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ON HARISH-CHANDRA'S μ-FUNCTION FOR p-ADIC GROUPS

BY

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ABSTRACT. The Harish-Chandra μ-function is, up to known constant factors, the Plancherel's measure associated to an induced series of representations. In this paper we show that, when the series is induced from special representations lifted to a parabolic subgroup, the μ-function is a quotient of translated μ-functions associated to series induced from supercuspidal representations. It is now known, in both the real and p-adic cases, that the μ-function is always an Euler factor.

This paper continues the development of the theory of harmonic analysis on reductive p-adic groups. In the first section we recall notation, terminology, and a few facts from [1] and [4]. The second section states and proves our main result (Theorem 1). In the proof of Theorem 1, Harish-Chandra's inequalities and theory of the Schwartz-space for p-adic groups ([4, §4.3], especially) play an essential role. The proof proceeds via three lemmas which show, in essence, that one can compute the c-functions (or intertwining operators) for induced from discrete series representations by analytically continuing the corresponding functions for certain induced from supercuspidal representations. These results are extensions and refinements of results of Harish-Chandra [1], [2]. They are analogues of results proved by Knapp and Stein [3] and Wallach [6] for real groups.

The major difference between the theory as developed in the references for real groups and the present paper lies in the fact that supercuspidal representations, in the p-adic case, may be associated to the Levi factor of any parabolic subgroup while, for real groups, one needs to study the analytic continuation of intertwining operators only for principal series representations, i.e., representations induced from a minimal parabolic subgroup. Since a Levi factor of a nonminimal parabolic subgroup is not compact modulo its center, our paper depends on the theory of the Schwartz space in a way not required for the corresponding discussion for real groups. Otherwise our methods are quite similar to those of Knapp and Stein and Wallach.

1. Preliminaries. Let Ω be a nonarchimedean local field of module q. Let G be the group of all Ω-points of a connected reductive algebraic group defined over Ω.

Throughout the paper we shall employ the convention of referring to algebraic groups defined over Ω and to their corresponding groups of Ω-points by the same capital letter, leaving it to the reader to interpret our meaning from the context.

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Fix a minimal p-pair (P_0, A_0) $(P_0 = M_0N_0)$ and an A_0-good maximal compact subgroup K [4, §0.6]. We use a subscript zero to denote various entities associated to A_0 (e.g., $\mathfrak{a}_0^*$, $\mathfrak{a}_0$, $W(A_0)$, etc.; see later) as opposed to the analogous entities for the more general tori and p-pairs considered below.

Let A be a standard torus of G and let $M = M_A$ denote its centralizer. We write $\mathcal{P}(A)$ for the set of all p-subgroups of G having A as a split component. Every $P \in \mathcal{P}(A)$ has a Levi decomposition of the form $P = MN_P = MN$. For every $P \in \mathcal{P}(A)$ there is a unique p-subgroup $\bar{P} \in \mathcal{P}(A)$ such that $P \cap \bar{P} = M$. We call $\bar{P}$ the opposite of P and write $\bar{N} = N_{\bar{P}}$ for its unipotent radical. Write δ_P for the modular function of $P \in \mathcal{P}(A)$.

Let $W(A)$ or $W(G/A)$ denote the "Weyl group of A", i.e., the finite factor group $\mathrm{Norm}_G(A)/\mathrm{Cent}_G(A)$.

Fix a smooth unitary double representation (V, τ) of K which satisfies associativity conditions [4, §1.12]. Let τ_M denote the restriction of τ to $K_M = K \cap M$. Given $P_1, P_2 \in \mathcal{P}(A)$ $(P_i = MN_i,\ i = 1, 2)$, we write $V_{P_1|P_2}$ for the subspace of V consisting of all $v \in V$ which satisfy $\tau(n_1)v\tau(n_2) = v$ for all $n_i \in N_i \cap K$ $(i = 1, 2)$. Then $V_{P_1|P_2}$ is τ_M-stable; we write $\tau_{P_1|P_2}$ for the subrepresentation of τ_M on $V_{P_1|P_2}$. Write $C(M, \tau_M)$ for the space of all functions f: $M \to V$ which satisfy $f(k_1mk_2) = \tau_M(k_1)f(m)\tau_M(k_2)$ $(m \in M;\ k_1, k_2 \in K_M)$. Write $C(M, \tau_{P_1|P_2})$ for the subspace of $C(M, \tau_M)$ consisting of all functions whose images lie in $V_{P_1|P_2}$.

We write $\mathcal{E}_{\mathbf{C}}(M)$ for the set of all classes of irreducible admissible representations of M, $\mathcal{E}(M)$ for the subset consisting of all unitary classes. We write ${}^0\mathcal{E}_{\mathbf{C}}(M)$ for the set of all classes of irreducible supercuspidal representations of M; ${}^0\mathcal{E}(M)$ denotes ${}^0\mathcal{E}_{\mathbf{C}}(M) \cap \mathcal{E}(M)$. Write $\mathcal{E}_2(M)$ for the set of all classes of discrete series representations of M and $\mathcal{E}_{\mathrm{sp}}(M) = \mathcal{E}_2(M) - {}^0\mathcal{E}(M)$.

