

## INNER PRODUCT SPACES ASSOCIATED WITH POINCARÉ COMPLEXES

BY

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**ABSTRACT.** We consider the homotopy type classification of a certain kind of Poincaré complex. First we define an inner product space associated with such a Poincaré complex and we investigate the relation between the inner product space and the homotopy type of the Poincaré complex. As an application, some results for manifolds are proved.

**0. Introduction.** Let  $M$  be an oriented closed manifold of dimension  $2n$ . It is well known that the bilinear map

$$H^n(M; Z) \times H^n(M; Z) \rightarrow Z, \quad ((x, y) \rightarrow \langle x \cup y, \mu_M \rangle)$$

makes  $H^n(M; Z)$  an inner product space (in the sense of [2]) if  $M$  has no torsion. This inner product space is closely related with the homotopy type of  $M$ . For example, J. Milnor has proved in [2, p. 103] the following (cohomology version).

**THEOREM.** *Let  $M_i$  ( $i = 1, 2$ ) be simply connected closed manifolds of dimension 4. Their inner product spaces are isomorphic to each other if and only if they have the same oriented homotopy type.*

We are interested in a generalization of Milnor's theorem to the case of torsion. In §1 we shall define an inner product space over  $Q/Z$  associated with an oriented Poincaré complex and in §§2–4 apply this to the homotopy type classification of Poincaré complexes  $K$  which satisfy the conditions

- (1)  $K$  is  $(n - 1)$ -connected and of dimension  $2n + 1$ ,
- (2)  $H_n(K; Z)$  is a finite abelian group  $G$  without 2-torsion ( $n > 3$ ).

We call  $K$  such as above a Poincaré complex of type  $P^n(G)$  and in §2 we shall discuss a special case  $G = \sum^s Z_{p^i}$ . In §3 we shall give a decomposition theorem.

**THEOREM A.** *Let  $K$  be a Poincaré complex of type  $P^n(G)$  and let  $G = \sum G_p^i$  be a direct sum decomposition of  $G$ , where  $G_p^i$  is the  $p^i$ -component of  $G$ . Then  $K$  has the same oriented homotopy type as the connected sum of Poincaré complexes of type  $P^n(G_p^i)$  ( $i = 1, 2, \dots$ , odd primes  $p$ ).*

Finally, in §4 we prove

**THEOREM B.** *Let  $K$  and  $K'$  be  $S$ -reducible Poincaré complexes of type  $P^n(G)$ . Then  $K$  has the same oriented homotopy type as  $K'$  if and only if their product spaces are isomorphic.*

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As an application we also prove

**THEOREM C.** *Let  $M$  be a closed smooth manifold whose underlying Poincaré complex is of type  $P^n(G)$ . Then if  $n \not\equiv 3 \pmod 4$  the oriented homotopy type of  $M$  is determined by the isomorphism classes of the inner product space associated with  $M$ .*

**REMARK.** In the case  $n \equiv 3 \pmod 4$ , Theorem C also holds for  $M$  with trivial Pontrjagin class  $P_{n+1/4}(M)$ .

In particular, for a manifold which is of type  $P^n(G)$  we have

**COROLLARY C-1.** *If  $n \equiv 0 \pmod 2$ , the oriented homotopy type of  $M$  is uniquely determined by  $H_n(M; Z)$ .*

**COROLLARY C-2.** *If  $n \equiv 1 \pmod 4$ ,  $M$  is the connected sum  $\#M_{p^i,k}$  up to oriented homotopy equivalence, where  $M_{p^i,k}$  is a closed manifold whose underlying Poincaré complex is of type  $P^n(Z_{p^i})$ .*

In §§1–4 we always assume that  $n > 3$  and  $p$  is an odd prime number.

**1. Inner product spaces.** Let  $K$  be a simply connected Poincaré complex with the fundamental class  $\mu_K \in H_{2n+1}(K; Z)$ . Let  $\delta: H^*(K; Q/Z) \rightarrow H^{*+1}(K; Z)$  be the connecting homomorphism associated with the exact sequence of coefficients,  $0 \rightarrow Z \rightarrow Q \rightarrow Q/Z \rightarrow 0$ .

