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EXTENDING COMBINATORIAL PIECEWISE LINEAR STRUCTURES ON STRATIFIED SPACES. II

BY

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ABSTRACT. Let X be a stratified space and suppose that both the complement of the n-skeleton and the n-stratum have been endowed with combinatorial piecewise linear (PL) structures. In this paper we investigate the problem of "fitting together" these separately given PL structures to obtain a single combinatorial PL structure on the complement of the $(n-1)$-skeleton. The first main result of this paper reduces the geometrically given "fitting together" problem to a standard kind of obstruction theory problem. This is accomplished by introducing a tangent bundle for the n-stratum and using immersion theory to show that the "fitting together" problem is equivalent to reducing the structure group of the tangent bundle of the n-stratum to an appropriate group of PL homeomorphisms. The second main theorem describes a method for computing the homotopy groups arising in the obstruction theory problem via spectral sequence methods. In some cases, the spectral sequences involved are fairly small and the first few differentials are described. This paper is an outgrowth of earlier work by the authors on this problem.

Let X be a stratified space and suppose X is locally triangulable. The problems that we wish to study in this paper are those of the existence and classification of piecewise linear (PL) structures on X which are compatible in a sense made more precise below, with the stratification of X. This paper is a continuation of our earlier papers [2] and [3], but may be read independently of them. We recall some definitions.

Let X be a space. A *stratification* of X is an increasing family of closed subsets of X, $\{X^{(n)} | n \geqslant -1\}$ such that $X^{(-1)} = \varnothing$, there exists a positive integer N such that $X^{(N)} = X$, and for every n, each component of $X^{(n)} - X^{(n-1)}$ is open in $X^{(n)} - X^{(n-1)}$. The space $X^{(n)}$ (respectively, $X^{(n)} - X^{(n-1)}$) is called the *n-skeleton* (respectively, *n-stratum*) of $\{X^{(n)}\}$; while N is called the *formal dimension* of X. If $X^{(n)} - X^{(n-1)}$ is a (possibly empty) topological (TOP) n-manifold without boundary, then $\{X^{(n)}\}$ is called a TOP *stratification* of X. A TOP stratified space is a pair $(X, \{X^{(n)}\})$ where $\{X^{(n)}\}$ is a TOP stratification of X. When no confusion will arise, we suppress explicit mention of the stratification and denote a stratified space simply by X.

Although the individual strata of a TOP stratified space are homogeneous when

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viewed as strata, they may not be homogeneous viewed as subsets of X. (Think of R^3 as having two strata: one consisting of a horned sphere, the other being its complement.) To rule out such pathologies, we will consider only locally cone-like TOP stratified spaces (i.e. CS spaces in the sense of Siebenmann [**26**]). A TOP stratified space is *locally cone-like* if for every $x \in X^{(n)} - X^{(n-1)}$, there exist a compact stratified space L and a stratum-preserving (see §1 for the definitions) open embedding h: $R^n \times cL \to X$ such that $h(0, v) = x$ where cL denotes the open cone on L and v is its vertex. The space L is called a *link* of x and h is called a *local chart*. We shall call a locally cone-like TOP stratified space a TOP CS *space*.

Let X be a TOP CS space. A PL structure on X is *combinatorial* if the following conditions hold:

(i) for every n, $X^{(n)}$ is a subpolyhedron of X;

(ii) for every n, $X^{(n)} - X^{(n-1)}$ with its induced PL structure is a PL manifold (i.e. $X^{(n)} - X^{(n-1)}$ is combinatorially triangulated);

(iii) for every n and every component C of $X^{(n)} - X^{(n-1)}$, there exists a polyhedron L depending on C such that for every $x \in C$, there exists a PL open embedding h: $R^n \times cL \to X$ such that $h(0, v) = x$.

In fact, we wish L to be a PL stratified polyhedron (see §1) and h to be stratum-preserving. Condition (iii) above is a compatibility condition. It is included to insure that the strata of X have the same homogeneity piecewise linearly as they do topologically. (The reader will note that this definition of combinatorial PL structure is stronger than our use of this term in [**2**].)

The problem we consider may now be stated somewhat more precisely as follows: Let X be a locally triangulable TOP CS space. Does X support a combinatorial PL structure? If so, how are these structures classified? Our approach is to proceed by induction down the strata of X under the following inductive hypotheses:

(a) the space X is a TOP CS space;

(b) there exists an integer $m \geqslant 0$ such that $X^{(m)} - X^{(m-1)}$ and $X - X^{(m)}$, respectively, are endowed with combinatorial PL structures α and β, respectively.

Condition (b) is a necessary condition for the existence of a combinatorial PL structure on $X - X^{(m-1)}$. By assuming that the manifold $X^{(m)} - X^{(m-1)}$ has a PL manifold structure α, we are assuming only that the well-understood obstructions of Kirby-Siebenmann (cf. [**13**] or [**14**]) or Lashof-Rothenberg (cf. [**17**] or [**18**]) type vanish.

Our main objective is to study the problem of extending the structures α on $X^{(m)} - X^{(m-1)}$ and β on $X - X^{(m)}$ to obtain a combinatorial PL structure γ on $X - X^{(m-1)}$. In order to have some chance of finding such a structure γ, we add the following condition to the inductive hypothesis:

(c) for every component C of $X^{(m)} - X^{(m-1)}$, there exists a PL stratified polyhedron L such that for every $x \in C$ there exists a stratum-preserving open embedding h: $R^m \times cL \to X - X^{(m-1)}$ such that $h(0, v) = x$, $h(R^m \times v) \subset X^{(m)} - X^{(m-1)}$, $h(R^m \times (cL - v)) \subset X - X^{(m)}$, and such that the restrictions of h to $R^m \times v$ and $R^m \times (cL - v)$ are PL.

Condition (c) amounts to assuming the existence of local triangulations of X that are compatible with the PL structures α and β. This is a necessary condition for the existence of a combinatorial PL structure γ on $X - X^{(m-1)}$ extending α and β. On the other hand, an example of Siebenmann [**25**, §3] shows that local triangulations satisfying (c) do not always exist.

The following results are our principal theorems.

PROPOSITION 3.1. *Let $(X, \{X^{(n)}\})$ be a* TOP CS *space. Let C be a component of $X^{(m)} - X^{(m-1)}, y \in C$, and L be a link of y. Then the diagram*

$$C \xrightarrow{\Delta} C \times X \xrightarrow{pr_1} C \tag{1}$$

where $\Delta(x) = (x, x)$ and $pr_1(x, z) = x$ defines a microbundle over C with fiber $cS^{m-1} * L$.

The proof of 3.1 is given in §3. In that section we also invoke the Kister-Mazur [**15**] arguments and replace the microbundle (1) with a fiber bundle with fiber $cS^{m-1} * L$. This bundle is called the *tangent bundle of C in X* and is denoted $t(C; X)$ or simply $t(C)$ when X is clear from the context.

In the presence of the inductive hypotheses (a)–(c), we may take L to be a polyhedron. The proof of 3.1 then shows that the structure group of $t(C)$ is the semi-simplicial complex $\mathrm{TOP}_m^+(cS^{m-1} * L)$ a k-simplex of which is a stratum-preserving homeomorphism h: $\Delta^k \times (cS^{m-1} * L) \to \Delta^k \times (cS^{m-1} * L)$ which commutes with projection on Δ^k such that $h|\Delta^k \times cS^{m-1}$ and $h|\Delta^k \times (cS^{m-1} * L - cS^{m-1})$ are PL. We require also that $h|\Delta^k \times v$ be the identity. Let $\mathrm{PL}(cS^{m-1} * L)$ be the subcomplex obtained by requiring h to be PL.

THEOREM A. *Let $(X, \{X^{(n)}\})$ satisfy the inductive hypotheses* (a)–(c) *above. Let C be a component of $X^{(m)} - X^{(m-1)}$ and suppose there exists a map τ_c making the following diagram commute*

$$\begin{tikzcd} & \mathrm{BPL}(cS^{m-1} * L) \arrow[d] \\ C \arrow[r, "t_c"'] \arrow[ur, dashed, "\tau_c"] & \mathrm{BTOP}_m^+(cS^{m-1} * L) \end{tikzcd}$$

where t_c is a classifying map for the tangent bundle of C in X. Then, if either $m \neq 4$ or $m = 4$ and C is an open manifold, there exists a combinatorial PL *structure γ on $(X - X^{(m)}) \cup C$ such that $\gamma|X - X^{(m)} = \beta$ and $\gamma|C = \alpha|C$.*

We remark that if $X - X^{(m)} \cup C$ has a combinatorial PL structure extending α and β, then there is a lift τ_c of t_c.

Let γ_0 and γ_1 be two combinatorial PL structures on $X - X^{(m)} \cup C$ that extend $\alpha|C$ and β. We say that γ_0 and γ_1 are *concordant* if there exists a combinatorial PL structure Γ on $[X - X^{(m)} \cup C] \times I$ such that $\Gamma|X - X^{(m)} \cup C \times \{i\} = \gamma_i$ $(i = 0, 1)$, $\Gamma|C \times I$ is PL homeomorphic to $\alpha|C \times I$, and $\Gamma|X - X^{(m)} \times I$ is PL homeomorphic to $\beta \times I$. We say that γ_0 is *isotopic* to γ_1 relative to $\alpha|C$ and β if

there exists a stratum-preserving topological isotopy h_t: $X - X^{(m)} \cup C \to X - X^{(m)} \cup C$ $(0 \leqslant t \leqslant 1)$ such that h_0 is the identity; h_1: $[X - X^{(m)} \cup C]_{\gamma_0} \to [X - X^{(m)} \cup C]_{\gamma_1}$ is a PL homeomorphism; and such that $h|X - X^{(m)} \times I$ and $h|C \times I$ are PL.

THEOREM B. *Let* $(X, \{X^{(n)}\})$ *satisfy the inductive hypotheses* (a)–(c) *above and suppose there exists a combinatorial* PL *structure* γ *on* $X - X^{(m)} \cup C$ *extending* β *on* $X - X^{(m)}$ *and* $\alpha|C$ *on* C *where* C *is a component of* $X^{(m)} - X^{(m-1)}$.

(i) *If* C *is an open manifold, then there is a one-to-one correspondence between isotopy classes of such extensions* γ *and vertical homotopy classes of lifts to* t_c: $C \to \mathrm{BTOP}_m^+(cS^{m-1} * L)$ *to* $\mathrm{BPL}(cS^{m-1} * L)$.

(ii) *If* C *is a closed manifold of dimension* $\neq 3$, *then there is a one-to-one correspondence between concordance classes of such extensions* γ *and vertical homotopy classes of* t_c: $C \to \mathrm{BTOP}_m^+(cS^{m-1} * L)$ *to* $\mathrm{BPL}(cS^{m-1} * L)$.

In order to give some bite to Theorems A and B, it is necessary to describe $\pi_*(\mathrm{TOP}_m^+(cS^{m-1} * L)/\mathrm{PL}(cS^{m-1} * L))$ where $\mathrm{TOP}_m^+(cS^{m-1} * L)/\mathrm{PL}(cS^{m-1} * L)$ is the fiber of $\mathrm{BPL}(cS^{m-1} * L) \to \mathrm{BTOP}_m^+(cS^{m-1} * L)$.

THEOREM C. *Let* $(L, \{L^{(m)}\})$ *be a* PL *stratified polyhedron of dimension* n. *Then there exists a spectral sequence* $\{\bar{E}^r, \bar{d}^r\}$ *converging to* $\pi_*(\mathrm{TOP}_m^+(cS^{m-1} * L)/\mathrm{PL}(cS^{m-1} * L))$ *for* $* \geqslant 1$ *such that, if we set* $G_{s,t} = \pi_{s+t}(\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)}))$, *then* $\bar{E}^1_{s,t} = G_{s,t}$ *for* $s + t \geqslant 2$, $\bar{E}^1_{s,t} \subset G_{s,t}$ *for* $s + t = 1$, *and* $\bar{E}^1_{s,t} = 0$ *for* $s + t \leqslant 0$.

Furthermore, there exists a second spectral sequence $\{E^r, d^r\}$ *converging to* $\pi_*(\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)}))$ *for* $* \geqslant 1$ *such that if* $p < 0$, *or* $p > m$, *or* $q > 0$, *then* $E^1_{p,q} = 0$. *In addition, if* $q \leqslant 0$ *and* $4 \leqslant n - s + 1 = t$, *then for* $p + q \geqslant 2$,

$$E^2_{p,q} = \begin{cases} H^{p+q+t}\big(Z_2; \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})\big) & \text{for } p < m, \\ \big\{x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) | x = (-1)^{p+q+t}x^*\big\} & \text{for } p = m, \end{cases}$$

and for $p + q = 1$,

$$E^2_{p,q} \subset \begin{cases} H^{t+1}\big(Z_2; \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})\big) & \text{for } p < m, \\ \big\{x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) | x = (-1)^{t+1}x^*\big\} & \text{for } p = m. \end{cases}$$

In stating this theorem, we have used $\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})$ as an abbreviation for

$$\mathrm{TOP}_m^+(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})$$

and $\tilde{K}_{q+1}G$ as an abbreviation for Wh G if $q = 0$, $\tilde{K}_0 G$ if $q = -1$, and $K_{q+1}G$ if $q < 1$ (cf. [4, Chapter XII]). Furthermore, $x \mapsto x^*$ denotes the usual duality involution on $\tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})$ and

$$H^r\big(Z_2; \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})\big) = \frac{\big\{x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) | x = (-1)^r x^*\big\}}{\big\{y + (-1)^r y^* | y \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})\big\}}.$$

We note that Theorem C is particularly complete in the case when L is a PL manifold of dimension $n \geqslant 5$. In this case, $\bar{E}^1_{s,t} = 0$ for $s \neq 1$ and the spectral

sequence $\{\bar{E}^r, \bar{d}^r\}$ collapses. On the other hand, the second spectral sequence lies entirely in the triangle sketched below. Furthermore, when $\pi_1(L)$ is finite, then $K_{q+1}\pi_1(L) = 0$ for $q \leqslant -3$ by a result of Carter [5]. In this case then, the spectral sequence lies entirely on the lines $q = 0, -1, -2$. In all these cases, we also have a description of the differential $d^2_{p,q}$.

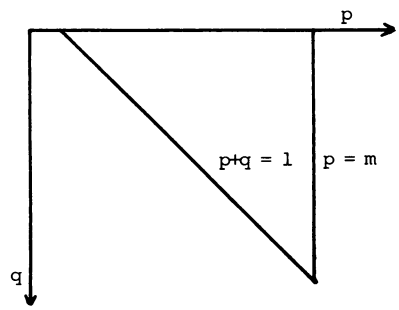

FIGURE 1

A note concerning the history of this paper might be in order. The first two theorems in this paper were found by the authors in October, 1976. A complete proof of Theorem C in the case when L is a PL manifold was known in July, 1977 and the essence of the general case was also known then. Because of other work, however, the writing of this paper has been rather delayed.