Let $X(A)$ and $X(M)$ denote, respectively, the group of all rational characters of A and the group of all rational characters of M which are defined over Ω. The group $X(A)$ is a free abelian group of finite rank and we may regard $X(M)$ as a subgroup of finite index. Write $\mathfrak{a}^* = X(A) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\mathfrak{a} = \mathrm{Hom}(X(A), \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{R}$, $\mathfrak{a}_{\mathbf{C}}^*$ and $\mathfrak{a}_{\mathbf{C}}$ for the respective complexifications. We regard $X(A)$ and $X(M)$ as identified with their images in $\mathfrak{a}^*$ and $\mathfrak{a}_{\mathbf{C}}^*$. There is a natural pairing $\langle\ ,\ \rangle$: $\mathfrak{a}_{\mathbf{C}}^* \times \mathfrak{a}_{\mathbf{C}} \to \mathbf{C}$ and a homomorphism H: $M \to \mathfrak{a}$ such that $\log_q|\chi(m)| = \langle \chi, H(m)\rangle$ for all $m \in M$ and $\chi \in X(M)$. We also define H_P: $G \to \mathfrak{a}$ by setting $H_P(kmn) = H(m)$ $(k \in K, m \in M, n \in N)$ for any $P \in \mathcal{P}(A)$. For any $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ we define a quasi-character χ_ν of M by setting $\chi_\nu(m) = q^{\sqrt{-1}\,\langle \nu, H(m)\rangle}$ $(m \in M)$; if $\nu \in \mathfrak{a}^*$, then χ_ν is a unitary character. For any $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ we write $\nu = \nu_{\mathbf{R}} + \sqrt{-1}\,\nu_I$ $(\nu_{\mathbf{R}}, \nu_I \in \mathfrak{a}^*)$ and set $\bar{\nu} = \nu_{\mathbf{R}} - \sqrt{-1}\,\nu_I$.

For any $P \in \mathcal{P}(A)$ we write ρ_P for the element of $\mathfrak{a}^*$ such that $q^{2\langle \rho_P, H_P(p)\rangle} = \delta_P(p)$ for all $p \in P$.

To any $P \in \mathcal{P}(A)$ there corresponds the set $\Sigma(P, A)$ of positive A-roots; the set $\Sigma(P, A)$ equals the set of images in $\mathfrak{a}^*$ of the nontrivial rational characters of A which occur in the adjoint representation of A acting on the Lie algebra of P. We fix a $W(A_0)$-invariant inner product on $\mathfrak{a}_0^*$ and use it to identify $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$, thus $\mathfrak{a}$ and $\mathfrak{a}^*$, too. Regard $\Sigma(P, A)$ as a subset of $\mathfrak{a}$. Let ${}^+\mathfrak{a}^*$ denote the open chamber in

$\mathfrak{a}^*$ consisting of all $\nu \in \mathfrak{a}^*$ such that $\langle \nu, \alpha \rangle > 0$ for all $\alpha \in \Sigma(P, A)$. Let $\mathfrak{F}_{\mathbf{C}}(P)$ be the set of all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ such that $\nu = \nu_{\mathbf{R}} + \sqrt{-1}\,\nu_I$ with $\nu_{\mathbf{R}} \in \mathfrak{a}^*$ and $\nu_I \in {}^+\mathfrak{a}^*$.

Normalize the Haar measure on $\bar{N}$ such that $\int_{\bar{N} \cap K} d\bar{n} = 1$. Set

$$\gamma(G/P) = \int_{\bar{N}} q^{-2\langle \rho_P, H_P(\bar{n}) \rangle} d\bar{n}.$$

It is known that $\gamma(G/P)$ does not depend on the choice of $P \in \mathcal{P}(A)$ [4, Corollary 5.4.3.4], so we write $\gamma(G/M) = \gamma(G/P)$. If the Haar measure dk on K is normalized (so that $1 = \int_K dk$), then dk and $\gamma^{-1}(G/M) q^{-2\langle \rho_P, H_P(\bar{n}) \rangle} d\bar{n}$ determine equal measures on the homogeneous space G/P.

We recall that both $\mathfrak{a}_{\mathbf{C}}^*$ and $W(A)$ act on $\mathfrak{S}_{\mathbf{C}}(M)$. For any $\sigma \in \omega \in \mathfrak{S}_{\mathbf{C}}(M)$ and $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ set $\sigma_\nu(m) = \sigma(m)\chi_\nu(m)$ $(m \in M)$. Call ω_ν the class of σ_ν. The groups $\mathfrak{a}_{\mathbf{C}}^*$ and $\mathfrak{a}^*$ stabilize ${}^0\mathfrak{S}_{\mathbf{C}}(M)$ and $\mathfrak{S}_2(M)$, respectively. For any $\sigma \in \omega \in \mathfrak{S}_{\mathbf{C}}(M)$ and any representative $y = y(s) \in G$ of $s \in W(A)$ set $\sigma^y(ymy^{-1}) = \sigma(m)$ $(m \in M)$. The class ω^s of σ^y depends only upon s.