We consider the bilinear map

$$\beta_K: H^n(K; Q/Z) \times H^n(K; Q/Z) \rightarrow Q/Z$$

defined by the composite map

$$\begin{aligned} H^n(K; Q/Z) \times H^n(K; Q/Z) &\xrightarrow{1 \times \delta} H^n(K; Q/Z) \times H^{n+1}(K; Z) \\ &\xrightarrow{1 \times D} H^n(K; Q/Z) \times H_n(K; Z) \xrightarrow{\langle \cdot, \cdot \rangle} Q/Z, \end{aligned}$$

where  $D$  is the Poincaré duality map and  $\langle \cdot, \cdot \rangle$  denotes the Kronecker pairing. The following is well known,

**LEMMA 1.1.**  $\beta_K(x, y) = (-1)^{n+1} \beta_K(y, x)$ .

Moreover assuming that  $H^n(K; Z)$  (and hence  $H^{n+1}(K; Z)$ ) are torsion, we have

**PROPOSITION 1.2.**  $\beta_K$  is a completely orthogonal pairing (cf. [2]) and symmetric, for odd  $n$ , and skew symmetric, for even  $n$ .

**PROOF.** Clearly the latter follows from Lemma 1.1. The former follows from the Poincaré duality theorem for torsion groups.

Thus we have the inner product space  $V(K) = \{H^n(K; Q/Z), \beta_K\}$  associated with a Poincaré complex  $K$ . Clearly this inner product space is an oriented homotopy type invariant for oriented Poincaré complexes and we are interested in the problem, “Is the converse true?” The next section investigates a special case.

**2. A special case  $G = \sum^s Z_{p^i}$ .** Let  $M(n, p^i)$  be the Moore space of type  $(n, Z_{p^i})$ . Note the following easy

**LEMMA 2.1.**  $\pi_{n+1}(M(n, p^i)) = 0 = \pi_{n+2}(M(n, p^i))$ .

To study the homotopy type classification of Poincaré complexes which are  $(n - 1)$ -connected and of dimension  $2n + 1$  having  $H_n(K; Z) = \sum^s Z_{p^i}$ , we must investigate the homotopy group  $\pi_{2n}(M_s)$  where  $M_s$  denotes the wedge sum of  $s$  copies of  $M(n, p^i)$ . Clearly a Poincaré complex  $K$  has the same homotopy type as the mapping cone of a map  $f: S^{2n} \rightarrow M_s$ . First we note the following two lemmas which are proved by standard arguments.

LEMMA 2.2. *The smash product  $M(n, p^i) \wedge M(n, p^i)$  is the wedge sum  $M(2n, p^i) \vee M(2n + 1, p^i)$  up to homotopy.*

LEMMA 2.3.  $\pi_{2n}(M_s) = \pi_{2n}(M_{s-1}) \oplus \pi_{2n}(M_1) \oplus \pi_{2n+1}(M_{s-1} \wedge M_1)$ .

Secondly we consider the special case  $s = 2$ . Let  $P$  be the natural map  $M(n, p^i) \rightarrow S^{n+1} = M(n, p^i)/S^n$  and consider the map

$$\hat{P} = \text{id} \vee P: M(n, p^i) \vee M(n, p^i) \rightarrow M(n, p^i) \vee S^{n+1}.$$

Then we have the commutative diagram,

$$\begin{array}{ccccccc} \pi_{2n}(M_2) & = & \pi_{2n}(M_1) \oplus & \pi_{2n}(M_1) \oplus & \pi_{2n+1}(M_1 \times M_1, M_2) \\ \downarrow \hat{P}_* & & \searrow \text{id} & \searrow P_* & \searrow (\text{id} \times P)_* \end{array}$$

$$\pi_{2n}(M_1 \vee S^{n+1}) = \pi_{2n}(M_1) \oplus \pi_{2n}(S^{n+1}) \oplus \pi_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1}),$$

in which  $(\text{id} \times P)_*$  is isomorphic because we have isomorphisms

$$\begin{array}{ccc} \pi_{2n+1}(M_1 \times M_1, M_1 \vee M_1) & \xrightarrow{\cong} & H_{2n+1}(M_1 \times M_1, M_1 \vee M_1) \\ \downarrow (\text{id} \times P)_* & & \downarrow \cong \\ \pi_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1}) & \xrightarrow{\cong} & H_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1}). \end{array}$$