This paper is organized as follows: §1 completes the list of definitions used in this paper; §2 outlines the proofs of Theorems A and B; the details of those proofs are given in §§3 through 7; the proof of Theorem C is contained in §§8 through 12; and §13 contains a description of the second differential of the second spectral sequence in Theorem C. A specific description of the ingredients of the proof of Theorem C is given at the beginning of §8.

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1. Some more definitions. In this section we complete the list of definitions of the basic terms used in this paper.

Let X be a stratified space. Let cX denote the open cone on X (i.e. the quotient space $X \times [0, \infty)/X \times 0$) and let $v \in cX$ be the vertex (i.e. the point $X \times 0$). Then setting $(cX)^{(n)} = v$ if $n = 0$, and $(cX)^{(n)} = cX^{(n-1)}$ if $n \geqslant 1$ gives a stratification of cX called the *cone stratification*. When X is a stratified space, unless otherwise stated, cX will denote the open cone on X equipped with the cone stratification.

If X and Y are stratified spaces, the *product stratification* on $X \times Y$ is defined by $(X \times Y)^{(n)} = \bigcup_{p+q=n} X^{(p)} \times X^{(q)}$. The *join stratification* on $X * Y$ is defined by $(X * Y)^{(n)} = \bigcup_{p+q=n-1} X^{(p)} * Y^{(q)} \cup X^{(n)} \cup Y^{(n)}$ where $X^{(n)}$ and $Y^{(n)}$ are included in $X * Y$ via the obvious inclusions of X and Y in $X * Y$. In the sequel, unless we state otherwise, $X \times Y$ (respectively, $X * Y$) will denote the product

$X \times Y$ (respectively, the join $X * Y$) equipped with the product (respectively, the join) stratification.

Let X and Y be stratified spaces. The map $f\colon X \to Y$ is *skeleton-* (respectively, *stratum-*) *preserving* if $f(X^{(n)}) \subset Y^{(n)}$ (respectively, $f(X^{(n)} - X^{(n-1)}) \subset Y^{(n)} - Y^{(n-1)}$) for every n. The map f is an *isomorphism of stratified spaces* if it is a stratum-preserving homeomorphism.

A *stratified polyhedron* is a pair $(X, \{X^{(n)}\})$ where X is a polyhedron and $\{X^{(n)}\}$ is a stratification of X such that $X^{(n)}$ is a subpolyhedron of X for every n. Since $X^{(n)} - X^{(n-1)}$ is an open subset of $X^{(n)}$, it inherits a PL structure. If $X^{(n)} - X^{(n-1)}$ with this PL structure is a PL manifold, then $(X, \{X^{(n)}\})$ is called a PL *stratified polyhedron*.

Let CAT denote either of the categories TOP or PL and let X be a CAT stratified space. We say X is a CAT CS *space* if for every n and every $x \in X^{(n)} - X^{(n-1)}$ there exist a stratified space (if CAT is TOP) or a stratified polyhedron (if CAT is PL), L, and a stratum-preserving open CAT embedding $h\colon R^n \times cL \to X$ such that $h(0, v) = x$ where R^n is given its *natural stratification* (i.e. $(R^n)^{(m)} = \varnothing$, if $m < n$; and $(R^n)^{(m)} = R^n$, if $m \geqslant n$) and, when CAT is PL, R^n, cL, and $R^n \times cL$ have their usual PL structures. The space L is called a CAT *link* of x in X. We remark that when CAT is TOP the examples of Milnor **[20]** or Stallings **[27]** show that x does not determine a CAT link uniquely. On the other hand, Corollary 3.3 below shows that a component C of a CAT CS space is homogeneous in the sense that given any two points x and y of C, the local charts h_x and h_y may be chosen so that L_x is CAT homeomorphic to L_y.

EXAMPLE 1.1. *A* PL *stratified polyhedron need not be a* PL CS *space*.

To construct such an example, let $m \geqslant 7$ and let W^{m-1} be a nontrivial PL h-cobordism. Cone off the two boundary components of W to obtain a polyhedron Y of the type considered by Stallings **[27]**. Let $X = S^0 * Y$ and notice that $S^0 * \{\text{cone points in } Y\}$ forms a circle $S^1 \subset X$. To complete the construction, stratify X by setting $X^{(n)} = \varnothing$, if $n < 1$; $X^{(n)} = S^1$, if $1 \leqslant n \leqslant m$; and $X^{(n)} = X$, if $m < n$. The remaining details are left to the reader.

We now introduce a category TOP_m^+ which is a mixture between TOP and PL. An object of TOP_m^+ is a stratified space $(X, \{X^{(n)}\})$ together with PL structures β on $X - X^{(m)}$ and α on $X^{(m)} - X^{(m-1)}$. A morphism in TOP_m^+ is a stratum-preserving map $f\colon X_1 \to X_2$ such that $f|X_1 - X_1^{(m)}$ (respectively, $f|X_1^{(m)} - X_1^{(m-1)}$) is PL with respect to the PL structures β_i on $X_i - X_i^{(m)}$ (respectively, α_i on $X_i^{(m)} - X_i^{(m-1)}$) $(i = 1, 2)$. A TOP_m^+ *stratified space* consists of a TOP stratified space X together with a PL manifold structure α on $X^{(m)} - X^{(m-1)}$ and a PL structure β on $X - X^{(m)}$ such that $(X - X^{(m)})_\beta$ is a PL stratified polyhedron.

A TOP_m^+ CS *space* consists of a TOP CS space X together with

(i) A PL manifold structure α on $X^{(m)} - X^{(m-1)}$;

(ii) A PL structure β on $X - X^{(m)}$ such that $(X - X^{(m)})_\beta$ is a PL CS space;

(iii) For each component C of $X^{(m)} - X^{(m-1)}$, a choice of the polyhedron L for the link and of coordinate charts h satisfying condition (c) of the inductive hypotheses given in the introduction.

2. The proofs of Theorems A and B. The proofs of Theorems A and B are based on immersion theory and follow the argument given by Lashof in **[17]** quite closely. Indeed, a careful reading of **[17]** shows that the conclusions of that paper are essentially formal consequences of the following results:

1. The existence of the tangent bundle (Chapter 1).
2. Lees Immersion Theorem (Chapter 2). This depends, in turn, on
3. The Isotopy Extension Theorem (Chapter 4), and
4. A suitable engulfing lemma (cf. pp. 142–143).
5. A collaring lemma (cf. pp. 159–160).

(All references in this list are to **[17]**.) In §§3 to 7 of this paper we shall establish the appropriate analogues of all these results. Theorems A and B then follow directly from obvious modifications of the definitions and of the arguments given in **[17]**.

A word about our style of presentation may be helpful to the reader. In addition to proving the needed analogues of the results cited above, we have also stated the analogues of some of the other key results used in the proofs given in **[17]**. This has been done to clarify for the reader what the appropriate analogues of those results are.

3. The tangent bundle of a stratum. Let X be a CAT CS space where CAT is one of the categories TOP, PL, or TOP_m^+. Let C be a component of $X^{(m)} - X^{(m-1)}$. In this section we shall show that C has a CAT tangent bundle in X. We recall first the following definition due to Milnor **[21]**.

Let X and F be spaces. A *microbundle over X with fiber cF* is a diagram

$$X \xrightarrow{s} E \xrightarrow{p} X \tag{2}$$

such that for every $x \in X$ there exist a space U_x and open embeddings $h_x\colon U_x \to X$ and $H_x\colon U_x \times cF \to E$ such that the diagram

$$\begin{array}{ccccc} U_x & \xrightarrow{s_1} & U_x \times cF & \xrightarrow{pr_1} & U_x \\ \downarrow h_x & & \downarrow H_x & & \downarrow h_x \\ X & \xrightarrow{s} & E & \xrightarrow{p} & X \end{array}$$

commutes where $s_1(y) = (y, v)$, v is the vertex of cF, and $pr_1(y, z) = y$. If all the spaces and maps appearing in this definition are in CAT, then (2) is called a CAT *microbundle*.

The main result of this section is the following proposition.

PROPOSITION 3.1. *Let X be a* CAT CS *space and let C be a component of $X^{(m)} - X^{(m-1)}$ in $X^{(m)} - X^{(m-1)}$. Let $y \in C$ and let L be a* CAT *link for y in X. Then the diagram*

$$C \xrightarrow{\Delta} C \times X \xrightarrow{pr_1} C \tag{3}$$

is a CAT *microbundle over C with fiber $cS^{m-1} * L$ where $\Delta(x) = (x, x)$ and $pr_1(x, z) = x$.*

We remark that when CAT is PL or TOP_m^+, then a CAT link for y is a polyhedron.

PROOF. Let $x \in C$. Then there exists a CAT chart h: $R^m \times cL \to X$ such that $h(0, v') = x$. (If CAT is TOP or PL, this follows from 3.3 below. If CAT is TOP_m^+, this is by definition.) Furthermore, there exists a CAT homeomorphism f: $cS^{m-1} * L \to R^m \times cL$ such that $f(v) = (0, v')$. Consider the following diagram

$$\begin{array}{ccccccc}
R^m & \xrightarrow{\mathrm{id}} & R^m & \xrightarrow{\mathrm{id}} & R^m & \xrightarrow{h|} & C \\
\downarrow s_1 & & \downarrow s_1' & & \downarrow \Delta' & & \downarrow \Delta \\
R^m \times (cS^{m-1} * L) & \xrightarrow{\mathrm{id}\times f} & R^m \times (R^m \times cL) & \xrightarrow{a} & R^m \times (R^m \times cL) & \xrightarrow{(h|)\times h} & C \times X \\
\downarrow pr_1 & & \downarrow pr_1 & & \downarrow pr_1 & & \downarrow pr_1 \\
R^m & \xrightarrow{\mathrm{id}} & R^m & \xrightarrow{\mathrm{id}} & R^m & \xrightarrow{h|} & C
\end{array}$$

where $s_1(y) = (y, v)$, $s_1'(y) = (y, 0, v')$, $\Delta'(y) = (y, y, v')$, pr_1 is projection on the first factor, and $a(y, z, w) = (y, y + z, w)$. It is easy to see that all squares commute and that the composite from the left-hand column to the right-hand column defines a CAT local trivialization of (3).

The reader will note that the proof above is just Milnor's proof [**21**, Lemma 2.1] that $M \xrightarrow{\Delta} M \times M \xrightarrow{p} M$ is an R^m microbundle when M is an m-manifold.

It follows from the proof of the Kister-Mazur Theorem [**15**] that the microbundle (3) determines a unique bundle over C with fiber $cS^{m-1} * L$. This bundle is called the CAT *tangent bundle* of C in X and is denoted $t_{\mathrm{CAT}}(C; X)$ or simply by $t(C)$ when CAT and X are clear from the context. The disjoint union $\amalg t_{\mathrm{CAT}}(C; X)$ where C runs over the components of $X^{(m)} - X^{(m-1)}$ is called the *tangent bundle of* $X^{(m)} - X^{(m-1)}$ in X. It is denoted by either $t_{\mathrm{CAT}}(X^{(m)} - X^{(m-1)}; X)$ or $t(X^{(m)} - X^{(m-1)})$. We note that the fibers of this bundle over points in different components may be different.

The proof of 3.1 shows a little more than its statement reveals. Namely, if we stratify S^{m-1} in the obvious way (let S^{m-1} be the $(m-1)$ stratum; let all other strata be empty) and, if we add a stratum to $R^m \times cL$ by setting $(R^m \times cL)^{(0)} = (0, v')$, then the CAT homeomorphism f: $cS^{m-1} * L \to R^m \times cL$ used in the proof of 3.1 can be taken to be a CAT isomorphism of CAT stratified spaces. It follows from this remark that the structure group of $t_{\mathrm{CAT}}(C; X)$ is the semi-simplicial complex $\mathrm{CAT}(cS^{m-1} * L)$ of CAT isomorphisms of $cS^{m-1} * L$ onto itself. More precisely, a k-simplex of $\mathrm{CAT}(cS^{m-1} * L)$ is a CAT isomorphism of stratified spaces f: $\Delta^k \times cS^{m-1} * L \to \Delta^k \times cS^{m-1} * L$ which commutes with projection on Δ^k where Δ^k is the standard k-simplex equipped with its *usual stratification* $(\Delta^k)^{(i)} = \{\text{faces of } \Delta^k \text{ of dimension } \leqslant i\}$. The face and degeneracy operators are defined in the obvious way.

Let X and Y be CAT stratified spaces and let f: $X \to Y$ be a CAT stratum-preserving map. Since $f(X^{(m)} - X^{(m-1)}) \subset Y^{(m)} - Y^{(m-1)}$, for every component C of $X^{(m)} - X^{(m-1)}$, there exists a component D of $Y^{(m)} - Y^{(m-1)}$ such that $f(C) \subset D$. Then the diagram

$$\begin{array}{ccccc}
C & \xrightarrow{\Delta} & C \times X & \xrightarrow{p} & C \\
f|\downarrow & & \downarrow f| \times f & & \downarrow f \\
D & \xrightarrow{\Delta} & D \times Y & \xrightarrow{p} & D
\end{array} \tag{4}$$

commutes and defines a microbundle map. This map, in turn, induces fiber-preserving maps df: $t(C) \to t(D)$ and, by taking disjoint unions, df: $t(X^{(m)} - X^{(m-1)}) \to t(Y^{(m)} - Y^{(m-1)})$. Either of the maps df is called the *differential* of f. We note that if f is a local homeomorphism, then the restriction of df to a fiber of $t(X^{(m)} - X^{(m-1)})$ is a homeomorphism.

We conclude this section with the proofs of the following lemma and corollary which were used in the proof of 3.1.

LEMMA 3.2. *Let X be a* CAT CS *space and let x and y be distinct points in the same component C of $X^{(m)} - X^{(m-1)}$. Then there exists a* CAT *isomorphism f: $X \to X$ such that $f(x) = y$. Furthermore, f may be taken to be* CAT *isotopic to the identity.*

PROOF. Since C is a CAT manifold, there exists a CAT ambient isotopy f_t': $C \to C$ $(0 \leqslant t \leqslant 1)$ supported on a compact subset K of C such that $f_0' = \mathrm{id}$ and $f_1'(x) = y$. For each $z \in C$, let h_z: $R^m \times cL_z \to X$ be a CAT chart about z and let $B_z = h_z(B^m)$ where $B^m = [-1, 1] \times \cdots \times [-1, 1]$ (m factors). Then the isotopy f_t' may be factored in CAT as a composite of ambient isotopies $f_{k,t}' f_{k-1,t}' \cdots f_{1,t}'$ where each $f_{j,t}'$ is supported by some set B_z. (If CAT is TOP, this follows from [**7**, Corollary 1.3]. If CAT is PL, this is well known (cf. [**29**]).) Clearly each $f_{j,t}'$ extends to a CAT ambient isotopy $f_{j,t}$ of isomorphisms of X. The proof is completed by setting $f_t = f_{k,t} f_{k-1,t} \cdots f_{1,t}$ and $f = f_1$.

COROLLARY 3.3. *Let X be a* CAT CS *space and let x and y be distinct points in the same component C of $X^{(m)} - X^{(m-1)}$. If L is a* CAT *link for x, then L is a* CAT *link for y.*

PROOF. Since L is a CAT link for x, there exists a CAT chart h: $R^m \times cL \to X$ such that $h(0, v') = x$. Let f: $X \to X$ be a CAT isomorphism such that $f(x) = y$. Then fh is a CAT chart about y and L is a CAT link for y.