For any $P \in \mathcal{P}(A)$ $(P = MN)$ and $\sigma \in \omega \in \mathfrak{S}_{\mathbf{C}}(M)$ we write $I(P, \sigma)$ for the admissible induced representation $\mathrm{Ind}_P^G(\delta_P^{1/2}\sigma)$, where $\sigma \in \omega$ is regarded as a representation of P which is trivial on N. We write $I(P, \omega)$ for the class of $I(P, \sigma)$.

Let $\sigma \in \omega \in \mathfrak{S}_{\mathbf{C}}(M)$ and let σ act in a vector space U. Let $\sigma^\sim$ denote the (admissible) contragredient of σ, acting in the space $U^\sim$, of smooth linear functionals on U. A function of the form $f(m) = \langle u^\sim, \sigma(m)u \rangle$ $(u \in U,\ u^\sim \in U^\sim;\ m \in M)$ is called a matrix coefficient of σ or ω. We write $\mathcal{Q}(\omega)$ for the vector space spanned by all the matrix coefficients of ω. We also write

$$\mathcal{Q}(\omega, \tau_M) = (\mathcal{Q}(\omega) \otimes V) \cap C(M, \tau_M), \quad \mathcal{Q}(\omega, \tau_{P_1|P_2}) = \mathcal{Q}(\omega, \tau_M) \cap \left(\mathcal{Q}(\omega) \otimes V_{P_1|P_2}\right)$$

for any $P_1, P_2 \in \mathcal{P}(A)$. For any $P \in \mathcal{P}(A)$, set $L(\omega, P) = \mathcal{Q}(\omega, \tau_{P|P})$ and $\mathfrak{L}(\omega, P) = \mathcal{Q}(\omega, \tau_{P|\bar{P}})$.

Let $P \in \mathcal{P}(A)$ and $\psi \in L(\omega, P)$. We extend ψ to a function on G by setting $\psi(kmn) = \tau(k)\psi(m)$ $(k \in K, m \in M, n \in N)$ and define the Eisenstein integral

$$E(P{:}\psi{:}\nu{:}x) = \int_K \psi(xk)\tau(k^{-1}) q^{\langle \sqrt{-1}\,\nu - \rho_P, H_P(xk) \rangle} dk \qquad (x \in G, \nu \in \mathfrak{a}_{\mathbf{C}}^*).$$

In this expression dk denotes a normalized Haar measure on K. The Eisenstein integral defines a surjective mapping of $L(\omega, P)$ on $\mathcal{Q}(I(P, \omega_\nu), \tau)$; the mapping is bijective if and only if $I(P, \omega_\nu)$ is irreducible.

From here on let $\sigma \in \omega \in \mathfrak{S}_2(M)$. Then $\mathcal{Q}(\omega, \tau_M)$ has the structure of a prehilbert space, the norm being defined by the formula $\|\psi\|^2 = \int_{M/A} \|\psi(m)\|_V^2\, dm^*$, where dm^* denotes a Haar measure on M/A.

For every $P_1, P_2 \in \mathcal{P}(A)$ and $s \in W(A)$ there is a meromorphic function of $\nu \in \mathfrak{a}_{\mathbf{C}}^*$,

$$c_{P_2|P_1}(s{:}\omega{:}\nu)\colon L(\omega, P_1) \to \mathfrak{L}(\omega^s, P_2)$$

defined such that for almost all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ the weak constant term is defined and satisfies the relation

$${}_wE_{P_2}(P_1{:}\psi{:}\nu{:}m) = \sum_{s \in W(A)} c_{P_2|P_1}(s{:}\omega{:}\nu)\psi(m)\chi_{s\nu}(m) \qquad (m \in M).$$

We also have the adjoint mappings $c_{P_2|P_1}(s:\omega:\nu)^*$: $\mathfrak{L}(\omega^s, P_2) \to L(\omega, P_1)$, defined relative to the pre-hilbert structure induced on these spaces as subspaces of $\mathcal{Q}(\omega, \tau_M)$. We may regard $c_{P_2|P_1}(s:\omega:\bar{\nu})^*$ as a meromorphic function of $\nu \in \mathfrak{a}_{\mathbf{C}}^*$. The functions $c_{P_2|P_1}(s:\omega:\nu)$ and $c_{P_2|P_1}(s:\omega:\bar{\nu})^*$ are bijective for almost all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$.

There is a scalar-valued meromorphic function $\mu(\omega:\nu) = \mu_G(\omega:\nu)$ on $\mathfrak{a}_{\mathbf{C}}^*$ with the following properties.

(1) μ is holomorphic on $\mathfrak{a}^*$ and assumes nonnegative real values there.

(2) $\mu(\omega:\nu)c_{P_2|P_1}(s:\omega:\bar{\nu})^*$ is the inverse of $c_{P_2|P_1}(s:\omega:\nu)$ for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ where each is defined (cf. [**1**, Theorem 20] and [**4**, Theorem 5.3.5.2]).

(3) Up to known constant factors $\mu(\omega:\nu)$ is the Plancherel's measure for the induced series $\{I(P, \omega_\nu)\ (\nu \in \mathfrak{a}^*)\}$ (cf. [**1**, Theorem 34] and [**4**, Theorem 5.5.2.1]).