Let  $\iota$  be the inclusion

$$S^n \hookrightarrow M_1 = M(n, p^i) = S^n \cup_{p^i} e^{n+1}.$$

Since it is easily seen that the Whitehead product  $[\iota, \iota_{n+1}]$  generates the group  $\pi_{2n+1}(M_1 \times S^{n+1}, M_1 \vee S^{n+1})$ , we can take the element  $\alpha = \hat{P}_*^{-1}([\iota, \iota_{n+1}])$  as a generator for the group  $\partial\pi_{2n+1}(M_1 \times M_1, M_2)$  using the above diagram. Let  $i_k$  be the inclusion of  $M_1$  into the  $k$ th factor of  $M_s$  and  $i_{k,l}$  be the map  $M_2 = M_1 \vee M_1 \rightarrow M_s$  ( $i_{k,l} = i_k \vee i_l$ ). Then by Lemmas 2.2 and 2.3, we have

PROPOSITION 2.4.  $\pi_{2n}(M_s) = \sum^s \pi_{2n}(M_1) \oplus \sum Z_{p^i}[\alpha_{k,l}]$ , where the second summation runs over  $1 \leq k < l \leq s$  and  $\alpha_{k,l} = i_{k,l*}(\alpha)$ .

Let  $x_0$  be the fixed generator of  $H^n(M(n, p^i); Q/Z) \approx Z_{p^i}$  and let  $p_k$  be the projection  $M_s \rightarrow M_1$  onto the  $k$ th factor of  $M_s$ . We take  $\{x_k = p_k^*(x_0)\}$  as a system of generators for  $H^n(M_s; Q/Z)$  and identify  $H^n(M_s; Q/Z)$  with  $H^n(K_f; Q/Z)$  for the mapping cone  $K_f$  of a map  $f: S^{2n} \rightarrow M_s$ . Now we define a homomorphism

$$D_s: \pi_{2n}(M_s) \rightarrow \{s \times s\text{-matrices over } Q/Z\}$$

by the formula

$$D_s(f) = (f_{k,l}), \quad f_{k,l} = \langle x_k, \mu_f \cap \delta x_l \rangle,$$

where  $\mu_f$  denotes the oriented generator of  $H_{2n+1}(K_f; Z)$ .

LEMMA 2.5. *The matrix  $(f_{k,l})$  is symmetric for odd  $n$  and skew symmetric for even  $n$ . Moreover,  $K_f$  is a Poincaré complex if and only if  $D_s(f) = (f_{k,l})$  is invertible.*

PROOF. The first part follows from Lemma 1.1. The second part is equivalent to Poincaré duality.

Let  $\varphi$  be a map  $M_s \rightarrow M_t$  and define the matrix  $U = (u_{k,l})$  ( $u_{k,l} \in Q/Z$ ) by  $\varphi^*(x_k) = \sum_l u_{k,l} x_l$  ( $k = 1, \dots, t$ ). Then by the definition of  $D_s$  and  $U$ , we have

LEMMA 2.6 (NATURALITY OF  $D_s$ ). *For a map  $\varphi: M_s \rightarrow M_t$ , we have*

$$D_t(\varphi_*(f)) = U \circ D_s(f) \circ {}^tU \quad (f \in \pi_{2n}(M_s)).$$

Now we prove a key lemma for our purpose.