4. Immersing a neighborhood of a stratum. In this section we define the notion of a CAT immersion near a stratum of a CAT CS space and we prove the analogue of the usual immersion theorem for manifolds.

Let X and Y be CAT CS spaces where CAT is one of the categories TOP, PL, or TOP_m^+. A stratum-preserving CAT map f: $X \to Y$ is a CAT *immersion* if for every $x \in X$ there exists a neighborhood V_x such that $f|V_x$: $V_x \to Y$ is an open embedding. A CAT *regular k-homotopy* is a CAT immersion F: $\Delta^k \times X \to \Delta^k \times Y$ commuting with projection on Δ^k (where Δ^k is the standard k-simplex with its usual stratification).

Let C be a component of $X^{(m)} - X^{(m-1)}$. A CAT *regular k-homotopy near C* is a CAT regular k-homotopy F: $\Delta^k \times U \to \Delta^k \times Y$ where U is an open neighborhood of C in X. (Here U becomes a CAT CS space by setting $U^{(n)} = U \cap X^{(n)}$ for all n.) We identify two CAT regular k-homotopies near C, F_i: $\Delta^k \times U_i \to \Delta^k \times Y$ ($i = 1, 2$) if $F_1|\Delta^k \times U = F_2|\Delta^k \times U$ for some open neighborhood U of C, $U \subset U_1 \cap U_2$. If $k = 0$, F is called a CAT *immersion near C*.

We note that if f is a CAT immersion near C, then we may form the diagram (4)

defining the differential of f. In this situation, when CAT is PL, df: $t(C) \to t(Y^{(m)} - Y^{(m-1)})$ is actually a map of bundles with structure group $\mathrm{PL}(c(S^{m-1} * L))$ where L is a PL link of some point $x \in C$. If CAT is TOP or TOP_m^+, let C' be the component of $Y^{(m)} - Y^{(m-1)}$ such that $f(C) \subset C'$. Then the structure group of $t(C)$ (respectively, $t(C')$) is $\mathrm{CAT}(cS^{m-1} * L)$ (respectively, $\mathrm{CAT}(cS^{m-1} * L')$) where L (respectively, L') is a CAT link of $x \in C$ (respectively, $f(x) \in C'$). In this case, although L may not be CAT homeomorphic to L', df is a bundle map in the sense that on each fiber it induces a CAT homeomorphism of $cS^{m-1} * L$ onto $cS^{m-1} * L'$. Similarly a regular k-homotopy near C, F: $\Delta^k \times U \to \Delta^k \times Y$ induces a bundle map dF: $\Delta^k \times t(C) \to \Delta^k \times t(Y^{(m)} - Y^{(m-1)})$ commuting with projection on Δ^k.

Let D be a closed subset of C and let f be a CAT immersion of a neighborhood of D into Y. We let CAT $\mathrm{Imm}_f(C; Y)$ be the semi-simplicial complex whose k-simplices are CAT regular k-homotopies near C, F: $\Delta^k \times U \to \Delta^k \times Y$, such that F equals $1 \times f$ on a neighborhood of $\Delta^k \times D$ where we identify two such maps F_i: $\Delta^k \times U_i \to \Delta^k \times Y$ $(i = 1, 2)$ if they agree on $\Delta^k \times U$ where U is an open neighborhood of C, $U \subset U_1 \cap U_2$. If $D = \varnothing$, we write simply CAT $\mathrm{Imm}(C; Y)$ and make no mention of f. Similarly, we let CAT $\mathrm{Rep}_{df}(t(C); t(Y^{(m)} - Y^{(m-1)}))$ be the semi-simplicial complex whose k-simplices are CAT bundle maps ϕ: $\Delta^k \times t(C) \to \Delta^k \times t(Y^{(m)} - Y^{(m-1)})$ commuting with projection on Δ^k such that ϕ equals $1 \times df$ on a neighborhood $\mathcal{V}$ of D in C; and we write CAT $\mathrm{Rep}(t(C); t(Y^{(m)} - Y^{(m-1)}))$ when $D = \varnothing$ and df plays no role.

It is easy to check that CAT $\mathrm{Imm}_f(C; Y)$ and CAT $\mathrm{Rep}_{df}(t(C); t(Y^{(m)} - Y^{(m-1)}))$ are Kan complexes and that the differential defines a semi-simplicial map

$$d\colon \mathrm{CAT}\ \mathrm{Imm}_f(C; Y) \to \mathrm{CAT}\ \mathrm{Rep}_{df}(t(C); t(Y^{(m)} - Y^{(m-1)})). \tag{5}$$

THEOREM 4.1. *In the situation described above suppose that the closed subset D of C is either compact or a locally flat submanifold. Suppose also that every component of $C - D$ is either noncompact or a manifold with boundary. Then the map d of* (5) *is a homotopy equivalence.*

PROOF. The arguments given in [9] or [17] show that this theorem is a more or less formal consequence of the following propositions:

PROPOSITION 4.2. *Let A be a locally flat codimension zero submanifold of C. Then the restriction map*

$$r_1\colon \mathrm{CAT}\ \mathrm{Rep}(t(C); t(Y^{(m)} - Y^{(m-1)})) \to \mathrm{CAT}\ \mathrm{Rep}(t(A); t(Y^{(m)} - Y^{(m-1)})) \tag{6}$$

is a Kan fibration.

In 4.2 CAT $\mathrm{Rep}(t(A); t(Y^{(m)} - Y^{(m-1)}))$ is the semi-simplicial complex of bundle maps of $t(U)$ into $t(Y^{(m)} - Y^{(m-1)})$ where U is an open neighborhood of A in X with its induced stratification and such bundle maps f_i: $\Delta^k \times t(U_i) \to \Delta^k \times t(Y^{(m)} - Y^{(m-1)})$ $(i = 1, 2)$ are identified if they agree on $\Delta^k \times t(U)$ where $U \subset U_1 \cap U_2$ is a smaller open neighborhood of A.

PROOF. This proposition follows easily from the Covering Homotopy Theorem.

PROPOSITION 4.3. *Let D be as in* 3.1. *Then the restriction map*

$$r_2\colon \mathrm{CAT\ Imm}(C;\ Y) \to \mathrm{CAT\ Imm}(D;\ Y) \tag{7}$$

is a Kan fibration.

In 4.3 a k-simplex of CAT Imm$(D;\ Y)$ is a regular k-homotopy $F\colon \Delta^k \times U \to \Delta^k \times Y$ where U is an open neighborhood of D in X where two such regular k-homotopies are identified if they agree on a smaller neighborhood of D.

The proof of 4.3 is given in §5.

PROPOSITION 4.4. *Let $X = R^m \times cL$ where L is a* TOP (*respectively*, PL) *stratified space* (*respectively*, *polyhedron*) *if* CAT *is* TOP (*respectively*, PL *or* TOP_m^+). *Then $R^m = R^m \times v = C$. Let $D = \varnothing$. Then*

$$d\colon \mathrm{CAT\ Imm}(R^m;\ Y) \to \mathrm{CAT\ Rep}(t(R^m);\ t(Y^{(m)} - Y^{(m-1)}))$$

is a homotopy equivalence.

PROOF. The proof of the corresponding statement given in [**9**, p. 84] or [**17**, pp. 138–139] works here. We need only replace the origin by the point $0 \times v$ of $R^m \times cL$.

5. The proof of Proposition 4.3. The arguments given in [**9**] and [**17**, pp. 144–147] show that 4.3 follows directly from the following propositions.

PROPOSITION 5.1. *Let X be a* CAT CS *space where* CAT *is* TOP, PL, *or* TOP_m^+. *Let D be a compact subset of $X^{(m)} - X^{(m-1)}$ and let U and V be open neighborhoods of D such that $\overline{U} \subset V \subset X - X^{(m-1)}$. Let $F\colon U \times I \to V \times I$ be a* CAT *isotopy. Then there exist a* CAT *isotopy $H\colon V \times I \to V \times I$ and an open neighborhood U_0 of D with $\overline{U}_0 \subset U$ such that*

$$\begin{array}{ccc} U_0 \times I & \xrightarrow{F_0 \times 1} & V \times I \\ \hookrightarrow & & \downarrow H \\ U \times I & \to & V \times I \end{array}$$

commutes and H is the identity outside some compact set.

PROOF. If CAT is TOP (respectively, PL), this proposition was established by Siebenmann in [**26**] (respectively, by Akin in [**1**]). The proof for the case when CAT is TOP_m^+ is based on the Handle Lemma below. We first set some notation.

In R^i, we let $rB^i = [-r, r] \times \cdots \times [-r, r]$ and $r\mathring{B}^i = (-r, r) \times \cdots \times (-r, r)$ where there are i factors in either case. If L is any space, the *closed* (respectively, *open*) *cone on L of radius r* is the image of $L \times [0, r]$ (respectively, $L \times [0, r)$) in cL. It is denoted $\bar{c}_r L$ (respectively, $c_r L$). The image of $L \times r$ in cL will be denoted L_r.

HANDLE LEMMA 5.2. *Let $X = R^{m-i} \times R^i \times cL$ where L is a polyhedron. Let $D = 0 \times B^i \times v$, U be a neighborhood of D in X, and suppose $F\colon U \times I \to X \times I$ is a* TOP_m^+ *isotopy such that $F|U_0 \times I \cup U \times 0$ is the identity where $U_0 \subset U$ is an*

open neighborhood of ∂B^i. *Then there exist an* $\varepsilon > 0$, *a* TOP_m^+ *isotopy* $H\colon X \times [0, \varepsilon] \to X \times [0, \varepsilon]$, *and a neighborhood* $U_1 \subset U$ *of* $0 \times B^i \times v$ *such that the diagram below commutes*

$$\begin{array}{ccc} U_1 \times [0, \varepsilon] & \xrightarrow{F_0 \times 1} & X \times [0, \varepsilon] \\ \hookrightarrow & & \downarrow H \\ U \times [0, \varepsilon] & \xrightarrow{F} & X \times [0, \varepsilon] \end{array}$$

and such that H *is the identity outside some compact set.*

PROOF. Without loss of generality we may assume that $U = 10\mathring{B}^{m-1} \times 10\mathring{B}^i \times c_{10}L$, that $U_0 = 10\mathring{B}^{m-i} \times (10\mathring{B}^i \times \frac{1}{2}B^i) \times c_{10}L$, and that $V = R^{m-i} \times R^i \times cL$. The proof now follows that of Siebenmann [**26**, §3] with some modifications. In particular, we construct a sequence of embeddings $F_1, F_2, F_3, F_4, F_5, F_6$ which are

(i) equal to F on $rB^{m-i} \times rB^i \times c_rL$ for some $r > 0$;

(ii) equal to the identity on the intersection of either $A = R^{m-i} \times (R^i - \frac{1}{2}B^i) \times cL \times I$ or $\hat{A} = T^{m-i} \times (T^i - \frac{1}{2}B^i) \times cL \times I$ with the domain of F_i (whichever makes sense); and

(iii) are level preserving (i.e. commute with projection on I).

We construct F_1 by using [**2**, Lemma 2.2] to wrap F around T^m. This involves restriction to a neighborhood W of $\bar{c}_1L \times 0$ in $c_{10}L \times I$. We may assume that W has the form $c_{t_1}L \times [0, \varepsilon_1]$ for some t_1, ε_1 ($1 < t_1 < 10$; $\varepsilon_1 > 0$). We obtain an embedding $F_1\colon T^{m-i} \times T^i \times c_{t_1}L \times [0, \varepsilon_1] \to T^{m-i} \times T^i \times cL \times I$ satisfying (i)–(iii) above such that F_1 is PL on $T^{m-i} \times T^i \times (c_{t_1}L - v) \times [0, \varepsilon_1]$ and on $T^{m-i} \times T^i \times v \times [0, \varepsilon_1]$ by Addendum (iii) to Lemma 2.2 of [**2**]. Note also that $F_1|T^{m-i} \times T^i \times c_{t_1}L \times 0$ is the identity.

To construct F_2, we first conjugate F_1 by a level-preserving homeomorphism of $T^{m-i} \times T^i \times c_{t_1}L \times I$ to insure that F_1 is the identity on a neighborhood of $\hat{A} \cap (T^{m-i} \times T^i \times L_{t_2} \times I)$ where $t_2 = t_1/2$. We regard the restriction of F_1 to $T^{m-i} \times T^i \times L_{t_2} \times [0, \varepsilon_1]$ as a PL isotopy into $T^{m-i} \times T^i \times (cL - v)$ which is the identity near $\hat{A} \cap (T^{m-i} \times T^i \times L_{t_2})$. By the Covering Isotopy Theorem for maps of polyhedra [**1**], there exists a PL isotopy G_1 of $T^{m-i} \times T^i \times (cL - v)$ onto itself such that for every $s \in [0, \varepsilon_1]$, $G_1^s(x) = G_1^sF_1^0(x) = F_1^s(x)$ for $x \in T^{m-i} \times T^i \times L_{t_2}$ where $G_1^s = G_1|T^{m-i} \times T^i \times (cL - v) \times s$ and F_1^s is defined similarly. Furthermore G_1 may be chosen to be the identity on $\hat{A}$ and outside a compact set containing $T^{m-i} \times T^i \times L_{t_2}$. Thus G_1 extends, via the identity, to an isotopy of $T^{m-i} \times T^i \times c_{t_1}L$ onto itself. We define F_2 by setting $F_2(x) = G_1^{-1}F_1(x)$ if $x \in T^{m-i} \times T^i \times c_{t_2}L \times [0, \varepsilon_1]$ and $F_2(x) = x$ otherwise. It follows immediately from the construction that there exists a $t_3 < t_1$ such that F_1 and F_2 agree on $T^{m-i} \times T^i \times \bar{c}_{t_3}L \times [0, \varepsilon_1]$. Clearly then F_2 satisfies (i)–(iii) above. Note also that F_2^0 is the identity.

We now define two auxiliary maps G_2 and H_2 as follows: $G_2 = (F_2|T^{m-i} \times T^i \times v \times [0, \varepsilon_1]) \times \mathrm{id}$ and $H_2 = G_2^{-1}F_2$ where id is the identity on cL.

Let F_3, G_3, and H_3 be obtained by lifting F_2, G_2 and H_2 to $R^{m-i} \times R^i \times c_{t_1}L \times [0, \varepsilon_1]$ and making the modifications described in [**26**, p. 136]. Note that all these

maps are the identity outside $R^{m-i} \times R^i \times c_{t_2}L \times [0, \varepsilon_1]$.