The function $\mu(\omega:\nu)$ has an analogue for real groups. This analogue has been shown to be expressible as a product of Γ-factors (cf. [**2**] and [**6**]). In our paper [**5**], we have shown that, for $\omega \in {}^0\mathfrak{E}(M)$, $\mu(\omega:\nu)$ is, after a change of variables, a product of rational functions. In the theorem of the next section we shall show that, if $\omega \in \mathfrak{E}_{\mathrm{sp}}(M)$, then $\mu(\omega:\nu)$ may be represented as a quotient of μ-functions associated to supercuspidal classes. It follows that $\mu(\omega:\nu)$ may always, through a natural change of variables, be transformed into a rational function. In this sense, the μ-functions, both for real and p-adic groups, are "Euler factors"–like the Euler factors corresponding to the Dirichlet series of number theory and the theory of automorphic forms. Our discussion below also implies that the c-functions are in essentially the same sense Eulerian, too.

2. The theorem. Let (P', A') $(P' = M'N')$ be a standard p-pair of G. Let $\sigma' \in \omega' \in \mathfrak{E}_{\mathrm{sp}}(M')$. Let (P, A) $(P = MN)$ be a standard p-pair of G such that $P \subset P'$ and $A' \subset A$. Write $({}^*P, A)$ $({}^*P = M^*N)$ for the standard p-pair $(P \cap M', A)$ $({}^*N = N \cap M')$ of M'. Let $\sigma \in \omega \in {}^0\mathfrak{E}(M)$ and assume (as we may, without loss of generality, according to results of Jacquet and Harish-Chandra [**4**, Corollary 2.4.2 and Theorem 4.4.4]) that ω' occurs as a quotient class of $I({}^*P, \omega_{\sqrt{-1}\,\nu_0})$ with $\nu_0 = \sum c_\alpha \alpha \in \mathfrak{a}^*$ (summed over $\alpha \in \Sigma({}^*P, A)$; $c_\alpha > 0$). In this case for any $\nu' \in \mathfrak{a}'_{\mathbf{C}}{}^*$, $I(P', \omega'_{\nu'})$ occurs as a quotient class of $I(P', I({}^*P, \omega_{\sqrt{-1}\,\nu_0+\nu'}))$ $= I(P, \omega_{\sqrt{-1}\,\nu_0+\nu'})$.

The result we wish to establish is

THEOREM 1. *With the above notations, define the meromorphic function*

$$\tilde{\mu}(\omega:\nu) = \frac{\mu_G\left(\omega:\sqrt{-1}\ \nu_0 + \nu\right)}{\mu_{M'}\left(\omega:\sqrt{-1}\ \nu_0 + \nu\right)} \qquad (\nu \in \mathfrak{a}_{\mathbf{C}}^*).$$

Then $\mu_G(\omega':\nu') = \tilde{\mu}(\omega:\nu')$ $(\nu' \in \mathfrak{a}'_{\mathbf{C}}{}^*)$.

The remainder of the paper is devoted to the proof of Theorem 1.

We introduce a slight variant of the set-up of [**4**, Theorem 5.3.5.4] (also see [**1**, §13]). First, fix a set of representatives $S \subset M$ for $K \backslash G/N$. Define the locally constant function μ: $G \to S$ and the continuous functions κ: $G \to K$ and η: $G \to N$ such that $x = \kappa(x)\mu(x)\eta(x)$ for all $x \in G$. Assume that $\kappa(x) \in K \cap M' = {}^*K$ and

$\eta(x) \in {}^*N$ for all $x \in M'$. Write $\eta(x) = {}^*\eta(x)\eta'(x)$ with ${}^*\eta(x) \in {}^*N$ and $\eta'(x) \in N'$, also $\mu'(x) = \mu(x){}^*\eta(x)$, i.e., $x = \kappa(x)\mu'(x)\eta'(x) = \kappa(x)\mu(x){}^*\eta(x)\eta'(x)$.

Let $Q \in \mathcal{P}(A)$ be such that $Q \subset P'$ and $Q \cap M' = {}^*\bar{P}$. For any $\nu \in \mathcal{F}_{\mathbf{C}}(Q)$ and any $\psi \in \mathcal{A}(\omega, \tau_M)$ set

$$J_{P'}(\nu)\psi(m) = \gamma(G/M')^{-1}\int_{\bar{N}'} \tau(\kappa(\bar{n}'))\psi(\mu(\bar{n}')m)q^{\langle\sqrt{-1}\,\nu-\rho_P, H_P(\bar{n}')\rangle}\,d\bar{n}',$$

$${}_{*P}J(\nu)\psi(m) = \gamma(M'/M)^{-1}\int_{{}^*\bar{N}} \psi\big(m\mu({}^*\bar{n})^{-1}\big)\tau\big(\kappa({}^*\bar{n})^{-1}\big)q^{\langle-\sqrt{-1}\,\nu-\rho_P, H_P({}^*\bar{n})\rangle}\,d{}^*\bar{n},$$

and

$$\begin{aligned} J_{\bar{Q}|P}(\nu)\psi(m) &= J_{P'}(\nu)\,{}_{*P}J(\nu)\psi(m) \\ &= {}_{*P}J(\nu)J_{P'}(\nu)\psi(m) \qquad (m \in M). \end{aligned}$$