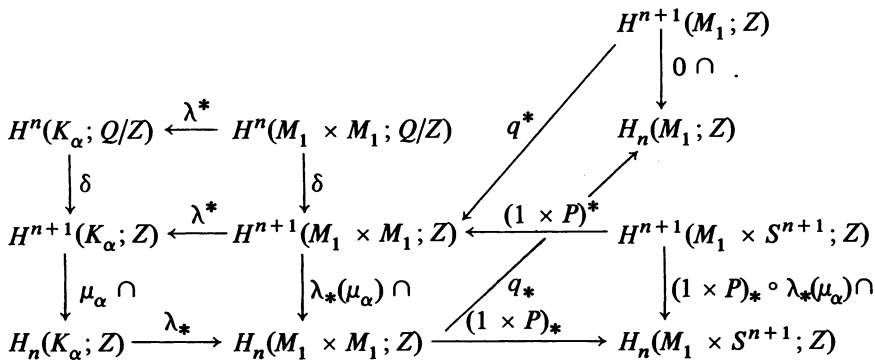
LEMMA 2.7. *Let  $f = \sum f_k + \sum a_{k,l}[\alpha_{k,l}]$  ( $k < l$ ) be the decomposition in Proposition 2.4. Then we have*

$$f_{k,l} = a_{k,l} \quad (k < l) \quad \text{and} \quad f_{k,k} = D_1(f'_k)$$

where  $D_s(f) = (f_{k,l})$  and  $i_{k*}(f'_k) = f_k$ .

PROOF. The equation  $f_{k,k} = D_1(f'_k)$  follows from the naturality for the projection  $p_k: M_s \rightarrow M_1$  ( $p_k(f) = f'_k$ ). For  $f_{k,l} = a_{k,l}$ , it is sufficient to prove  $D_s(\alpha_{k,l}) = \{(k, l)$  and  $(l, k)$  components are  $\pm 1$  and others are 0}, and this is equivalent to  $D_2(\alpha) = (\pm 1 \ 0; \ 0 \ \pm 1)$  by the naturality. And so we consider as follows:

Now there exists a map  $\lambda: K_\alpha \rightarrow M_1 \times M_1$  such that  $\lambda|_{M_2} = \text{id}_{M_2}: M_2 \rightarrow M_2 \subset M_1 \times M_1$  and  $\lambda_*(\mu_\alpha)$  is a generator of  $H_{2n+1}(M_1 \times M_1; Z) \approx Z_p$ . Then we consider the following diagram:



where  $P$  denotes the natural collapsing map  $M_1 \rightarrow M_1/S_n$  and  $q$  denotes the projection  $M_1 \times M_1 \rightarrow M_1$  onto the first factor. Then it is easily verified that  $D_2(\alpha) = (\pm 1 \ 0; \ 0 \ \pm 1)$  using the standard generators of  $H_n(K_\alpha; Z)$  and  $H^{n+1}(K_\alpha; Z)$ . Thus the proof of the lemma is completed.

Here we recall that for two matrices  $A$  and  $B$  over a ring  $R$   $A \equiv B$  if and only if there exists an invertible matrix  $U$  such that  $U \circ A \circ {}^tU = B$ . Then the preceding argument shows

LEMMA 2.8.  *$D_s(f) \equiv D_s(g)$  for  $f, g \in \pi_{2n}(M_s)$  if and only if there exists a homotopy equivalence  $\varphi: M_s \rightarrow M_s$  such that  $\varphi_*(f) \equiv g \text{ mod } D_s^{-1}(0)$ .*

Let  $K_i$  ( $i = 1, 2$ ) be two Poincaré complexes of type  $P^n(\Sigma^s Z_p)$  and we shall prove the main proposition in this section.

**PROPOSITION 2.9.** *If  $K_i$  are both  $S$ -reducible, then  $K_1$  has the same homotopy type as  $K_2$  if and only if  $V(K_1)$  is isomorphic to  $V(K_2)$ .*

Let

$$E^N: \pi_{2n}(M_s) \rightarrow \pi_{2n+N}(E^N M_s) \quad (N \rightarrow \infty)$$

denote the suspension homomorphism. Specially we use the restricted homomorphism

$$E^N: D_s^{-1}(0) \rightarrow \pi_{2n+N}(E^N M_s).$$

Since  $D_s^{-1}(0)$  is decomposed into the direct sum  $\sum i_k \cdot D_1^{-1}(0)$  by Proposition 2.4, it is sufficient for us to consider the case  $s = 1$ . Then from Diagram 1.6 of [4] ( $h_n$  is essentially  $D_1$ ), we get