We now construct H_4 by applying the "horn device" of [**26**, §3] to H_3. Thus, we let β: $[0, \infty) \to [0, t_1]$ be the PL function that satisfies $\beta(t) = t_1$ if $0 \leqslant t \leqslant 4$, $\beta(n) = 4t_1/n$ if $n \geqslant 4$ is an integer, and that maps $[n, n+1]$ linearly onto $[4t_1/(n+1), 4t_1/n]$. Let Γ: $R^{m-i} \times R^i \times cL \to R^{m-i} \times R^i \times cL$ be a PL homeomorphism such that $\Gamma = \mathrm{id}$ outside $R^{m-i} \times R^i \times c_{t_1}L$ and on $4B^{m-i} \times 4B^i \times c_{t_1}L$; $\Gamma(R^{m-i} \times R^i \times \bar{c}_{t_2}L) = C = \{(x, y, z, u) \in R^{m-i} \times R^i \times cL | u \leqslant \beta(|x, y|)\}$ where we have parametrized cL via the collapsing map $L \times [0, \infty) \to cL$ and $|x, y| = \max\{|x_1|, \ldots, |x_{m-i}|, |y_1|, \ldots, |y_i|)\}$ if $x = (x_1, \ldots, x_{m-i})$ and $y = (y_1, \ldots, y_i)$; and Γ commutes with projection on $R^{m-i} \times R^i$. Note that C is subpolyhedron of $R^{m-i} \times R^i \times cL$. We set $H_4 = (\Gamma \times \mathrm{id})H_3(\Gamma \times \mathrm{id})^{-1}$ where id is the identity map on $[0, \varepsilon_1]$. Note that H_4 is the identity outside C and on $R^{m-i} \times R^i \times v \times [0, \varepsilon_1]$.

We also construct a map G_4 as follows: We regard the restriction of G_3 (or, what is the same thing, F_3) to $4B^{m-i} \times 4B^i \times v \times [0, \varepsilon_1]$ to be a PL isotopy of $4B^{m-i} \times 4B^i$ in R^m. Since $G_3^0 = F_3^0$ is the identity, there exists an $\varepsilon_2 \leqslant \varepsilon_1$ such that $G_3^s(4B^{m-i} \times 4B^i) \subset 5\mathring{B}^{m-i} \times 5\mathring{B}^i$ for $0 \leqslant s \leqslant \varepsilon_2$. But then there exists a PL isotopy $\bar{G}_4$: $R^{m-i} \times R^i \times [0, \varepsilon_2] \to R^{m-i} \times R^i \times [0, \varepsilon_2]$ such that $\bar{G}_4^0$ is the identity; for every s $(0 \leqslant s \leqslant \varepsilon_2)$, $\bar{G}_4^s(x) = G_3^s(x) = F_3^s(x)$ for $x \in 4B^{m-i} \times 4B^i$; and $\bar{G}_4^s$ is the identity outside $5\mathring{B}^{m-i} \times 5\mathring{B}^i$. We set $G_4 = \bar{G}_4 \times \mathrm{id}$ where id is the identity on cL.

We now define F_4 to be the composite $G_4H_4|R^{m-i} \times R^i \times c_{t_1}L \times [0, \varepsilon_2]$. We note that $H_4 = H_3 = G_3^{-1}F_3$ on $4B^{m-i} \times 4B^i \times c_{t_1}L \times [0, \varepsilon_2]$ and that $G_4 = G_3$ on the same set. Since the restrictions of G_4 and H_4 to $3B^{m-i} \times 3B^i \times c_{t_1}L \times 0$ are the identity it follows that there exists a neighborhood N of $3B^{m-i} \times 3B^i \times v \times 0$ contained in $4B^{m-i} \times 4B^i \times c_{t_1}L \times [0, \varepsilon_2]$ such that $H_4(N) \subset 4B^{m-i} \times 4B^i \times c_{t_1}L \times [0, \varepsilon_2]$. If $x \in N$, it follows that $F_4(x) = F_3(x) = F(x)$. Thus F_4 satisfies (i). That F_4 satisfies (ii) and (iii) is obvious.

The construction of F_5 requires a map J similar to that of [**26**, §3]. To describe J, we identify $R^{m-i} \times R^i$ and $5\mathring{B}^{m-i} \times 5\mathring{B}^i$ with R and $5\mathring{B}^m$ in the obvious way and write R^m and $5\mathring{B}^m$ respectively as the union of the annuli

$$A_n = \{x \in R^m | n \leqslant |x| \leqslant n + 1\}$$

and

$$A_n' = \{x \in 5\mathring{B}^m | 5 - 1/2^{n-4} \leqslant |x| \leqslant 5 - 1/2^{n-3}\}$$

respectively ($n = 4, 5, 6, \ldots$). We can then construct a PL embedding J': $R^m \times cL \to R^m \times cL$ mapping $A_n \times cL$ onto $A_n' \times cL$ with the following properties:

(a) $J'|4\mathring{B}^m \times cL = \mathrm{id}$;

(b) $J'(C) = \{(x, z, u) | u \leqslant \gamma|x|\}$ where γ: $[0, 5) \to (0, t_1]$ is the PL function that satisfies $\gamma(t) = t_1$ for $0 \leqslant t \leqslant 4$, $\gamma(5 - 1/2^{n-4}) = 4t_1/n$ if $n \geqslant 4$ is a positive integer, and γ maps $[5 - 1/2^{n-4}, 5 - 1/2^{n-3}]$ linearly onto $[4t_1/(n+1), 4t_1/n]$;

(c) J' preserves the level in the cone coordinate; and

(d) J' preserves rays through the origin in R^m.

Let $J = J' \times \mathrm{id}$ where id denotes the identity on $[0, 1]$.

We now define H_5 by setting $H_5(x) = JH_4J^{-1}(x)$ if $x \in 5\mathring{B}^m \times c_{t_1}L \times [0, \varepsilon_2]$ and $H_5(x) = x$ otherwise. Since $H_4(x)$ is bounded and H_4 is the identity outside C, H_5 is continuous. Furthermore, the restriction of H_5 to $R^m \times (cL - v) \times [0, \varepsilon_2]$ is PL since J and H_4 are PL on that set. Also $H_5|R^m \times v \times [0, \varepsilon_2]$ is the identity since H_4 is the identity there. Hence $H_5|R^m \times v \times [0, \varepsilon_2]$ is PL.

We now set $F_5 = G_4H_5$. Since H_5 agrees with H_4 on $4B^m \times c_{t_1}L \times [0, \varepsilon_2]$, the argument given above shows that F_5 agrees with F on a neighborhood N of $3B^m \times v \times 0$. Clearly conditions (ii) and (iii) hold and F_5 is the identity outside $5\mathring{B}^m \times c_{t_1}L \times [0, \varepsilon_2]$. Finally the restrictions of F_5 to $R^m \times (cL - v) \times [0, \varepsilon_2]$ and $R^m \times v \times [0, \varepsilon_2]$ are PL since the restrictions of H_5 to these sets are PL and G_4 is PL.

By an argument similar to the one used to construct F_2, we may replace F_5 with a new TOP_m^+ map F_6: $R^{m-i} \times R^i \times cL \times [0, \varepsilon_2] \to R^{m-i} \times R^i \times cL \times [0, \varepsilon_2]$ such that F_6 agrees with F on N, has properties (ii) and (iii) and is the identity outside a compact set containing $0 \times B^i \times v$. Choose $r < t_1$ and $\varepsilon > 0$ so that $rB^{m-i} \times rB^i \times c_rL \times [0, t] \subset N$. Then F_6 satisfies (i).

We complete the proof of extending F_6 to H defined on $X \times [0, \varepsilon]$ by setting $H = \mathrm{id}$ outside $R^{m-i} \times R^i \times cL \times [0, \varepsilon]$.

ADDENDUM TO 5.2. *Lemma* 5.2 *holds even if* $F|U \times 0$ *is not the identity.*

PROOF. This case can be reduced to the case when $F|U \times 0$ is the identity by a standard argument.

PROOF OF 5.1. Let $t \in [0, 1]$. Standard arguments using the Handle Lemma and its Addendum show that there exist an $\varepsilon(t) > 0$ and an isotopy G_t: $V \times [t - \varepsilon(t), t + \varepsilon(t)] \to V \times [t - \varepsilon(t), t + \varepsilon(t)]$ such that for every $s \in [t - \varepsilon(t), t + \varepsilon(t)]$ $G_t^sF^t(x) = F^s(t)$ where $G_t^s = G_t|V \times s$ and $F^s = F|U \times s$. The G_t may now be spliced together to yield the desired isotopy H by another standard argument.

6. An engulfing lemma. In this section we prove the following lemma.

LEMMA 6.1. *Let X be a* CAT CS *space where* CAT *is one of the categories* TOP, PL, *or* TOP_m^+. *Let C be a noncompact component of $X^{(m)} - X^{(m-1)}$ and let f: $[0, \infty) \to C$ be a proper locally flat embedding. Then there exists a* CAT *ambient isotopy ϕ_t $(0 \leqslant t \leqslant 1)$ of $X - X^{(m-1)}$ such that $\phi_1(X - X^{(m-1)}) \subset X - X^{(m-1)} - f[0, \infty)$. Furthermore ϕ_t may be taken to be the identity outside a neighborhood of $f[0, \infty)$.*

The proof of this lemma requires the following lemma.

LEMMA 6.2. (i) *Let X, C, and f: $[0, \infty) \to C$ be as in the hypotheses of* 6.1. *Then there exist an open neighborhood U of $f[0, \infty)$ in X and a* CAT *homeomorphism h: $(U, f[0, \infty)) \to (R^1 \times R^{m-1} \times cL, [0, \infty) \times 0 \times v)$ where L is a* CAT *link of $f(0)$ in X.*

(ii) *In fact, if we stratify U by setting $U^{(n)} = U \cap X^{(n)}$, then h is an isomorphism of* CAT CS *spaces.*

PROOF. The argument given by Lacher **[16]** to establish this result for manifolds extends trivially to establish this case also.

PROOF OF 6.1. By 6.2 there exist a neighborhood U of $f[0, \infty)$ in X and a CAT homeomorphism h: $U \to R^1 \times R^{m-1} \times cL$. Clearly there exists an ambient isotopy ψ_t $(0 \leqslant t \leqslant 1)$ of $R^1 \times R^{m-1} \times cL$ such that $\psi_1(R^1 \times R^{m-1} \times cL) \subset R^1 \times R^{m-1} \times cL - [0, \infty) \times 0 \times v$ and such ψ_t is the identity outside $(-1, \infty) \times \mathring{B}^{m-1} \times c_1 L$. Set $\phi_t(x) = h^{-1}\psi_t h(x)$ if $x \in U$ and $\phi_t(x) = x$ otherwise.

7. A collaring lemma. In this section we prove the following lemma.

LEMMA 7.1. *Let X be a* TOP^+_{m-1} CS *space. Let C be a noncompact component of $X^{(m)} - X^{(m-1)}$ such that C is homeomorphic to $N \times R^1$ where N is a closed topological $(m-1)$-manifold. Let M be a locally flat* PL *submanifold of C such that the inclusion $M \hookrightarrow C$ is a homotopy equivalence. If $m \geqslant 5$, then there exist a closed neighborhood V of C in X, a* PL *stratified polyhedron $Y \subset V$ such that $M = Y^{(m-1)} - Y^{(m-2)}$, and a* PL *homeomorphism h: $V \to Y \times R^1$ such that h maps Y identically onto $Y \times 0$.*

PROOF. We note first that M separates C into two submanifolds C_1 and C_2 with $\partial C_1 = M = \partial C_2$. Let W be a collar neighborhood of M in C_1 and let U be a regular neighborhood of W in $X - X^{(m-1)}$ relative to $\partial W = M \cup M'$ (cf. [**6**]) where M' is PL homeomorphic to M. Let Y (respectively, Y') be a regular neighborhood of M (respectively, M') in Fr U where the frontier is taken in $X - X^{(m-1)}$. (Notice that this is the same as the frontier of U in X.) Then there exists a PL homeomorphism g_1: $(U; Y, Y') \to (Y \times [0, 1]; Y \times 0, Y \times 1)$ such that g_1 maps Y identically onto $Y \times 0$. The argument given in [**17**, p. 159] based on Siebenmann's thesis [**23**] and Stallings engulfing [**12**] clearly extends to produce a PL homeomorphism f_1: $(U - Y'; Y) \to (V_1; Y)$ which is the identity on Y where V_1 is a relative regular neighborhood of C_1 modulo M in $X - X^{(m-1)}$. The map $g_1 f_1^{-1}$ followed by a change of scale yields a PL homeomorphism $h_1(V_1; Y) \to (Y \times [0, +\infty); Y \times 0)$ such that h_1 maps Y identically onto $Y \times 0$.

By a similar argument, there exists a PL homeomorphism h_2: $(V_2; Y) \to (Y \times (-\infty, 0]; Y \times 0)$ which is the identity on Y. The desired PL homeomorphism h is obtained by patching h_1 and h_2 together.

8. The proof of Theorem C. In this section we begin the proof of Theorem C by constructing the first of the two spectral sequences described in it. The second spectral sequence is constructed in §9, its E^1 term is calculated in §§10 and 11, and its E^2 term is determined in §12 at which point the proof of Theorem C is complete. Before stating the main results of this section, we set some notation.

Throughout the remainder of this paper, CAT will denote any one of the three categories TOP, PL, or TOP^+_m. Let (X, A) be a pair of spaces in CAT–when CAT is TOP^+_m, regard A as being a substratified space of X (i.e. $A^{(n)} = A \cap X^{(n)}$) and $A - A^{(m)}$ as being a subpolyhedron of $X - X^{(m)}$. We let $\mathrm{CAT}(X, A)$ (respectively, $\mathcal{P}\,\mathrm{CAT}(X, A)$) denote the semi-simplicial complex of CAT homeomorphisms (respectively, CAT pseudo-isotopies) of X that restrict to the identity on A. Thus a k-simplex of $\mathrm{CAT}(X, A)$ (respectively, $\mathcal{P}\,\mathrm{CAT}(X, A)$) is a CAT homeomorphism f: $\Delta^k \times X \to \Delta^k \times X$ (respectively, F: $\Delta^k \times I \times X \to \Delta^k \times I \times X$) that commutes

with projection on Δ^k and that satisfies $f|\Delta^k \times A = \mathrm{id}$ (respectively, that commutes with projection on Δ^k and that satisfies $F|\Delta^k \times I \times A \cup \Delta^k \times 0 \times X = \mathrm{id}$ and $F(\Delta^k \times 1 \times X) = \Delta^k \times 1 \times X$). When CAT is TOP_m^+, we require that the restrictions of f (respectively, F) to $\Delta^k \times X^{(m)} - X^{(m-1)}$ and $\Delta^k \times X - X^{(m)}$ (respectively, to $\Delta^k \times I \times X^{(m)} - X^{(m-1)}$ and $\Delta^k \times I \times X - X^{(m)}$) be PL.

To simplify our notation, we shall let $\mathrm{TOP}_m^+/\mathrm{PL}(X, A)$ and $\mathcal{P}\,\mathrm{TOP}_m^+/\mathcal{P}\,\mathrm{PL}(X, A)$, respectively, denote $\mathrm{TOP}_m^+(X, A)/\mathrm{PL}(X, A)$ and $\mathcal{P}\,\mathrm{TOP}_m^+(X, A)/\mathcal{P}\,\mathrm{PL}(X, A)$ respectively; and we shall follow this convention for the rest of this paper.