The above integrals are absolutely and compact uniformly convergent (at least) on $\mathcal{F}_{\mathbf{C}}(Q)$. By [4, Theorem 5.3.5.4]

$$\begin{aligned} J_{\bar{Q}|P}(\nu)\psi &= c_{\bar{Q}|P}(1:\omega:\nu)\psi \qquad (\psi \in L(\omega, P)), \\ &= c_{Q|\bar{P}}(1:\omega:\bar{\nu})^*\psi \qquad (\psi \in \mathfrak{L}(\omega, Q)); \\ {}_{*P}J(\nu)\psi &= c_{*P|*P}(1:\omega:\nu)\psi \qquad (\psi \in L(\omega, {}^*P)), \\ &= c_{*\bar{P}|*\bar{P}}(1:\omega:\bar{\nu})^*\psi \qquad \big(\psi \in \mathfrak{L}(\omega, {}^*\bar{P})\big). \end{aligned}$$

Since the c-functions and their adjoints are meromorphic functions on $\mathfrak{a}_{\mathbf{C}}^*$, it follows that $J_{\bar{Q}|P}$ and ${}_{*P}J$, regarded as operators with the respective domains $L(\omega, P)$ or $\mathfrak{L}(\omega, Q)$, $L(\omega, {}^*P)$ or $\mathfrak{L}(\omega, {}^*\bar{P})$, have meromorphic extensions to $\mathfrak{a}_{\mathbf{C}}^*$. Similarly, for suitably chosen ψ, the functions $J_{P'}(\nu)\psi$ are also meromorphic on $\mathfrak{a}_{\mathbf{C}}^*$. Each of these operators has an adjoint with respect to the given pre-hilbert structure on $\mathcal{A}(\omega, \tau_M)$. (We always denote the adjoint of an operator T by T^*.) We shall also use, without explicitly defining them, the operators $J_{\bar{P}'}$, ${}_{*\bar{P}}J$, $J_{Q|\bar{P}}$, and their adjoints, i.e., the analogous operators associated to the opposite parabolic subgroups.

LEMMA 2. *Let $\psi \in L(\omega, {}^*P)$ and assume that*

$$E\big({}^*P{:}\psi{:}\sqrt{-1}\,\nu_0\big) = \phi \in \mathcal{A}(\omega', \tau_{M'}).$$

(1) *If $\phi \in L(\omega', P')$, then*

$$c_{\bar{P}'|P'}(1{:}\omega'{:}\nu')\phi = E\big({}^*P{:}\,J_{P'}\big(\sqrt{-1}\,\nu_0 + \nu'\big)\psi{:}\,\sqrt{-1}\,\nu_0\big).$$

(2) *If $\phi \in \mathfrak{L}(\omega', P')$, then*

$$c_{P'|\bar{P}'}(1{:}\omega'{:}\bar{\nu}')^*\phi = E\big({}^*P{:}\,J_{P'}\big(\sqrt{-1}\,\nu_0 + \nu'\big)\psi{:}\,\sqrt{-1}\,\nu_0\big).$$

In both cases, the two sides of the equality and the operator $J_{P'}(\sqrt{-1}\,\nu_0 + \nu')$ are meromorphic functions of ν' on $\mathfrak{a}'^*_{\mathbf{C}}$ whose only singularities lie in hyperplanes.

PROOF. We prove only (1), as (2) goes similarly. By [4, Theorem 5.3.5.4], (analogue for $\omega' \in \mathcal{E}_2(M')$ of [1, Theorem 23])

$$\gamma(G/M')c_{\bar{P}'|P'}(1:\omega':\nu')\phi(m)\chi_{\nu'+\sqrt{-1}\,\rho_{P'}}(m)$$

$$= \int_{\bar{N}'}\int_{{}^*K} \tau(\kappa(\bar{n}'))\psi(\mu'(\bar{n}')mk)\tau(k^{-1})q^{\langle\sqrt{-1}\,\nu'-\rho_{P'},H_{P'}(\bar{n}'m)\rangle}$$

$$\cdot\, q^{\langle -\nu_0-\rho_{{}^*P},H_{{}^*P}(\mu'(\bar{n}')mk)\rangle}\,dk\,d\bar{n}'$$

$$= \int_{\bar{N}'}\int_{{}^*K} \tau(\kappa(\bar{n}'))\psi(\mu'(\bar{n}')mk)\tau(k^{-1})q^{\langle\sqrt{-1}\,\nu'-\nu_0-\rho_P,H_P(\bar{n}'mk)\rangle}\,dk\,d\bar{n}',$$

$$m \in M.$$

For all $\nu' \in \mathfrak{F}_{\mathbf{C}}(P')$ the iterated integral converges (again, [4, Theorem 5.3.5.4]).