**LEMMA 2.10.**  *$E^N: D_1^{-1}(0) \rightarrow \pi_{2n+N}(E^N M_1)$  is injective.*

Now we prove Proposition 2.9. Since  $K_1$  has the same homotopy type as the mapping cone  $K_{f_1}$  for a map  $f_1: S^{2n} \rightarrow M_s$ , the inner product space  $V(K_1)$  is isomorphic to  $V(K_{f_1})$ . If  $V(K_{f_1})$  is isomorphic to  $V(K_{f_2})$  for another map  $f_2: S^{2n} \rightarrow M_s$ , there exists a homotopy equivalence  $\varphi: M_s \rightarrow M_s$  such that  $\varphi_*(f_1) \equiv f_2 \pmod{D_s^{-1}(0)}$ , by Lemma 2.8. Since the  $S$ -reducibility of  $K_i$  means  $E^N(f_i) = 0$  ( $N \rightarrow \infty$ ), the proof is completed from Lemma 2.10.

**COROLLARY 2.11.** *Let  $K$  be a Poincaré complex of type  $P^n(\Sigma^s Z_p)$ . Then if  $n$  is odd,  $K$  is the connected sum of Poincaré complexes of type  $P^n(Z_p)$  up to homotopy.*

**COROLLARY 2.12.** *If  $n$  is even, the homotopy type for  $S$ -reducible Poincaré complex of type  $P^n(\Sigma^s Z_p)$  is uniquely determined by  $s$ .*

**PROOF.**  $K$  is the mapping cone  $K_f$  up to homotopy and we may assume that  $D_s(f)$  is a diagonal matrix if  $n$  is odd. Hence  $f$  has the decomposition  $f = \sum f_k$  in Proposition 2.4. This means that  $K_f$  is the connected sum  $\#^s K_{f_k}$  up to homotopy. Since  $K_{f_k}$  is a Poincaré complex of type  $P^n(Z_p)$ , the proof of Corollary 2.11 is completed. Moreover since we know that  $D_s(f)$  is unique up to equivalence if  $n$  is even, Corollary 2.12 follows from Proposition 2.9.

**3. Decomposition (up to homotopy).** First we define a generalized Hopf homomorphism  $H_G: \pi_{2n}(M(n, G)) \rightarrow \text{End } G$  by the formula

$$H_G(f) = \mu_f \cap : H^{n+1}(K_f; Z) = G \rightarrow H_n(K_f; Z) = G$$

where  $\mu_f \in H_{2n+1}(K_f; Z)$  denotes the oriented generator of the mapping cone  $K_f$  for a map  $f: S^{2n} \rightarrow M(n, G)$ . Let  $\rho: G \rightarrow H$  be a homomorphism and  $\hat{\rho}: M(n, G) \rightarrow M(n, H)$  be the map induced by  $\rho$  ( $\rho = \hat{\rho}_*: H_n(M(n, G); Z) = G \rightarrow H_n(M(n, H); Z) = H$ ). If we denote by  ${}'\rho$  the homomorphism

$$\rho^*: H^{n+1}(M(n, H); Z) = H \rightarrow H^{n+1}(M(n, G); Z) = G,$$

then we have

LEMMA 3.1.  $H_H(\hat{\rho} \circ f) = \rho \circ H_G(f) \circ \rho'$  for  $f \in \pi_{2n}(M(n, G))$ .

PROOF. Let  $\hat{\rho}_f: K_f \rightarrow K_{\hat{\rho} \circ f}$  be the natural extension of  $\hat{\rho}$  and consider the diagram

$$\begin{array}{ccc} H^{n+1}(K_f; Z) = G & \xrightarrow{\mu_f \cap} & H_n(K_f; Z) = G \\ \uparrow \rho_f^* & & \downarrow \rho_f \\ H^{n+1}(K_{\hat{\rho} \circ f}; Z) = H & \xrightarrow{\mu_{\hat{\rho} \circ f} \cap} & H_n(K_{\hat{\rho} \circ f}; Z) = H \end{array}$$

Then the proof follows from the commutativity of the diagram.