LEMMA 8.1. *There are homotopy equivalences*
(i) $\mathrm{TOP}_m^+/\mathrm{PL}(cS^{m-1} * L, cS^{m-1}) \simeq \mathrm{TOP}_m^+/\mathrm{PL}(cS^{m-1} * L)$,
(ii) $\mathrm{TOP}_m^+/\mathrm{PL}(cS^{m-1} * L, cS^{m-1}) \simeq \mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m)$.

The proof is given later in this section.

It follows from 8.1 that it is sufficient for us to determine the homotopy of $\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m)$.

PROPOSITION 8.2. *Let L be a* PL *stratified polyhedron of dimension n. Then there exists a spectral sequence $\{\bar{E}^r, \bar{d}^r\}$ converging to $\pi_*(\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m))$ for $* \geqslant 1$ such that*

$$\bar{E}^1_{s,t} = \begin{cases} \pi_{s+t}(\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})) & \text{if } s + t \geqslant 2, \\ \operatorname{Im} r_s & \text{if } s + t = 1, \\ 0 & \text{if } s + t \leqslant 0, \end{cases}$$

where

$$r_s\colon \pi_1(\mathrm{TOP}_m^+(S^m * L, S^m * L^{(n-s)})) \to \pi_1(\mathrm{TOP}_m^+(S^m * L^{(n-s+1)}, S^m * L^{(n-s)}))$$

is induced by restriction.

PROOF. The main step in constructing this spectral sequence is the following proposition whose proof will be given later in this section.

PROPOSITION 8.3. *There exists a fibration*

$$\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m * L^{(n-s+1)}) \xrightarrow{i} \mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m * L^{(n-s)}) \xrightarrow{r} \mathrm{TOP}_m^+/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})$$

where r and i are induced by restriction and inclusion respectively.

Assuming 8.3, the proof of 8.2 proceeds by setting $X = \mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m)$ and $X_s = \mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m * L^{(n-s)})$. Then $X_s = X$ for $s \geqslant n + 1$ and X_0 is a point. The spectral sequence of 8.2 is just the spectral sequence of the homotopy exact couple of the tower of $X_0 \to X_1 \to \cdots \to X_{n-1} = X$ of fibrations where we have truncated the homotopy exact sequences at the fundamental group level by

$$\to \pi_1(\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m * L^{(n-s+1)})) \to \pi_1(\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m * L^{(n-s)})) \to \operatorname{Im} r_s \to 1.$$

LEMMA 8.4. *In the following diagram of semi-simplicial groups and homomorphisms*

$$\begin{array}{ccccc} H' & \xrightarrow{h_0} & H & \xrightarrow{h_1} & H'' \\ \downarrow i' & & \downarrow i & & \downarrow i'' \\ G' & \xrightarrow{g_0} & G & \xrightarrow{g_1} & G'' \end{array}$$

suppose that the top row is a fibration onto, the bottom row is a fibration (onto), and the vertical maps are monomorphisms. Then

$$G'/H' \xrightarrow{f_0} G/H \xrightarrow{f_1} G''/H''$$

is a fibration (onto) where f_j is the map induced by g_j and h_j ($j = 0, 1$).

PROOF. This lemma is well known.

PROOF OF 8.1. The first part of 8.1 follows directly from 8.4 upon observing that the diagram

$$\begin{array}{ccccc} \mathrm{PL}(cS^{m-1} * L, cS^{m-1}) & \to & \mathrm{PL}(cS^{m-1} * L) & \xrightarrow{r} & \mathrm{PL}(cS^{m-1}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{TOP}_m^+(cS^{m-1} * L, cS^{m-1}) & \to & \mathrm{TOP}_m^+(cS^{m-1} * L) & \xrightarrow{r} & \mathrm{PL}(cS^{m-1}) \end{array}$$

satisfies the hypotheses of 8.4 where r is the restriction map.

To prove the second part of 8.1, let $\mathrm{ECAT}(cS^{m-1} * L, cS^{m-1})$ denote the semi-simplicial complex of germs at the cone point of CAT embeddings of $cS^{m-1} * L$ into itself that are the identity on cS^{m-1}. The restriction map r yields a fibration

$$\begin{aligned} \mathrm{CAT}(cS^{m-1} * L, \bar{c}S^{m-1} * L \cup cS^{m-1}) &\to \mathrm{CAT}(cS^{m-1} * L, cS^{m-1}) \\ &\xrightarrow{r} \mathrm{ECAT}(cS^{m-1} * L, cS^{m-1}) \end{aligned}$$

where $\bar{c}X$ denotes a closed cone on X. The Alexander trick shows that $\mathrm{CAT}(cS^{m-1}, \bar{c}S^{m-1} * L \cup cS^{m-1})$ is contractible. If we now apply 8.4 to the diagram obtained from the above fibration by setting CAT equal to PL and TOP_m^+, we obtain a homotopy equivalence

$$r\colon \mathrm{TOP}_m^+/\mathrm{PL}(cS^{m-1} * L, cS^{m-1}) \to \mathrm{ETOP}_m^+/\mathrm{EPL}(cS^{m-1} * L, cS^{m-1}).$$

Similarly the restriction map r' yields a fibration onto

$$\begin{aligned} \mathrm{CAT}(D^m * L, S^{m-1} * L \cup D^m) &\to \mathrm{CAT}(S^m * L, S^m) \\ &\xrightarrow{r'} \mathrm{ECAT}(cS^{m-1} * L, cS^{m-1}) \end{aligned}$$

whose fiber is contractible by the Alexander trick. Hence r' induces a homotopy equivalence

$$r'\colon \mathrm{TOP}_m^+/\mathrm{PL}(S^m * L, S^m) \to \mathrm{ETOP}_m^+/\mathrm{EPL}(cS^{m-1} * L, cS^{m-1}).$$

The composite of r and an inverse for r' is the desired homotopy equivalence.

PROOF OF 8.3. Proposition 8.3 is an immediate consequence of 8.4 and the observation that there is a fibration

$$\mathrm{CAT}(S^m * L, S^m * L^{(n-s+1)}) \xrightarrow{i} \mathrm{CAT}(S^m * L, S^m * L^{(n-s)}) \xrightarrow{r} \mathrm{CAT}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})$$

where r is the restriction map and i is the inclusion.

9. Another spectral sequence. In this section we construct a spectral sequence that converges to $\bar{E}^1$ of the spectral sequence 8.2. In particular, we prove the following proposition.

PROPOSITION 9.1. *Let L be a* PL *stratified polyhedron of dimension n. Then there exists a spectral sequence $\{E^r, d^r\}$ converging to*

$$\pi_*(\mathrm{TOP}_m^+/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})) \quad \textit{for } * \geqslant 1$$

such that $E^1_{p,q} = 0$ if $p < 0$ or $p > m$ or $q > 0$, and if $q \leqslant 0$ and $0 \leqslant p \leqslant m$, then

$$E^1_{p,q} = \begin{cases} \pi_{p+q-1}(\mathcal{P}\,\mathrm{TOP}_{p-1}^+/\mathcal{P}\,\mathrm{PL}(S^{p-1} * L^{(n-s+1)}, S^{p-1} * L^{(n-s)})) & \textit{for } p+q \geqslant 2, \\ \mathrm{Im}\,\partial_p & \textit{for } p+q = 1, \\ 0 & \textit{for } p+q \leqslant 0, \end{cases}$$

where

$$\partial_p\colon \pi_1(\mathrm{TOP}_p^+/\mathrm{PL}(S^p * L^{(n-s+1)}, S^p * L^{(n-s)})) \to \pi_0(\mathcal{P}\,\mathrm{TOP}_{p-1}^+/\mathcal{P}\,\mathrm{PL}(S^{p-1} * L^{(n-s+1)}, S^{p-1} * L^{(n-s)}))$$

is defined after 9.2 *below.*

PROOF. Let $X = \mathrm{TOP}_m^+/\mathrm{PL}(S^m * L^{(n-s+1)}, S^m * L^{(n-s)})$ and for $0 \leqslant p \leqslant m+1$ set $X_p = \mathrm{TOP}_{p-1}^+/\mathrm{PL}(S^{p-1} * L^{(n-s+1)}, S^{p-1} * L^{(n-s)})$ where for $p = 0$, $S^{p-1} * A$ is just A. Then $X_{m+1} = X$ and $X_0 = \mathrm{TOP}_{-1}^+(L^{(n-s+1)}, L^{(n-s)})/\mathrm{PL}(L^{(n-s+1)}, L^{(n-s)})$ is a point by the definition of TOP_{-1}^+. By 9.2 below there is a tower $X_0 \to X_1 \to \cdots \to X_{m+1} = X$ of fibrations whose homotopy exact couple, truncated at the point

$$\cdots \to \pi_1(\mathrm{TOP}_{p-1}^+/\mathrm{PL}(S^{p-1} * L^{(n-s+1)}, S^{p-1} * L^{(n-s)})) \to \pi_1(\mathrm{TOP}_p^+/\mathrm{PL}(S^p * L^{(n-s+1)}, S^p * L^{(n-s)})) \xrightarrow{\partial_p} \mathrm{Im}\,\partial_p \to 1,$$

yields the desired spectral sequence.

PROPOSITION 9.2. *Let (X, A) be a polyhedral pair. Then there exists a fibration*

$$\mathcal{P}\,\mathrm{TOP}_p^+/\mathcal{P}\,\mathrm{PL}(S^p * X, S^p * A) \xrightarrow{\rho} \mathrm{TOP}_p^+/\mathrm{PL}(S^p * X, S^p * A) \xrightarrow{\Sigma} \mathrm{TOP}_{p+1}^+/\mathrm{PL}(S^{p+1} * X, S^{p+1} * A)$$

where Σ is induced by suspension and ρ is induced by restricting a pseudo-isotopy F: $\Delta^k \times I \times (S^p * X) \to \Delta^k \times I \times (S^p * X)$ to $\Delta^k \times 1 \times (S^p * X)$.

The function ∂_p above is just the connecting function in the homotopy exact sequence of this fibration.

PROOF. Proposition 9.2 is an immediate consequence of the following lemma and 8.4.

LEMMA 9.3. *Let (X, A) be a polyhedral pair. Then there is a fibration*

$$\mathcal{P}\,\mathrm{CAT}(S^p * X, S^p * A) \xrightarrow{\rho} \mathrm{CAT}(S^p * X, S^p * A) \xrightarrow{\Sigma} \mathrm{CAT}(S^{p+1} * X, S^{p+1} * A)$$

where Σ is the suspension map and ρ is the restriction of a pseudo-isotopy to time $t = 1$.

PROOF. The existence of this fibration is established by an argument similar to one given in [**19**, §8]. Namely, consider the following homotopy commutative diagram

$$\begin{array}{ccccc}
\mathcal{P}\,\mathrm{CAT}(S^p * X, S^p * A) & \xrightarrow{\rho} & \mathrm{CAT}(S^p * X, S^p * A) & \xrightarrow{\Sigma} & \mathrm{CAT}(S^{p+1} * X, S^{p+1} * A) \\
\downarrow \mathrm{id} & & \downarrow c & & \downarrow r' \\
\mathcal{P}\,\mathrm{CAT}(S^p * X, S^p * A) & \xrightarrow{i} & \overline{\mathrm{CAT}}(\bar{c}S^p * X, \bar{c}S^p * A) & \xrightarrow{r} & \mathrm{ECAT}(cS^p * X, cS^p * A)
\end{array}$$

where $\overline{\mathrm{CAT}}(\bar{c}S^p * X, \bar{c}S^p * A)$ is the semi-simplicial complex of CAT homeomorphisms of the closed cone $\bar{c}S^p * X$ onto itself that carry the base of the cone $S^p * X$ into itself and that restrict to the identity on $\bar{c}S^p * A$; c is the map obtained by coning; r and r' are restriction maps; and i is the map defined by $i(F) = F \cup \mathrm{id}$ where $F\colon \Delta^k \times [0, 1] \times (S^p * X) \to \Delta^k \times [0, 1] \times (S^p * X)$ is a k-simplex of $\mathcal{P}\,\mathrm{CAT}(S^p * X, S^p * A)$, id is the identity map of $\Delta^k \times \bar{c}\{0 \times (S^p * X)\}$ and we have identified $\bar{c}\{0 \times (S^p * X)\} \cup [0, 1] \times (S^p * X)$ with $\bar{c}S^p * X$ via a (fixed) CAT homeomorphism.

Let $r''\colon \overline{\mathrm{CAT}}(\bar{c}S^p * X, \bar{c}S^p * A) \to \mathrm{CAT}(S^p * X, S^p * A)$ be the restriction map. Then r'' is a fibration onto whose fiber $\mathrm{CAT}(\bar{c}S^p * X, \bar{c}S^p * A \cup S^p * X)$ is contractible by the Alexander trick. Thus r'' is a homotopy equivalence. Since $r''c = \mathrm{id}$, c is also a homotopy equivalence. Similarly r' is a fibration with contractible fiber $\mathrm{CAT}(\bar{c}S^p * X, \bar{c}S^p * A \cup S^p * X)$. Hence r' is a homotopy equivalence.

Since the bottom row of the diagram above is obviously a fibration, so is the upper row and the proof of 9.3 is completed.

10. The determination of $E^1_{p,q}$. It is the object of this section and the next to identify most of the E^1 term of the spectral sequence described in §9. In this section, we prove the following proposition.

PROPOSITION 10.1. *If $q > 0$, then $E^1_{p,q} = 0$. More precisely,*

$$\pi_{p+q-1}\big(\mathcal{P}\,\mathrm{TOP}^+_{p-1}/\mathcal{P}\,\mathrm{PL}(S^{p-1} * L^{(n-s+1)}, S^{p-1} * L^{(n-s)})\big) = 0.$$

The following lemmas are the main tools used in proving 10.1.

LEMMA 10.2. *Let $k \leqslant p$ and (X, A) be a pair of polyhedra. Then there are homotopy equivalences*

$$\sigma\colon \Omega^k \operatorname{CAT}(S^{p-1} * X, S^{p-1} * A) \to \operatorname{CAT}((S^{p-1-k} * X) \times I^k, \partial)$$

and

$$\omega\colon \Omega^k \mathcal{P}\operatorname{CAT}(S^{p-1} * X, S^{p-1} * A) \to \mathcal{P}\operatorname{CAT}((S^{p-1-k} * X) \times I^k, \partial)$$

where $\partial = (S^{p-1-k} * X) \times \mathring{I}^k \cup (S^{p-1-k} * A) \times I^k$ and $S^{-1} * X = X$. Furthermore, σ and ω homotopy commute with the maps induced by the inclusions of categories $i_1\colon \mathrm{PL} \hookrightarrow \mathrm{TOP}^+_{p-1}$, $i_2\colon \mathrm{PL} \hookrightarrow \mathrm{TOP}$, $i_3\colon \mathrm{TOP}^+_{p-1} \hookrightarrow \mathrm{TOP}$.

The proof of 10.2 uses arguments similar to those in [**19**, §8] and will be given later in this section. This result is also known to Hatcher [**10**] and [**11**].