First, let us see that iterating in the opposite order yields

$$\gamma(G/M')E\big({}^*P: J_{P'}\big(\sqrt{-1}\,\nu_0+\nu'\big)\psi:\sqrt{-1}\,(\nu_0+\rho_{P'})+\nu':m\big).$$

(We leave aside, for the moment, questions of convergence.) Setting

$${}^*\bar{n} = \kappa({}^*\bar{n})\mu({}^*\bar{n})\eta({}^*\bar{n}) \qquad ({}^*\bar{n} \in {}^*\bar{N} = \bar{N}\cap M')$$

and changing variables, we obtain

$$\gamma(M'/M)\int_{{}^*K}\int_{\bar{N}'} \tau(\kappa(\bar{n}'))\psi(\mu'(\bar{n}')mk)\tau(k^{-1})q^{\langle\sqrt{-1}\,\nu'-\nu_0-\rho_P,H_P(\bar{n}'mk)\rangle}\,d\bar{n}'\,dk$$

$$= \int_{{}^*\bar{N}}\int_{\bar{N}'} \tau(\kappa(\bar{n}'))\psi(\mu'(\bar{n}')m{}^*\bar{n}\mu^{-1}({}^*\bar{n}))\tau(\kappa({}^*\bar{n}))^{-1}$$

$$\cdot\, q^{\langle\sqrt{-1}\,\nu'-\nu_0-\rho_P,H_P(\bar{n}'m{}^*\bar{n}\mu^{-1}({}^*\bar{n}))\rangle-2\langle\rho_P,H_P({}^*\bar{n})\rangle}\,d\bar{n}'\,d{}^*\bar{n}$$

$$= \int_{{}^*\bar{N}}\int_{\bar{N}'} \tau(\kappa(\bar{n}'))\psi(\mu'(\bar{n}'){}^*\bar{n}_1m\mu^{-1}({}^*\bar{n}))\tau(\kappa({}^*\bar{n}))^{-1}$$

$$\cdot\, q^{\langle\sqrt{-1}\,\nu'-\nu_0-\rho_P,H_P(\bar{n}'{}^*\bar{n}_1m\mu^{-1}({}^*\bar{n}))\rangle-2\langle\rho_P,H_P({}^*\bar{n})\rangle}\,d\bar{n}'d{}^*\bar{n} \qquad ({}^*\bar{n}_1 = m{}^*\bar{n}m^{-1})$$

$$= \int_{{}^*\bar{N}}\int_{\bar{N}'} \tau(\kappa({}^*\bar{n}_1))\tau(\kappa(\bar{n}''))\psi(\mu(\bar{n}'')\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n}))\tau(\kappa({}^*\bar{n}))^{-1}$$

$$\cdot\, q^{\langle\sqrt{-1}\,\nu'-\nu_0-\rho_P,H_P(\bar{n}''\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n}))\rangle-2\langle\rho_P,H_P({}^*\bar{n})\rangle}\,d\bar{n}'\,d{}^*\bar{n}$$

$$(\bar{n}'' = \kappa^{-1}({}^*\bar{n}_1)\bar{n}'\kappa({}^*\bar{n}_1))$$

$$= \int_{{}^*\bar{N}} \tau(\kappa({}^*\bar{n}_1))\gamma(G/M')J_{P'}\big(\sqrt{-1}\,\nu_0+\nu'\big)$$

$$\cdot\, \psi(\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n}))\tau(\kappa({}^*\bar{n}))^{-1}$$

$$\cdot\, q^{\langle\sqrt{-1}\,\nu'-\nu_0-\rho_P,H_P(\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n}))\rangle-2\langle\rho_P,H_P({}^*\bar{n})\rangle}\,d{}^*\bar{n}$$

$$= \gamma(M'/M)\gamma(G/M')E\big({}^*P:J_{P'}\big(\sqrt{-1}\,\nu_0+\nu'\big)\psi:\sqrt{-1}\,(\nu_0+\rho_{P'})+\nu':m\big),$$

as claimed.

We claim that, to prove Lemma 2(1), it is enough to show that the double integral exists for all ν' in an open subset of $\mathfrak{a}'_{\mathbf{C}}{}^*$. Since the singularities of $J_{P'}(\nu)$, by the product formula [4, §5.4.3], lie in hyperplanes in $\mathfrak{a}^*_{\mathbf{C}}$, the existence of the double integral on an open subset of $\mathfrak{a}'_{\mathbf{C}}{}^*$ implies that $J_{P'}(\sqrt{-1}\,\nu_0+\nu')$ is meromorphic as

a function of ν' on $\mathfrak{a}'_{\mathbf{C}}{}^*$ with singularities lying in hyperplanes of $\mathfrak{a}'_{\mathbf{C}}{}^*$. Since the Eisenstein integral obviously preserves meromorphicity, both sides of the integral formula are meromorphic on $\mathfrak{a}'_{\mathbf{C}}{}^*$ with hyperplane singularities. (The equality of the two sides combined with the product formula for $c_{\bar{P}'|P'}(1:\omega':\nu')$ also implies this.)