Let  $G = G_1 \oplus G_2$  be a direct sum decomposition of  $G$ . From Lemma 3.1 we have the diagram

$$\begin{array}{ccccccc} \pi_{2n}(M(n, G)) & = & \pi_{2n}(M(n, G_1)) & \oplus & \pi_{2n}(M(n, G_2)) & \oplus & \pi_{2n+1}(M(n, G_1) \times M(n, G_2), \vee) \\ \downarrow H_G & & \downarrow H_{G_1} & & \downarrow H_{G_2} & & \searrow \wedge \\ \text{End } G & = & \text{End } G_1 & \oplus & \text{End } G_2 & \oplus & \text{Hom}(G_1, G_2) \oplus \text{Hom}(G_2, G_1). \end{array} \tag{3.2}$$

LEMMA 3.3. *The restriction  $H_G|_{\partial\pi_{2n+1}(M(n, G_1) \times M(n, G_2), \vee)}$  is injective.*

PROOF. Let  $G_1 = \sum Z_{p'}[x_{p,i,l}]$  and  $G_2 = \sum Z_{p'}[y_{p,j,k}]$  be a direct sum decomposition of  $G_1$  and  $G_2$  respectively. Since we have decompositions

$$\begin{aligned} \pi_{2n+1}(M(n, G_1) \times M(n, G_2), M(n, G_1) \vee M(n, G_2)) \\ = \pi_{2n+1}(M(n, Z_{p'}) \times M(n, Z_{p'}), M(n, Z_{p'}) \vee M(n, Z_{p'})) \end{aligned}$$

and  $\text{Hom}(G_1, G_2) \oplus \text{Hom}(G_2, G_1) = \sum \text{Hom}(Z_{p'}, Z_{p'}) \oplus \text{Hom}(Z_{p'}, Z_{p'})$ , it is sufficient for us to consider the case  $G_1 = Z_{p'}[x]$  and  $G_2 = Z_{p'}[y]$ . Thus the proof follows by an argument similar to the proof of Lemma 2.7.

REMARK. We note that the proof also shows  $H_G(f_3)|_{G_2} = (H_G(f_3)|_{G_1})$  for  $f = f_1 + f_2 + f_3$  in diagram (3.2).

LEMMA 3.4. *Let  $f = f_1 + f_2 + f_3$  be the decomposition of  $f$  in diagram (3.2). If  $H_{G_1}(f_1)$  is contained in  $\text{Aut } G_1$ , then there exists a homotopy equivalence  $h: M(n, G) \rightarrow M(n, G)$  such that  $h_*(f)$  is contained in  $\pi_{2n}(G_1) \oplus \pi_{2n}(G_2)$  and  $H_{G_1}(f_1) = H_{G_1}(f'_1)$  where  $h_*(f) = f'_1 + f'_2$ .*

PROOF. If we show that there exists an isomorphism  $\lambda: G \rightarrow G$  such that  $\lambda \circ H_G(f) \circ \lambda|_{\text{End } G_1} = 0$  and  $\lambda \circ H_G(f) \circ \lambda|_{\text{Hom}(G_1, G_2)} = H_{G_1}(f_1)$ , then the proof follows from Lemmas 3.1, 3.3 and the above remark.

For homomorphisms  $\lambda_{i,j}: G_i \rightarrow G_j$  ( $i, j = 1, 2$ ), we define a homomorphism  $\lambda: G \rightarrow G$  by  $\lambda(g) = \lambda_{1,1}(g_1) + \lambda_{2,1}(g_2) + \lambda_{1,2}(g_1) + \lambda_{2,2}(g_2)$  for  $g = g_1 + g_2$ . Then from easy computation, we can get

$$\begin{aligned} \lambda \circ h_G(f) \circ \lambda|_{\text{End } G_1} &= \lambda_{1,1} \circ H_{G_1}(f_1) \circ \lambda_{1,1} + \lambda_{2,1} \circ H_G(f_3)|_{G_1} \circ \lambda_{1,1} \\ &\quad + \lambda_{1,1} \circ H_G(f_3)|_{G_2} \circ \lambda_{2,1} + \lambda_{2,1} \circ H_{G_2}(f_2) \circ \lambda_{2,1} \end{aligned}$$

and

$$\lambda \circ H_G(f) \circ \lambda | \text{Hom}(G_1, G_2) = \lambda_{1,2} \circ H_{G_2}(f_1) \circ \lambda_{1,1} + \lambda_{2,2} \circ H_G(f_3) | G_1 \circ \lambda_{1,1} \\ + \lambda_{1,2} \circ H_G(f_3) | G_2 \circ \lambda_{2,1} + \lambda_{2,2} \circ H_{G_2}(f_2) \circ \lambda_{2,1}.$$