LEMMA 10.3. *Let (X, A) be a polyhedral pair. Then*

$$\pi_k(\mathcal{P}\mathrm{TOP}^+_{p-1}/\mathcal{P}\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) = 0 \quad \textit{for } k \geqslant p.$$

PROOF. Assuming 10.2 we proceed as follows. Consider the exact sequence

$$\begin{aligned} \to \pi_k(\mathcal{P}\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) &\to \pi_k(\mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1} * X, S^{p-1} * A)) \\ &\to \pi_k(\mathcal{P}\mathrm{TOP}^+_{p-1}/\mathcal{P}\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) \to . \end{aligned}$$

By 10.2 for $k \geqslant p$, there is a commutative diagram

$$\begin{array}{ccc} \pi_k(\mathcal{P}\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) & \xrightarrow{i_{1*}} & \pi_k(\mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1} * X, S^{p-1} * A)) \\ \approx \downarrow \partial & & \approx \downarrow \partial \\ \pi_{k-p}(\Omega^p \mathcal{P}\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) & \xrightarrow{i_{1*}} & \pi_{k-p}(\Omega^p \mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1} * X, S^{p-1} * A)) \\ \approx \downarrow \omega_* & & \approx \downarrow \omega_* \\ \pi_{k-p}(\mathcal{P}\mathrm{PL}(X \times I^p, \partial)) & \xrightarrow{i_{1*}} & \pi_{k-p}(\mathcal{P}\mathrm{TOP}^+_{p-1}(X \times I^p, \partial)) \end{array}$$

where ∂ is the natural homomorphism and ω is the map of 10.2. Since $\mathcal{P}\mathrm{TOP}^+_{p-1}(X \times I^p, \partial)$ is identical with $\mathcal{P}\mathrm{PL}(X \times I^p, \partial)$, the bottom map is an isomorphism. Hence so is the top map. In the next section we shall show that $\pi_k(\mathcal{P}\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) = 0$ for $k \leqslant p - 1$. It now follows that $\pi_k(\mathcal{P}\mathrm{TOP}^+_{p-1}/\mathcal{P}\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) = 0$ for $k \geqslant p$.

PROOF OF 10.1. Proposition 10.1 is an obvious corollary of 10.3.

PROOF OF 10.2. It suffices to construct homotopy equivalences

$$\sigma\colon \Omega \operatorname{CAT}((S^{p-l} * X) \times I^l, \partial_l) \to \operatorname{CAT}((S^{p-l-1} * X) \times I^{l+1}, \partial_{l-1})$$

and

$$\omega\colon \Omega\mathcal{P}\operatorname{CAT}((S^{p-l} * X) \times I^l, \partial_l) \to \mathcal{P}\operatorname{CAT}((S^{p-l-1} * X) \times I^{l+1}, \partial_{l+1})$$

which homotopy commute with the inclusions i_j $(j = 1, 2, 3)$ where $\partial_l = (S^{p-l} * X) \times \dot{I}^l \cup (S^{p-l} * A) \times I^l$. Since the construction of ω is similar to that of σ, we describe only the latter.

Let $\operatorname{CAT}G((R^{p-l} * X) \times I^l, \partial_l)$ be the semi-simplicial complex of germs at $O \times I^l$ of CAT homeomorphisms of $(R^{p-l} * X) \times I^l$ that restrict to the identity

on $\partial_l = (R^{p-l} * X) \times \dot{I}^l \cup (R^{p-l} * A) \times I^l$ where $O \in R^{p-l}$ is the origin. If we identify (R^{p-l}, O) with (S^{p-l} – north pole, south pole) the operation r of taking the germ gives a fibration

$$\mathrm{CAT}((D^{p-l} * X) \times I^l, \partial') \to \mathrm{CAT}((S^{p-l} * X) \times I^l, \partial) \to \mathrm{CAT}G((R^{p-l} * X) \times I^l, \partial)$$

where $\partial' = (S^{p-l-1} * X) \times I^l \cup (D^{p-l} * X) \times \dot{I}^l \cup (D^{p-l} * A) \times I^l$. Since $\mathrm{CAT}((D^{p-l} * X) \times I^l, \partial')$ is contractible by the Alexander trick, r is a homotopy equivalence. Clearly r commutes with i_j ($j = 1, 2, 3$).

Similarly, taking germs gives a fibration

$$\mathrm{CAT}((S^{p-l-1} * X) \times I^{l+1}, \partial) \to \mathrm{CAT}((D^{p-l} * X) \times I^l, \partial') \to \mathrm{CAT}G((R^{p-l} * X) \times I^l, \partial)$$

whose total space is contractible. It follows that the map

$$\lambda\colon \Omega\, \mathrm{CAT}G((R^{p-l} * X) \times I^k, \partial) \to \mathrm{CAT}((S^{p-l-1} * X) \times I^{l+1}, \partial)$$

from the loops on the base of the above fibration to the fiber is a homotopy equivalence. Clearly λ commutes with i_j ($j = 1, 2, 3$).

The desired homotopy equivalence σ is just $\lambda(\Omega r)$.

COROLLARY 10.4. *Let $k \leqslant p$. Then there are homotopy equivalences*

$$\sigma\colon \Omega^k\, \mathrm{TOP}^+_{p-1}/\mathrm{PL}(S^{p-1} * X, S^{p-1} * A) \to \mathrm{TOP}^+_{p-1}/\mathrm{PL}((S^{p-k} * X) \times I^k, \partial)$$

and

$$\omega\colon \Omega^k \mathcal{P}\, \mathrm{TOP}^+_{p-1}/\mathcal{P}\, \mathrm{PL}(S^{p-1} * X, S^{p-1} * A) \to \mathcal{P}\, \mathrm{TOP}^+_{p-1}/\mathcal{P}\, \mathrm{PL}((S^{p-k} * X) \times I^k, \partial)$$

where $\partial = (S^{p-k} * X) \times \dot{I}^k \cup (S^{p-k} * A) \times I^k$.

PROOF. This follows immediately from 10.2 and 8.4.

COROLLARY 10.5. *Let $k \leqslant p$. If the functor Ω^k is applied to the fibration of* 9.3 *or* 9.2 *respectively, we obtain the fibrations*

$$\mathcal{P}\, \mathrm{CAT}((S^{p-k} * X) \times I^k, \partial) \to \mathrm{CAT}((S^{p-k} * X) \times I^k, \partial) \to \mathrm{CAT}((S^{p-k+1} * X) \times I^k, \partial)$$

and

$$\mathcal{P}\, \mathrm{TOP}^+_p/\mathcal{P}\, \mathrm{PL}((S^{p-k} * X) \times I^k, \partial) \xrightarrow{\rho} \mathrm{TOP}^+_p/\mathrm{PL}((S^{p-k} * X) \times I^k, \partial) \to \mathrm{TOP}^+_{p+1}/\mathrm{PL}((S^{p-k+1} * X) \times I^k, \partial)$$

respectively, where ρ is the restriction map of 9.2 *and* $\partial = (S^{p-k} * X) \times \dot{I}^k \cup (S^{p-k} * A) \times I^k$.

PROOF. This is obvious.

For any space X, let $\eta\colon \pi_k(X) \to \pi_0(\Omega^k X)$ be the natural isomorphism.

Let i: $\mathrm{TOP}_p^+((S^{p-k} * X) \times I^k, \partial) \to \mathcal{P}\mathrm{TOP}_{p-1}^+((S^{p-k} * X) \times I^{k-1}, \partial)$ be the map obtained by noting that since a k-simplex in the first complex is the identity on $\Delta^k \times (S^{p-k} * X) \times I^{k-1} \times 0$, it is also in the second complex.

COROLLARY 10.6. *There is a commutative diagram*

$$\begin{array}{ccc}
\pi_0(\mathcal{P}\mathrm{TOP}_p^+((S^{p-k}*X) \times I^k, \partial)) & \xrightarrow{\omega * \eta} & \pi_k(\mathcal{P}\mathrm{TOP}_p^+(S^p*X, S^p*A)) \\
\downarrow \rho_* & & \downarrow \rho_* \\
\pi_k(\mathrm{TOP}_p^+(S^p*X, S^p*A)) & \xrightarrow{\sigma * \eta} & \pi_0(\mathrm{TOP}_p^+((S^{p-k}*X) \times I^k, \partial)) \\
\downarrow \partial_* & & \downarrow i_* \\
\pi_{k-1}(\mathcal{P}\mathrm{TOP}_{p-1}^+(S^{p-1} * X, S^{p-1} * A)) & \xrightarrow{\omega * \eta} & \pi_0(\mathcal{P}\mathrm{TOP}_{p-1}^+((S^{p-k} * X) \times I^{k-1}, \partial))
\end{array}$$

where ∂_* *is the connecting homomorphism of the homotopy exact sequence of the fibration* 9.3.

11. The rest of $E_{p,q}^1$. In this section, we identify most of the remaining $E_{p,q}^1$ terms of the spectral sequence of §9.

For any space X, we shall let $\mathrm{Wh}\, \pi_1(X) = \sum \mathrm{Wh}\, \pi_1(X_\alpha)$ where the summation runs over the path components of X and $\mathrm{Wh}\, G$ denotes the Whitehead group of the group G. The groups $\tilde{K}_0\pi_1(X)$ and $K_i\pi_1(X)$ for $i < 0$ are defined similarly where the latter groups refer to the "lower" algebraic K-theoretic groups of [**4**, Chapter XII].

THEOREM 11.1. *Let L be a compact* PL *stratified polyhedron of dimension n. Let $t = n - s + 1 \leqslant n$ and suppose $5 \leqslant p + m$. Then for $p + q \geqslant 2$ in the spectral sequence* 9.1,

$$E_{p,q}^1 = \begin{cases} K_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) & \text{if } q \leqslant -2, \\ \tilde{K}_0\pi_1(L^{(t)} - L^{(t-1)}) & \text{if } q = -1, \\ \mathrm{Wh}\, \pi_1(L^{(t)} - L^{(t-1)}) & \text{if } q = 0, \end{cases}$$

and for $p + q = 1$,

$$E_{p,q}^1 \subset \begin{cases} K_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) & \text{if } q \leqslant -2, \\ \tilde{K}_0\pi_1(L^{(t)} - L^{(t-1)}) & \text{if } q = -1, \\ \mathrm{Wh}\, \pi_1(L^{(t)} - L^{(t-1)}) & \text{if } q = 0. \end{cases}$$

We shall show in §12 that when $p + q = 1$ then $E_{p,q}^1$ is actually a subgroup of $\{x \in K_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) | x = (-1)^{t+1}x^*\}$ where "star" denotes the usual duality involution.

We also remark that when L has no strata of dimensions $\leqslant 4$, then 11.1 completes the description of $E_{p,q}^1$ for $p + q \geqslant 2$.

PROOF. This follows directly from the exact sequence

$$\to \pi_k(\mathcal{P}\,\mathrm{PL}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)}))$$
$$\to \pi_k(\mathcal{P}\,\mathrm{TOP}^+_{p-1}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)}))$$
$$\to \pi_k(\mathcal{P}\,\mathrm{TOP}^+_{p-1}/\mathcal{P}\,\mathrm{PL}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)})) \to$$

and the following propositions.

PROPOSITION 11.2. *Let L be a* PL *stratified polyhedron of dimension n, $t \leqslant n$ and suppose $5 \leqslant p + t$. Then*

$$\pi_k(\mathcal{P}\,\mathrm{TOP}^+_{p-1}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)}))$$
$$= \begin{cases} K_{k-p+2}\pi_1(L^{(t)} - L^{(t-1)}) & \text{if } 0 \leqslant k \leqslant p-3, \\ \tilde{K}_0\pi_1(L^{(t)} - L^{(t-1)}) & \text{if } k = p-2, \\ \mathrm{Wh}\,\pi_1(L^{(t)} - L^{(t-1)}) & \text{if } k = p-1. \end{cases}$$

PROPOSITION 11.3. *Let (X, A) be a polyhedral pair. Then $\pi_k(\mathcal{P}\,\mathrm{PL}(S^{p-1} * X, S^{p-1} * A)) = 0$ for $k \leqslant p - 1$.*

PROOF OF 11.2. Since $k \leqslant p - 1$, by 10.2 it suffices to determine $\pi_0(\mathcal{P}\,\mathrm{TOP}^+_{p-1}((S^{p-1-k} * L^{(t)}) \times I^k, \partial))$ where $\partial = (S^{p-1-k} * L^{(t)}) \times \dot{I}^k \cup (S^{p-1-k} * L^{(t-1)}) \times I^k$. The techniques used to prove [3, Theorem 4], however, may be easily adapted to show that this group is $K_{k-p+2}\pi_1(L^{(t)} - L^{(t-1)})$ if $0 \leqslant k \leqslant p - 3$, $\tilde{K}_0\pi_1(L^{(t)} - L^{(t-1)})$ if $k = p - 2$, and $\mathrm{Wh}\,\pi_1(L^{(t)} - L^{(t-1)})$ if $k = p - 1$.

The proof of 11.3 is rather tedious. In order to indicate the idea of the proof, we prove first the following special case.

LEMMA 11.4. *Let X be a polyhedron. Then $\pi_0(\mathcal{P}\,\mathrm{PL}(S^0 * X, S^0)) = 0$.*

PROOF. Let F_0: $I \times (S^0 * X) \to I \times (S^0 * X)$ be a 0-simplex of $\mathcal{P}\,\mathrm{PL}(S^0 * X, S^0)$. Then $F_0|0 \times (S^0 * X) \cup I \times S^0 = \mathrm{id}$. Let $N = I \times (c_+ * X)$ where $S^0 = \{c_-, c_+\}$. Then N is a regular neighborhood of $0 \times (c_+ * X) \cup I \times c_+$ relative to $0 \times X \cup 1 \times c_+$. Hence, so is $F_0(N)$. By uniqueness of relative regular neighborhoods [6], there exists a PL isotopy of F_0 through pseudo-isotopies in $\mathcal{P}\,\mathrm{PL}(S^0 * X, S^0)$ to a new pseudo-isotopy F_1 that satisfies $F_1(N) = N$, $F_1(I \times X) = I \times X$, and $F_1[1 \times (c_+ * X)] = 1 \times (c_+ * X)$.

Let $G = F_1|I \times X$ and notice that G is a 0-simplex of $\mathcal{P}\,\mathrm{PL}(X)$. Hence, G is pseudo-isotopic to the identity; i.e. there exists a PL homeomorphism H: $I \times [1/2, 1] \times X \to I \times [1/2, 1] \times X$ such that $H|I \times 1/2 \times X = \mathrm{id}$, $H|I \times 1 \times X = G$, and $H|0 \times [1/2, 1] \times X = \mathrm{id}$. We now define a pseudo-isotopy F_2 of $S^0 * X$ by setting $F_2|I \times (c_- * X) = F_1|I \times (c_- * X)$, $F_2|I \times [1/2, 1] \times X = H$, and $F_2|I \times \bar{c}_{1/2}X = \mathrm{id}$ where we have identified $c_+ * X$ with $\bar{c}_{1/2}X \cup [1/2, 1] \times X$. An Alexander isotopy shows that F_2 is isotopic to F_1 in $\mathcal{P}\,\mathrm{PL}(S^0 * X, S^0)$. Similarly F_2 is isotopic to the identity in $\mathcal{P}\,\mathrm{PL}(S^0 * X, S^0)$.

