2$. If K can be empty then E is the fibre join of an oriented orthogonal circle bundle with a* 0-*sphere bundle.* (*Of course the converse is contained in Lemma* 1(ii).)

The next result is a consequence of Lemma 1(i) and §2, Lemma 3(i).

THEOREM 2. *Suppose that* dim $B \leq 2q - 6$. *If K can be empty then E is the fibre join of an oriented orthogonal circle bundle with an orthogonal $q - 2$-sphere bundle.* (*Of course the converse is contained in Lemma* 1(ii).)

COROLLARY. *Let E be orientable.*

(i) *If $q \geqslant 6$ and if K can be empty then for some $a \in H^2(B; \mathbf{Z})$ the sum*

$$W_q + aW_{q-2} + a^2W_{q-4} + \cdots$$

is zero.

(ii) *If* $\dim B \leqslant q$ *and if for some a the sum in* (i) *is zero then K can be empty.*

Here W_i is the ith Stiefel-Whitney class of E and we work in H^*B with coefficients $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z}$ accordingly as q is odd or even. To prove (i) we apply the Whitney duality theorem with Theorem 2. To prove (ii) we apply [**6**, Proposition 2.1] together with Lemma 1(ii).

THEOREM 3. *Let $q = 3, 4, 5$ and let $j = [(\dim B + 1)/2] - q + 4$. If K can be empty then $\Sigma^j E$ is the fibre join of an oriented orthogonal circle bundle with a bundle which is fibre homotopy equivalent to the j-fold fibre suspension of a homotopy $q - 2$-sphere bundle.* (*Of course a converse is contained in Lemma* 1(ii).)

To prove Theorem 3 we compare Lemma 1(i) with §3, Lemma 3.

COROLLARY. *Let $q = 3, 4, 5$. If K can be empty then for some $a \in H^2(B; \mathbf{Z})$ the* mod 2 *reduction of the sum in the first part of the corollary to Theorem* 2 *is zero.* (*Of course a converse is contained in the second part of the corollary to Theorem* 2.)

To prove this we apply the mod 2 Whitney duality theorem with Theorem 3.

EXAMPLES. (i) If B is a sphere then K can be empty if and only if E has two orthogonal cross-sections. This follows from Lemma 1 together with the remark in §2.

(ii) Let B be a closed oriented 4-manifold such that for all $a \in H^2(B; \mathbf{Z})$ the mod 2 reduction of a^2 is zero. For instance $S^2 \times S^2$ would do. Let $g: B \to S^4$ be a degree 1 map. Let H be the Hopf 3-sphere bundle S^7 over S^4. Then let E be the pullback $g^*\Sigma H$ of the fibre suspension ΣH. We have $W_0 = 1$, $W_1 = W_2 = W_3 = 0$ and $W_4 \neq 0$. Therefore by the corollary to Theorem 3 the catastrophe space K must be nonempty.

There remains only an appendix for the proof of Lemma 1(ii). I wish to thank Dr. A. du Plessis and Professor M. G. Barratt for their helpful comments. I am especially grateful to Professor Barratt for his hospitality during the winter of 1976-77.

Appendix. Let E be a C^r bundle with fibre a closed C^r manifold M ($r \geqslant 1$) over a connected finite CW-complex B. Let A' be an oriented orthogonal circle bundle over B. The purpose of this appendix is to prove the following result.

PROPOSITION. *If there is a fibrewise C^r embedding of A' in E then K can be empty with $A_0 = A'_0$.*

The proof is modelled on that of the corresponding result in §1 for point attractors. As in §1 we form the vector bundles TFE, TFA' over E, A'. We identify A' with the image in E. Then the line bundle TFA' is a subbundle of $TFE|A'$. We choose a Riemannian metric on $TFE|A'$.

If dim $M = 1$ then M is a disjoint union of circles and the proposition holds trivially. If dim $M > 1$ we define NFA' to be the orthogonal complement to TFA' in $TFE|A'$. Then NFA' is a Riemannian vector bundle over A'. Using the compactness of B we choose $\delta > 0$ so that the fibrewise exponential map ε: $NFA' \to E$ maps the open disc bundle B_δ of radius δ homeomorphically into E. Then $N = \varepsilon(B_\delta)$ is a neighbourhood of A' in E.

We identify NFA' with its own fibrewise tangent bundle and define a family R_b ($b \in B$) of fibrewise C^{r-1} vector fields on N by means of the identity $R_b(e) = -d\varepsilon_b(t)$ where $e \in E_b$ and $e = \varepsilon(t)$ for $t \in B_\delta$. The orientation defines a family of fibrewise C^{r-1} unit vector fields on A'. We extend this to a family C_b ($b \in B$) of fibrewise C^{r-1} vector fields on N. Then for $e \in E_b$ we define $V_b(e)$ to be either 0 or

$$C_b(e) + R_b(e)\exp(-\sec\theta)$$

accordingly as $e \notin N$ or $e = \varepsilon(t) \in N$ for $t \in B_\delta$. Here $\theta = \pi\|t\|/2\delta$. Taking $A_b = A'_b$ we see that K is empty. This completes the proof.

References

1. A. Dold, *Über fasernweise Homotopieaquivälenz von Faserräumen*, Math. Z. **62** (1955), 111–136.

2. E. L. Lima, *On the local triviality of the restriction map for embeddings*, Comment. Math. Helv. **38** (1964), 163–164.

3. N. E. Steenrod, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, N. J., 1951.

4. A. S. Švarc, *Homology of spaces of smooth embeddings*, Soviet Math. Dokl. **1** (1966), 339–342.

5. R. Thom, *Structural stability and morphogenesis*, Benjamin, New York, 1975.

6. P. E. Thomas, *Fields of tangent k-planes on manifolds*, Invent. Math. **3** (1967), 334–347.

7. A. M. Vinogradov, *Some homotopic properties of the space of embeddings of a circle into a sphere or ball*, Soviet Math. Dokl. **5** (1964), 737–740.

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