Hence if we take  $\lambda_{1,1} = \text{id}_{G_1}$ ,  $\lambda_{2,2} = \text{id}_{G_2}$ ,  $\lambda_{2,1} = 0$  and

$$\lambda_{1,2} = -H_G(f_3) | G_1 \circ H_{G_1}(f_1)^{-1},$$

then  $\lambda$  satisfies our conditions.

**LEMMA 3.5.** *Suppose  $G = G_1 \oplus G_2$  where  $G_1 = \sum Z_{p^m}$  and  $p^{m-1}G_2$  has a trivial  $p$ -primary component. If  $H_G(f)$  is an automorphism for  $f = f_1 + f_2 + f_3$ , then  $H_{G_1}(f_1)$  is also an automorphism.*

**PROOF.** Consider the following exact ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker p^{m-1} & \longrightarrow & G_1 & \xrightarrow{p^{m-1}} & G_1 & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow & 0 \\ & & \downarrow \approx & & \downarrow i_1 & & \downarrow i_1 & & \downarrow \approx & & \\ 0 & \longrightarrow & \ker p^{m-1} & \longrightarrow & G & \xrightarrow{p^{m-1}} & G & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow H_G(f) & & \downarrow H_G(f) & & \downarrow & & \\ 0 & \longrightarrow & \ker p^{m-1} & \longrightarrow & G & \xrightarrow{p^{m-1}} & G & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow & 0 \\ & & \downarrow \approx & & \downarrow p_1 & & \downarrow p_1 & & \downarrow \approx & & \\ 0 & \longrightarrow & \ker p^{m-1} & \longrightarrow & G_1 & \longrightarrow & G_1 & \longrightarrow & \sum Z_{p^{m-1}} & \longrightarrow & 0 \end{array}$$

where  $i_1$  is the inclusion  $G_1 \hookrightarrow G$  and  $p_1$  is the projection  $G \rightarrow G_1$  and  $p^{m-1}$  denotes the map multiplied by  $p^{m-1}$ . The right and left vertical maps are isomorphisms because  $H_G(f)$  is also. Hence it follows that  $H_{G_1}(f_1) = p_1 \circ H_G(f) \circ i_1$  is surjective and so an isomorphism. Now Theorem A follows from Lemmas 3.4 and 3.5.

**4. Proofs of Theorems B and C.**

**LEMMA 4.1.** *Let  $K_i$  ( $i = 1, 2$ ) be two Poincaré complexes of dimension  $m$ . Then the connected sum  $K_1 \# K_2$  is  $S$ -reducible if and only if  $K_1$  and  $K_2$  are both  $S$ -reducible.*

**PROOF.** This is clear because  $K_1 \# K_2$  has a CW-decomposition  $(K_1^{(m-1)} \vee K_2^{(m-1)}) \cup_f e^m$ , where  $f = f_1 + f_2$  for  $K_i = K_i^{(m-1)} \cup_{f_i} e^m$ .

Now we prove Theorem B. Let  $K$  and  $K'$  be two Poincaré complexes of type  $P^n(G)$ . Trivially  $V(K) \cong V(K')$  if  $K$  and  $K'$  have the same oriented homotopy type. Assume that  $V(K) \cong V(K')$ . By Theorem A, we may assume that  $K$  and  $K'$  are the connected sums  $\# K_{p,i}$  and  $\# K'_{p,i}$  (corresponding to the  $Z_{p^i}$ -component of  $G$ ) respectively. Let  $h_i: K \rightarrow K_{p,i}$  and  $h'_i: K' \rightarrow K'_{p,i}$  be the projection onto each factor. Since we may consider  $\beta_K = \sum \beta_{K_i} | \sum Z_{p^i}$  and  $\beta_{K'} = \sum \beta_{K'_i} | \sum Z_{p^i}$  by Lemmas 3.4 and 3.5,  $h_i$  ( $h'_i$ ) induces an isomorphism  $V(K) | \sum Z_{p^i} \cong V(K_{p,i})$  ( $V(K') | \sum Z_{p^i} \cong V(K'_{p,i})$ ) and so  $V(K_{p,i})$  is isomorphic to  $V(K'_{p,i})$ . Thus  $K_{p,i}$  has the same oriented