It now follows that F_0 is isotopic to the identity in $\mathcal{P}\,\mathrm{PL}(S^0 * X, S^0)$ from which 11.4 follows.

The above proof is sharp enough to show the following addendum.

ADDENDUM TO 11.4. *If $A \subset X$ is a subpolyhedron, then* $\pi_0(\mathcal{P}\mathrm{PL}(S^0 * X, S^0 * A) = 0$.

In order to prove 11.3 in general, we need the following lemma.

LEMMA 11.5. *Let X be a polyhedron. Let $J = \{(t_1, \ldots, t_k) \in \dot{I}^k | t_i = 0$ for some $i\}$. Then*

(i) *There is a* PL *homeomorphism of $(S^0 * X) \times I^k$ onto $S^0 * (X \times I^k) \cup v_+ * [(c_+ * X) \times J] \cup v_- * [(c_- * X) \times J]$ where the notation is explained in the proof. Furthermore, $v_\pm * [(c_\pm * X) \times J] \cap [S^0 * (X \times I)] = v_\pm * (X \times J)$.*

(ii) *If $A \subset X$ is a subpolyhedron, then $(S^0 * A) \times I^k$ is carried onto $S^0 * (A \times I^k) \cup v_+ * [(c_+ * A) \times J] \cup v_- * [(c_- * A) \times J]$.*

(iii) *The subspace $(S^0 * X) \times J$ is mapped onto $(c_+ * X) \times J \cup (c_- * X) \times J \subset v_+ * [(c_+ * X) \times J] \cup v_- * [(c_- * X) \times J]$.*

(iv) *Let $K = \{(t_1, \ldots, t_k) \in \dot{I}^k | t_i = 1$ for some $i\}$. Then $(S^0 * X) \times K$ is mapped onto $v_+ * [(c_+ * X) \times (J \cap K)] \cup v_- * [(c_- * X) \times (J \cap K)] \cup S^0 * (X \times K)$.*

PROOF. Let $S^0 = \{c_+, c_-\}$. Then $(S^0 * X) \times I^k = (c_+ * X) \times I^k \cup (c_- * X) \times I^k$ and the intersection of these two pieces is $X \times I^k$. But also $I^k = w * J$ where $w = (1, \ldots, 1) \in \dot{I}^k$. Thus

$$(c_\pm * X) \times I^k = (c_\pm * X) \times (w * J) = v_\pm * \left[(c_\pm * X) \times J \cup X \times (w * J)\right]$$
$$= v_\pm * (c_\pm * X) \times J \cup v_\pm * (X \times I^k).$$

where $v_\pm$ is the point $(c_\pm, w) \in (c_\pm * X) \times (w * J)$. Hence,

$$\begin{aligned}(S^0 * X) \times I^k &= v_+ * (X \times I^k) \cup v_+ * \left[(c_+ * X) \times J\right] \\ &\quad \cup v_- * (X \times I^k) \cup v_- * \left[(c_- * X) \times J\right] \\ &= S^0 * (X \times I) \cup v_+ * \left[(c_+ * X) \times J\right] \cup v_- * \left[(c_- * X) \times J\right]\end{aligned}$$

where S^0 is now $\{v_-, v_+\}$. A simple inspection of this construction completes the proof of the lemma.

PROOF OF 11.3. By 10.2 it suffices to show that $\pi_0(\mathcal{P}\mathrm{PL}((S^{p-1-k} * X) \times I^k, \partial)) = 0$. Let F_0: $I \times (S^{p-1-k} * X) \times I^k \to I \times (S^{p-1-k} * X) \times I^k$ be a 0-simplex of $\mathcal{P}\mathrm{PL}((S^{p-1-k} * X) \times I^k, \partial)$. By 11.5, we can write $(S^{p-1-k} * X) \times I^k = S^0 * [(S^{p-2-k} * X) \times I^k] \cup v_+ * B_+ \cup v_- * B_-$ where $B_\pm = c_\pm * (S^{p-2-k} * X) \times J$. Under this identification $(S^{p-1-k} * X) \times \dot{I}^k$ corresponds to $S^0 * [(S^{p-2-k} * X) \times J'] \cup B_+ \cup B_-$ where $J' = \{(t_1, \ldots, t_k) \in I^k | t_i = 1$ for some $i\}$, and $(S^{p-1-k} * A) \times I^k$ corresponds to $S^0 * [(S^{p-2-k} * A) \times I^k] \cup v_+ * C_+ \cup v_- * C_-$ where $C_\pm = c_\pm * (S^{p-2-k} * A) \times J$. Thus, $F_0 | I \times (B_\pm \cup v_\pm * C_\pm) = \mathrm{id}$.

Now let $N_\pm = I \times (v_\pm * B_\pm)$. Then $N_\pm$ is a regular neighborhood of $0 \times (v_\pm * B_\pm) \cup I \times (v_\pm * C_\pm)$ relative to $0 \times v_\pm * [(S^{p-2-k} * X) \times J] \cup 1 \times (v_\pm * C_\pm)$. By the uniqueness of relative regular neighborhoods, F_0 is isotopic in $\mathcal{P}\mathrm{PL}((S^{p-1-k} * X) \times I^k, \partial)$ to a pseudo-isotopy F_1 such that $F_1(N_\pm) = N_\pm$. Since $F_0 | I \times (B_\pm \cup v_\pm * C_\pm) = \mathrm{id}$, we have $F_1 | I \times (B_\pm \cup v_\pm * C_\pm) = \mathrm{id}$; and also $F_1 | 0 \times (v_\pm * B_\pm) = \mathrm{id}$.

If we now regard $I \times (v_\pm * B_\pm)$ as the cone on $0 \times (v_\pm * B_\pm) \cup I \times B_\pm$ with vertex $1 \times v_\pm$, we may isotope $F_1|N_\pm$ to the identity via an Alexander isotopy. By the isotopy extension theorem, this extends to an isotopy in $\mathcal{P}\mathrm{PL}((S^{p-1-k} * X) \times I^k, \partial)$ from F_1 to a pseudo-isotopy F_2 such that $F_2|N_\pm = \mathrm{id}$.

Consider now the map $G_0 = F_2|I \times S^0 * [(S^{p-2-k} * X) \times I^k]$. Then $G_0 \in \mathcal{P}\mathrm{PL}(S^0 * Y, S^0 * D)$ where $Y = (S^{p-2-k} * X) \times I^k$ and $D = (S^{p-2-k} * L) \times I^k \cup (S^{p-2-k} * A) \times I^k$. But now 11.4 and its addendum show that G_0 is isotopic to the identity in $\mathcal{P}\mathrm{PL}(S^0 * Y, S^0 * D)$. We extend this isotopy via the identity to an isotopy of F_2 to the identity in $\mathcal{P}\mathrm{PL}((S^{p-1-k} * X) \times I^k, \partial)$. It now follows that F_0 is isotopic to the identity in $\mathcal{P}\mathrm{PL}((S^{p-1-k} * X) \times I^k, \partial)$ and 11.3 is established.

12. The E^2 term. In this section we shall obtain a description of the E^2 term of the spectral sequence 9.1 by determining the differential d^1.

To simplify our notation in this section, we shall use $\tilde{K}_i G$ to denote, respectively, Wh G if $i = 1$, $\tilde{K}_0 G$ if $i = 0$, and $K_i G$ if $i < 0$. Let w: $G \to Z_2 = \{\pm 1\}$ be a homomorphism and denote by $x \mapsto x^*$ the involution on $\tilde{K}_i G$ induced by the anti-involution of ZG that sends $\sum n_g g$ to $\sum w(g) n_g g^{-1}$. These involutions are natural on the category of pairs (G, w) and define a $Z(Z_2)$ module structure on $\tilde{K}_i(G)$. We let

$$H^r(Z_2; \tilde{K}_i G) = \{x \in \tilde{K}_i G | x = (-1)^r x^*\} / \{y + (-1)^r y^* | y \in \tilde{K}_i G\}.$$

The main result of this section is the following proposition.

PROPOSITION 12.1. *Let L be a* PL *stratified polyhedron of dimension n and suppose $4 \leqslant t = n - s + 1 \leqslant n$. Then, in the spectral sequence* 9.1, *for $p + q \geqslant 2$ we have*

$$E^2_{p,q} = \begin{cases} H^{p+q+t}(Z_2; \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})) & \text{for } p < m, \\ \{x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) | x = (-1)^{p+q+t} x^*\} & \text{for } p = m. \end{cases}$$

Furthermore, if $p + q = 1$, then

$$E^2_{p,q} \subset \begin{cases} H^{t+1}(Z_2; \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})) & \text{for } p < m, \\ \{x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) | x = (-1)^{t+1} x^*\} & \text{for } p = m. \end{cases}$$

PROOF. By the arguments of the three previous sections, for $p + q \geqslant 2$ we may identify $E^1_{p,q}$ with $\pi_{p+q-1}(\mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)})$ and $d^1_{p,q}$ with the composite

$$\begin{aligned} &\pi_{p+q-1}(\mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)})) \\ &\qquad \to \pi_{p+q-1}(\mathrm{TOP}^+_{p-1}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)})) \\ &\qquad \to \pi_{p+q-2}(\mathcal{P}\mathrm{TOP}^+_{p-2}(S^{p-2} * L^{(t)}, S^{p-2} * L^{(t-1)})) \end{aligned}$$

coming from the fibration 9.3. The study of this composite is based on the commutative diagram on the following page.

$$
\begin{array}{ccccc}
\pi_{p+q-1}(\mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1}*L)) & \to & \pi_{p+q-1}(\mathrm{TOP}^+_{p-1}(S^{p-1}*L)) & \to & \pi_{p+q-2}(\mathcal{P}\mathrm{TOP}^+_{p-2}(S^{p-2}*L)) \\
\eta\downarrow\approx & & \eta\downarrow\approx & & \eta\downarrow\approx \\
\pi_1(\Omega^{p+q-2}\mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1}*L)) & \to & \pi_1(\Omega^{p+q-2}\mathrm{TOP}^+_{p-1}(S^{p-1}*L)) & \to & \pi_0(\Omega^{p+q-2}\mathcal{P}\mathrm{TOP}^+_{p-2}(S^{p-2}*L)) \\
\omega_*\downarrow\approx & & \sigma_*\downarrow\approx & & \omega_*\downarrow\approx \\
\pi_1(\mathcal{P}\mathrm{TOP}^+_{p-1}((S^{-q+1}*L)\times I^{p+q-2})) & \to & \pi_1(\mathrm{TOP}^+_{p-1}((S^{-q+1}*L)\times I^{p+q-2})) & \to & \pi_0(\mathcal{P}\mathrm{TOP}^+_{p-2}((S^{-q}*L)\times I^{p+q-2})) \\
\lambda\downarrow\approx & & \lambda\downarrow\approx & \overset{i_*}{\nearrow} & \tau\downarrow\approx \\
 & & & & \tilde{K}_{q+1}\pi_1(L^{(t)}-L^{(t-1)}) \\
\pi_0(\mathcal{P}\mathrm{TOP}^+_{p-1}((S^{-q}*L)\times I^{p+q-1})) & \xrightarrow{r_*} & \pi_0(\mathrm{TOP}^+_{p-1}((S^{-q}*L)\times I^{p+q-2}\times I)) & \xrightarrow{\rho} & \\
\tau\downarrow\approx & & & & \alpha\uparrow \\
\tilde{K}_{q+1}\pi_1(L^{(t)}-L^{(t-1)}) & & & & \\
\delta\downarrow & & & & \\
\mathrm{Wh}(\pi_1(L^{(t)}-L^{(t-1)})\times Z^{-q}) & & \to & & \mathrm{Wh}(\pi_1(L^{(t)}-L^{(t-1)})\times Z^{-q})
\end{array}
$$

Warning. To simplify notation in this diagram we have suppressed mention of subspaces and strata dimensions as much as possible. Thus $\mathcal{P}\mathrm{TOP}^+_r(S^r * L)$ and $\mathcal{P}\mathrm{TOP}^+_r((S^{-q} * L) \times I^{q+r})$ are used to denote $\mathcal{P}\mathrm{TOP}^+_r(S^r * L^{(t)}, S^r * L^{(t-1)})$ and $\mathcal{P}\mathrm{TOP}^+_r((S^{-q} * L^{(t)}) \times I^{q+r}, \partial)$ $(q = p-1, p)$ where $\partial = (S^{-q} * L^{(t)}) \times \dot{I}^{q+r} \cup (S^{-q} * L^{(t-1)}) \times I^{q+r}$, etc.

In this diagram, η is the natural isomorphism, ω_* and σ_* are induced by the maps of 10.2, and i_* is the map of 10.6. Furthermore τ is the isomorphism of 11.1 and [3, §8] and $\rho = \tau i_*$ can be regarded as being obtained by the same device of wrapping around T^{-q} that was used to construct τ. Finally δ and α are the inclusion and projection coming from the direct sum decomposition [4, Chapter XII]

$$\mathrm{Wh}(G \times Z^r) = \mathrm{Wh}\, G \oplus r\tilde{K}_0 G + \sum_{s=2}^{r} \binom{r}{s} K_{-s+1} G \oplus \text{Nil terms}$$

for $r = -q$ in which a single copy of the group $K_{q+1}G$ appears.

It follows from the diagram that d^1 may be identified with $\rho r_* \tau^{-1}$. We recall that for $x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})$, $\tau^{-1}x$ is obtained as follows: Construct the relative h-cobordism W with base $(L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times I$ and torsion $\delta(x)$ where N is a regular neighborhood of $L^{(t-1)}$ in $L^{(t)}$. (See Figure 2 below. Notice that $(L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times I$ is a manifold with boundary

$$B = (L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times \dot{I} \cup (L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times \dot{I}^{p+q-1} \times I \cup \partial N \times T^{-q} \times I^{p+q-1} \times I$$

and that the part of W over B is the product h-cobordism V. Thus $\partial W = \partial_+ W \cup (L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times I \cup V$.) Glue a copy of $N \times T^{-q} \times I^{p+q-1} \times I \times I$ to W along the face $\partial N \times T^{-q} \times I^{p+q-1} \times I \times I$ to obtain an "invertible" cobordism C (cf. [24]) with base $L^{(t)} \times T^{-q} \times I^{p+q-1} \times I$. The arguments of [23, Chapter IX] and [3, §8] may now be applied to C to obtain a pseudo-isotopy representing $\tau^{-1}(x)$.