We shall check that the double integral exists for all $\nu' \in \mathfrak{a}'_{\mathbf{C}}{}^*$ such that $\nu' = \nu'_{\mathbf{R}} + \sqrt{-1}\,\nu'_I$ ($\nu'_{\mathbf{R}}, \nu'_I \in \mathfrak{a}'^*$) with $\langle \nu'_I + \nu_0, \alpha\rangle > 0$ for all $\alpha \in \Sigma(P, A) - \Sigma(^*P, A)$. To see this, let Ξ_M be the Harish-Chandra Ξ-function [4, §4.2] associated to the group M and the torus A_0. Extend Ξ_M to a function on G by setting $\Xi_M(kmn) = \Xi_M(m)$ ($k \in K$, $m \in M$, $n \in N$). Then [4, Corollary 4.3.10] implies that

$$\int_{\bar{N}'} q^{-(1+\varepsilon)\langle \rho_P, H_P(\bar{n}')\rangle} \Xi_M(\bar{n}')\, d\bar{n}'$$

converges absolutely for every $\varepsilon > 0$. We shall change variables and observe that the double integral can be, essentially, split into the product of two integrals, one of which is obviously finite and the other of which is dominated by the above integrand.

Since ω is a discrete series class (being in ${}^0\mathfrak{E}(M)$), $\psi \in \mathcal{C}_*(M, \tau_M)$ (the Schwartz space with respect to M/A). Therefore, there exists a constant $C > 0$ such that

$$\|\tau(\kappa(\bar{n}'))\psi(\mu'(\bar{n}')mk)\tau(k^{-1})\|_V |q^{\langle \sqrt{-1}\,\nu' - \nu_0 - \rho_P, H_P(\bar{n}'mk)\rangle}|$$
$$\leq C\Xi_M(\bar{n}'mk)q^{\langle -\nu'_I - \nu_0 - \rho_P, H_P(\bar{n}'mk)\rangle}, \quad \text{for all } \bar{n}' \in \bar{N}',\ m \in M, \text{ and } k \in {}^*K.$$

Changing variables as before, we have

$$\begin{aligned}
\gamma(M'/M) \int_{\bar{N}'}\int_{{}^*K} \Xi_M(\bar{n}'mk)q^{\langle -\nu'_I - \nu_0 - \rho_P, H_P(\bar{n}'mk)\rangle} dk\, d\bar{n}'
&= \int_{\bar{N}'}\int_{{}^*\bar{N}} \Xi_M(\bar{n}'m{}^*\bar{n}\mu^{-1}({}^*\bar{n})) \\
&\qquad \cdot q^{\langle -\nu'_I - \nu_0 - \rho_P, H_P(\bar{n}'m{}^*\bar{n}\mu^{-1}({}^*\bar{n}))\rangle - 2\langle \rho_P, H_P({}^*\bar{n})\rangle} d{}^*\bar{n}\, d\bar{n}' \\
&= \int_{\bar{N}'}\int_{{}^*\bar{N}} \Xi_M(\bar{n}''\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n})) \\
&\qquad \cdot q^{\langle -\nu'_I - \nu_0 - \rho_P, H_P(\bar{n}''\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n}))\rangle - 2\langle \rho_{{}^*P}, H_P({}^*\bar{n})\rangle} d{}^*\bar{n}\, d\bar{n}'' \\
&\qquad \left(\bar{n}'' = \kappa({}^*\bar{n}_1)^{-1}\bar{n}'\kappa({}^*\bar{n}_1);\ {}^*\bar{n}_1 = m{}^*\bar{n}m^{-1}\right) \\
&= \int_{\bar{N}'}\int_{{}^*\bar{N}} \Xi_M(\bar{n}''\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n}))q^{\langle -\nu'_I - \nu_0 - \rho_P, H_P(\bar{n}'')\rangle} \\
&\qquad \cdot q^{\langle -\nu_0 - \rho_P, H_P({}^*\bar{n}_1)\rangle} q^{\langle \nu_0 - \rho_P, H_P({}^*\bar{n})\rangle} q^{\langle -\nu'_I - \nu_0 - \rho_P, H_P(m)\rangle} d{}^*\bar{n}\, d\bar{n}''.
\end{aligned}$$

By [4, Lemma 4.3.4], $|\langle \rho_{P_0}, H_{P_0}({}^*\bar{n}_1) - H_{P_0}({}^*\bar{n})\rangle|$ is bounded for all ${}^*\bar{n} \in {}^*\bar{N}$; similarly, $|\langle \nu_0, H_P({}^*\bar{n}_1) - H_P({}^*\bar{n})\rangle|$. It follows from [4, Theorem 4.2.1] that $\Xi_M(\bar{n}'')$ and $\Xi_M(\bar{n}''\mu({}^*\bar{n}_1)m\mu^{-1}({}^*\bar{n}))$ have the same order of magnitude, independent of ${}^*\bar{n} \in {}^*\bar{N}$. Furthermore, $q^{-2\langle \rho_P, H_P({}^*\bar{n})\rangle}$ dominates $q^{-\langle \rho_P, H_P({}^*\bar{n}_1) + H_P({}^*\bar{n})\rangle}$. Thus, since $\int_{{}^*\bar{N}} q^{-2\langle \rho_P, H_P({}^*\bar{n})\rangle} d{}^*\bar{n}$ and $\int_{\bar{N}'} \Xi_M(\bar{n}')q^{-(1+\varepsilon)\langle \rho_P, H_P(\bar{n}')\rangle} d\bar{n}'$ converge, the double integral converges, provided $\langle \nu'_I + \nu_0, \alpha\rangle > 0$ for all $\alpha \in \Sigma(P, A) - \Sigma(^*P, A)$. Indeed, since $H_P(\bar{n}')$ is a nonnegative (essentially integer) linear combination of elements of