homotopy type as  $K'_{p,i}$  by Proposition 2.9 and Lemma 4.1, and hence  $K$  has the same oriented homotopy type as  $K'$ . Thus the proof is completed.

From Theorem A and Corollaries 2.11 and 2.12, we have

**COROLLARY B-1.** *Let  $K$  be a Poincaré complex of type  $P^n(G)$ . Then  $K$  has the same homotopy type as the connected sum of Poincaré complexes of type  $P^n(Z_{p^i})$  for various  $p$  and  $i$  if  $n$  is odd.*

**COROLLARY B-2.** *If  $n$  is even, the oriented homotopy type for  $S$ -reducible Poincaré complexes of type  $P^n(G)$  is uniquely determined by  $n$  and  $G$ .*

Let  $M$  be a closed smooth manifold of dimension  $2n + 1$  whose underlying Poincaré complex is of type  $P^n(G)$  and let  $\nu_M: M \rightarrow BO$  be the stable normal bundle for  $M$ . Since  $M$  has the same homotopy type as the mapping cone  $K_f$  for a map  $f: S^{2n} \rightarrow M(n, G)$ , we may replace  $M$  with  $K_f$ .

**LEMMA 4.2.** *If  $n \not\equiv 3 \pmod 4$  then  $\nu_M: K_f \rightarrow BO$  is trivial.*

**PROOF.** First consider the restriction  $\nu_M|_{M_1}$ ,

$$K_f \supset M(n, G) \supset M_1 = S^n \cup_{p^i} e^{n+1} \rightarrow BO.$$

From Puppe’s sequence and  $\pi_n(BO) = 0, Z$  or  $Z_2$ , we obtain that  $\nu_M|_{M_1}$  is trivial, and so  $\nu_M|M(n, G)$  is trivial. Hence  $\nu_M$  is decomposed as follows

$$\nu_M: K_f \xrightarrow{P} S^{2n+1} \xrightarrow{\nu} BO$$

where  $P$  denotes the natural map  $K_f \rightarrow K_f/M(n, G) = S^{2n+1}$ . Since  $P$  induces the map  $T_P: T(\nu_M) \rightarrow T(\nu)$  of degree 1,  $T(\nu)$  is reducible and  $\nu$  is fiber homotopically trivial, that is,  $J(\nu) = 0$  in this case. Hence  $\nu$  is trivial by Adams’ Theorem. This completes the proof.

Thus since Lemma 4.2 shows that  $M$  is  $S$ -reducible, Theorem C and Corollary C-1 follow from Theorem B and Corollary B-2 respectively. Corollary C-2 is also obtained from Lemmas 4.1 and 4.2 and Corollary B-1.

For the case  $n \equiv 3 \pmod 4$ , the situation is more complicated and we could not obtain simple theorems. As an example we list

**PROPOSITION 4.3.** *Suppose  $(\text{order } G, \text{order } J(S^{n+1})) = 1$ . For any element  $x$  of  $H^{n+1}(M(n, G); Z) \approx G$ , there exists a closed smooth manifold  $M^{2n+1}$  with  $P_{n+1/4}(M) = x$  which is of type  $P^n(G)$  as a Poincaré complex and is a  $\pi$ -manifold up to homotopy.*

**PROOF.** The proof follows from the case  $G = Z_{p^i}$ , using the connected sum operation. On the other hand, the condition  $(p, \text{order } J(S^{n+1})) = 1$  means that  $J(K_f) = 0$  for any  $f \in \pi_{2n}(M(n, Z_{p^i}))$ . Since we can take  $f$  such that  $D_1(f) = 1$  and  $E^N(f) = 0$ , the proof is completed.



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