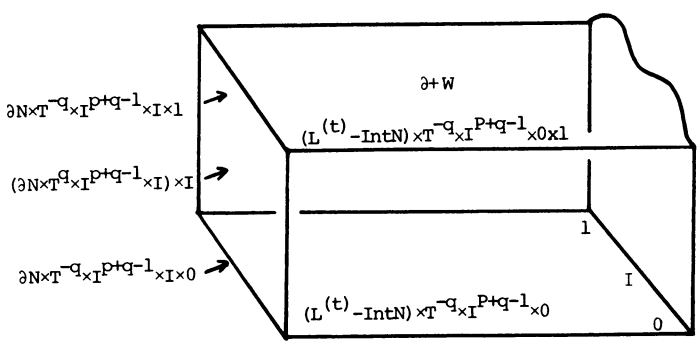

FIGURE 2

Since r_* is the restriction of a pseudo-isotopy to time $t = 1$ which, in turn, corresponds to the h-cobordism $\partial_+ W$ on the top of the above picture, it follows easily that $\rho r_* \tau^{-1}(x) = \alpha\tau_+$ where $\tau_+ = \tau(\partial_+ W, (L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times 0 \times 1)$. But $\tau_+ = \delta(x) + (-1)^{t+p-1}\delta(x)^*$; hence $d^1_{p,q}(x) = \rho r_* \tau^{-1}(x) = x + (-1)^{t+p+q-1}x^*$ since α: $\mathrm{Wh}(G \times Z^{-q}) \to \tilde{K}_{q+1}G$ satisfies $\alpha(x^*) = (-1)^q(\alpha(x))^*$ and $\alpha\delta = \mathrm{id}$. This completes the proof of the first part of 12.1.

The second part of 12.1 (when $p + q = 1$) is established by a similar argument.

It is sufficient to note that in this case, it is still possible to define a function

$$\rho\colon \pi_0\big(\mathrm{TOP}^+_{p-1}(S^{p-1} * L^{(t)}, S^{p-1} * L^{(t-1)})\big) \to \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})$$

$(p + q = 1)$ such that $\rho r_* \tau^{-1}(x) = x + (-1)^t x^*$. Now consider the diagram

$$\begin{tikzcd}
\pi(\mathrm{TOP}^+_p(S^p * L)) \arrow[r] \arrow[d] & \pi_1(\mathrm{TOP}^+_p/\mathrm{PL}(S^p * L)) \arrow[d, "\partial_p"] \\
\pi_0(\mathcal{P}\mathrm{TOP}^+_{p-1}(S^{p-1} * L)) \arrow[r, "\approx"] \arrow[d, "r^*"'] \arrow[dr, "r"', "\approx"] & \pi_0(\mathcal{P}\mathrm{TOP}^+_{p-1}/\mathcal{P}\mathrm{PL}(S^{p-1} * L)) \arrow[d, "\bar{r}^*"] \arrow[dl, "\approx"', "\bar{\tau}"] \\
\pi_0(\mathrm{TOP}^+_{p-1}(S^{p-1} * L)) \arrow[r] \arrow[dr, "\rho"'] & \pi_0(\mathrm{TOP}^+_{p-1}/\mathrm{PL}(S^{p-1} * L)) \arrow[dl, "\bar{\rho}"] \\
& \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})
\end{tikzcd}$$

where we have denoted the pair $(S^r * L^{(t)}, S^r * L^{(t-1)})$ by $S^r * L$ $(r = p, p-1)$ to simplify notation. It is easy to see that ρ induces a function $\bar{\rho}$ making the bottom triangle commute. Since the middle triangle also commutes, we now see that $\bar{\rho}\bar{r}_*\bar{\tau}^{-1}x = x + (-1)^t x^*$.

Since $E^1_{p,q} = \operatorname{Im} \partial_p$ for $p + q = 1$, $\bar{\rho}\bar{r}_*(y) = 0$ for any $y \in E^1_{p,q}$. Hence

$$E^1_{p,q} \subset \left\{x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)}) \mid x = (-1)^{t+1}x^*\right\}$$

as we asserted in §11. Since the argument for the first case identifies the boundaries in $E^1_{p,q}$ as $\{y + (-1)^{t+1}y^* \mid y \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})\}$, the proof of 12.1 is now complete.

We remark that the function ρ described in the case $p + q = 1$ above is not in general a homomorphism. The problem is that a homeomorphism $f\colon S^{p-1} * L^{(t)} \to S^{p-1} * L^{(t)}$, induces a homotopy equivalence $g\colon L^{(t)} - L^{(t-1)} \to L^{(t)} - L^{(t-1)}$ which may not induce the identity on fundamental groups. The most we can expect then is that ρ be a crossed homomorphism. On the image of r_* this problem does not arise and $\rho|\operatorname{Im} r_*$ is a homomorphism.

13. The second differential. In this section we obtain a description of the differential $d^2_{p,q}$ in the spectral sequence 9.1. In order to do this, we recall that the direct sum decomposition

$$\mathrm{Wh}(G \times Z^a) = \mathrm{Wh}\, G \oplus a\tilde{K}_0 G \oplus \sum_{b=2}^{a} \binom{a}{b} K_{-b+1}G \oplus \text{Nil terms}$$

leads to a direct sum decomposition

$$H^c(Z_2; \mathrm{Wh}(G \times Z^a)) = H^c(Z_2; \mathrm{Wh}\, G) \oplus aH^{c+1}(Z_2; \tilde{K}_0 G) \oplus \sum_{b=2}^{a} \binom{a}{b} H^{c+b}(Z_2; K_{-b+1}G).$$

In particular, there is a well-defined projection

$$\alpha\colon H^c(Z_2; \mathrm{Wh}(G \times Z^a)) \to H^{c+a}(Z_2; K_{-a+1}G).$$

There is a similar direct sum decomposition of $H^c(Z_2; \tilde{K}_0(G \times Z^a))$ which leads to

an inclusion

$$\delta\colon H^{c-a}(Z_2;\, K_{-a}G) \to H^c(Z_2;\, \tilde{K}_0(G \times Z^a)).$$

Finally, we recall the Rothenberg exact sequences

$$\to L^h_{c-1}(G \times Z^a, w) \xrightarrow{\beta} H^{c-1}(Z_2;\, \mathrm{Wh}(G \times Z^a)) \to L^s_{c-2}(G \times Z^a, w) \to$$

and

$$\to L^p_c(G \times Z^a, w) \to H^c(Z_2;\, \tilde{K}_0(G \times Z^a)) \xrightarrow{\gamma} L^h_{c-1}(G \times Z^a, w) \to .$$

PROPOSITION 13.1. *Let L be a* PL *stratified polyhedron of dimension n and suppose $5 \leqslant t = n - s + 1 \leqslant n$. Then for $p + q \geqslant 2$, the differential $d^2_{p,q}\colon E^2_{p,q} \to E^2_{p-2,q+1}$ of the spectral sequence* 9.1 *is given by*

$$d^2_{p,q} = \begin{cases} \alpha\beta\gamma\delta\varepsilon & \text{if } p = m, \\ \alpha\beta\gamma\delta & \text{if } 0 < p < m, \end{cases}$$

where $\varepsilon\colon E^2_{m,q} \to H^{m+q+t}(Z_2;\, K_{q+1}\pi_1(L^{(t)} - L^{(t-1)}))$ is the natural surjection. In all other cases $d^2_{p,q} = 0$.

We remark that although $H^*(Z_2;\, C)$ is periodic of period 2 for any coefficient module C, the periodicity of the Wall groups makes the maps β and γ in the Rothenberg sequence periodic of period 4. The appropriate value of c to use in computing $d^2_{p,q}$, then, is $p + t + 1$.

PROOF. Let $\xi \in E^2_{p,q}$. We first give a geometric interpretation of $d^2_{p,q}(\xi)$. Namely, let $x \in \tilde{K}_{q+1}\pi_1(L^{(t)} - L^{(t-1)})$ represent ξ and construct the relative PL h-cobordism W with base $(L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times I$ and torsion $\delta(x) \in \mathrm{Wh}(\pi_1(L^{(t)} - L^{(t-1)}) \times Z^{-q})$ as in Figure 2 of §12. The proof of 12.1 now shows that $\partial_+ W$ is a trivial h-cobordism. Hence there exists a PL homeomorphism h: $(L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times I \to \partial_+ W$ such that h identifies the subset S of the domain with $S \times 1 \subset \partial_+ W$ in the obvious way for

$$S = (L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times (I^{p+q-1} \times 0 \cup \dot{I}^{p+q-1} \times I)$$
$$\cup\, \partial N \times T^{-q} \times I^{p+q-1} \times I$$

and such that h maps $(L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times 1$ PL homeomorphically onto the back face of $\partial_+ W$, $(L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times 1 \times 1$. We extend $h|(L^{(t)} - \mathrm{Int}\, N) \times T^{-q} \times I^{p+q-1} \times 1$ to a PL homeomorphism $g\colon L^{(t)} \times T^{-q} \times I^{p+q-1} \to L^{(t)} \times T^{-q} \times I^{p+q-1}$ by setting $g = \mathrm{id}$ on $N \times T^{-q} \times I^{p+q-1}$.

Since g induces the identity map on fundamental groups, g may be covered by a homeomorphism $\bar{g}$: $L^{(t)} \times R^{-q} \times I^{p+q-1} \to L^{(t)} \times R^{-q} \times I^{p+q-1}$ which is bounded in the R^{-q} direction (i.e. there exists a positive number M such that $\|p_2\bar{g}(x, y, z) - y\| \leqslant M$ for all $(x, y, z) \in L^{(t)} \times R^{-q} \times I^{p+q-1}$ where p_2 is projection on the R^{-q} factor). If $L^{(t)} \times R^{-q} \times I^{p+q-1}$ is identified with $S^{-q-1} * (L^{(t)} \times I^{p+q-1}) - S^{-q-1}$ with a little care, then setting $\bar{g}|S^{-q-1} = \mathrm{id}$ extends $\bar{g}$ to a homeomorphism

$$\bar{g}\colon S^{-q-1} * (L^{(t)} \times I^{p+q-1}) \to S^{-q-1} * (L^{(t)} \times I^{p+q-1}).$$

LEMMA 13.2. *Let X be a polyhedron and suppose $0 \leqslant r \leqslant s$. Then there is an inclusion*

$$S^r * \left[\left(S^{s-r-1} * X\right) \times I^t\right] \subset S^{r-1} * \left[\left(S^{s-r} * X\right) \times I^t\right]$$

whose complement is the union of two cones.

PROOF. By 11.5 there is an inclusion $S^0 * [(S^{s-r-1} * X) \times I^t] \subset (S^{s-r} * X) \times I^t$ with complement the union of two cones. Join this inclusion with S^{r-1}.

It follows from 13.2 that there are inclusions

$$\begin{aligned} S^{-q-1} * (L^{(t)} \times I^{p+q-1}) &\subset S^{-q-2} * \left[\left(S^0 * L^{(t)}\right) \times I^{p+q-1}\right] \subset \cdots \\ &\subset S^{-q-r} * \left[\left(S^{r-2} * L^{(t)}\right) \times I^{p+q-1}\right] \subset \cdots \\ &\subset \left(S^{-q-1} * L^{(t)}\right) \times I^{p+q-1} \end{aligned}$$

whose successive complements are cones. A tedious but straightforward argument shows that $\bar{g}$ may be extended at each stage by coning to produce a homeomorphism G: $(S^{-q-1} * L^{(t)}) \times I^{p+q-1} \to (S^{-q-1} * L^{(t)}) \times I^{p+q-1}$. In fact G is the identity outside $S^{-q-1} * (L^{(t)} \times I^{p+q-1})$ and on $(S^{-q-1} * L^{(t)}) \times \dot{I}^{p+q-1} \cup (S^{-q-1} * L^{(t-1)}) \times I^{p+q-1}$. Since g is PL, G is PL off $S^{-q-1} \times I^{p+q-1}$ and we may regard G as a 0-simplex of $\mathrm{TOP}^+_{p-2}((S^{-q-1} * L^{(t)}) \times I^{p+q-1}, \partial)$, and, hence, as a 0-simplex of $\mathcal{P}\ \mathrm{TOP}^+_{p-2}((S^{-q-1} * L^{(t)}) \times I^{p+q-2}, \partial)$. Clearly $\tau(G)$ represents $d^2_{p,q}(\xi)$.

It is easy to describe $\tau(G)$ in terms of g. In particular, let g' be the lift of g to the infinite cyclic cover $(L^{(t)} - \operatorname{Int} N) \times R^1 \times T^{-q-1} \times I^{p+q-1}$ of $(L^{(t)} - \operatorname{Int} N) \times T^{-q} \times I^{p+q-1}$ corresponding to the first factor of S^1 in T^{-q}. For a sufficiently large integer M the closure of the region between $(L^{(t)} - \operatorname{Int} N) \times 0 \times T^{-q-1} \times I^{p+q-1}$ and $g'((L^{(t)} - \operatorname{Int} N) \times M \times T^{-q-1} \times I^{p+q-1})$ is a relative h-cobordism V such that the component of $\tau(V, (L^{(t)} - \operatorname{Int} N) \times 0 \times T^{-q-1} \times I^{p+q-1})$ in $\tilde{K}_{q+2}\pi_1(L^{(t)} - L^{(t-1)})$ is $\tau(G)$. In fact, we may assume that the torsion of this h-cobordism actually equals $\tau(G)$; i.e. that its components in all other summands of $\mathrm{Wh}(\pi_1(L^{(t)} - L^{(t-1)}) \times Z^{-q-1})$ vanish.

Now let F: $W \to (L^{(t)} - \operatorname{Int} N) \times T^{-q} \times I^{p+q-1} \times I \times I$ be a homotopy equivalence such that $F|S = \mathrm{id}$ where

$$\begin{aligned} S = (L^{(t)} - \operatorname{Int} N) \times T^{-q} \times \left\{I^{p+q-1} \times (\dot{I} \times I \cup I \times 0) \cup \dot{I}^{p+q-1} \times I \times I\right\} \\ \cup\ \partial N \times T^{-q} \times I^{p+q-1} \times I \times I. \end{aligned}$$

By an obvious modification of the argument in [**28**, Lemma 4], we may find an embedding of V in $\partial_+ W$ such that V meets the front and back faces of $\partial_+ W$ in $(L^{(t)} - \operatorname{Int} N) \times T^{-q-1} \times I^{p+q-1}$ where $T^{-q-1} \subset T^{-q}$ is the subtorus consisting of the last $-q-1$ copies of S^1. Furthermore, we may assume that $F(V) = (L^{(t)} - \operatorname{Int} N) \times T^{-q-1} \times I^{p+q-1} \times I \times 1$ and that F is split along V (cf. [**8**]). If we now make F transverse to $(L^{(t)} - \operatorname{Int} N) \times T^{-q-1} \times I^{p+q-1} \times I \times I$, relative to the boundary, and let U be the inverse image of this set, we obtain a normal cobordism U from $(L^{(t)} - \operatorname{Int} N) \times T^{-q-1} \times I^{p+q-1} \times I \times 0$ to V. It is immediate from the definitions in [**22**] that the class that U represents in $L^h_{p+t}(\pi_1(L^{(t)} - L^{(t-1)}) \times Z^{-q-1})$ is just $\gamma\delta(\xi)$. Since $\tau(V, (L^{(t)} - \operatorname{Int} N) \times 0 \times T^{-q-1} \times I^{p+q-1}) = \beta(\gamma\delta(\xi))$, we now see that $d^2_{p,q}(\xi) = \alpha\beta\gamma\delta(\xi)$ as claimed. This completes the proof of 13.1.

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