$\Sigma(P, A) - \Sigma({}^*P, A)$, we see that $\langle \nu_I' + \nu_0, H_P(\bar{n}')\rangle \geqslant \varepsilon\langle \rho_P, H_P(\bar{n}')\rangle$ for $\varepsilon > 0$, assuming the above condition on $\nu_I' + \nu_0$. This completes the proof of Lemma 2(1).

LEMMA 3. *For all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ where these operators are defined $J_{P'}(\bar{\nu})^* = J_{\bar{P}'}(\nu)$ on $J_{P'}(\nu)L(\omega, P)$; furthermore*

$$\mu_G(\omega : \nu)\mu_{M'}(\omega : \nu)^{-1}J_{\bar{P}'}(\nu)J_{P'}(\nu)$$

is the identity operator on $L(\omega, P)$.

PROOF. By [1, Theorem 20], for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$ and $\psi \in L(\omega, P)$

$$\begin{aligned}\psi &= \mu_G(\omega : \nu)c_{\bar{Q}|P}(1 : \omega : \bar{\nu})^* c_{\bar{Q}|P}(1 : \omega : \nu)\psi \\ &= \mu_G(\omega : \nu)J_{\bar{Q}|P}(1 : \omega : \bar{\nu})^* J_{\bar{Q}|P}(1 : \omega : \nu)\psi \\ &= \mu_G(\omega : \nu)J_{P'}(\bar{\nu})^* c_{*P|*P}(1 : \omega : \bar{\nu})^* c_{*P|*P}(1 : \omega : \nu)J_{P'}(\nu)\psi \\ &= \mu_G(\omega : \nu)\mu_{M'}(\omega : \nu)^{-1}J_{P'}(\bar{\nu})^* J_{P'}(\nu)\psi.\end{aligned}$$

It follows from [4, Theorem 5.3.5.4] that

$$J_{\bar{Q}|P}(\bar{\nu})^*\psi = J_{Q|\bar{P}}(\nu)\psi \quad \text{for all } \psi \in \mathfrak{L}(\omega, Q);$$

similarly

$${}_{*P}J(\bar{\nu})^*\psi = {}_{*\bar{P}}J(\nu)\psi \quad \text{for all } \psi \in \mathfrak{L}(\omega, {}^*\bar{P}).$$

Since, moreover,

$$J_{Q|\bar{P}}(\nu)\psi = {}_{*\bar{P}}J(\nu)J_{\bar{P}'}(\nu)\psi = J_{\bar{P}'}(\nu){}_{*\bar{P}}J(\nu)\psi = J_{P'}(\bar{\nu})^*{}_{*P}J(\bar{\nu})^*\psi$$

for all $\psi \in \mathfrak{L}(\omega, \bar{Q})$, we see that $J_{\bar{P}'}(\nu)\psi = J_{P'}(\bar{\nu})^*\psi$ for all $\psi \in J_{P'}(\nu)L(\omega, P)$ and that

$$\mu_G(\omega : \nu)\mu_{M'}(\omega : \nu)^{-1}J_{\bar{P}'}(\nu)J_{P'}(\nu)$$

is the identity on $L(\omega, P)$ for all $\nu \in \mathfrak{a}_{\mathbf{C}}^*$.

LEMMA 4. $\tilde{\mu}(\omega : \nu') = \mu(\omega' : \nu')$.

PROOF. Let ψ and ϕ be as in Lemma 2. By [4, Theorem 5.3.5.2] (generalization to ω' of [1, Theorem 20])

$$\phi = \mu(\omega' : \nu')c_{\bar{P}'|P'}(1 : \omega' : \bar{\nu}')^* c_{\bar{P}'|P'}(1 : \omega' : \nu')\phi.$$

By Lemma 2,

$$\phi = E\big({}^*P : \mu(\omega' : \nu')J_{\bar{P}'}\big(\sqrt{-1}\,\nu_0 + \nu'\big)J_{P'}\big(\sqrt{-1}\,\nu_0 + \nu'\big)\psi : \sqrt{-1}\,\nu_0\big)$$

and, by Lemma 3,

$$\phi = E\big({}^*P : \tilde{\mu}(\omega : \nu')J_{\bar{P}'}\big(\sqrt{-1}\,\nu_0 + \nu'\big)J_{P'}\big(\sqrt{-1}\,\nu_0 + \nu'\big)\psi : \sqrt{-1}\,\nu_0\big).$$

Since $\mu(\omega' : \nu')$ is meromorphic on $\mathfrak{a}_{\mathbf{C}}'^*$, the lemma and Theorem 1 follow.

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