TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 260, Number 2, August 1980

A $\mathbf{Z} \times \mathbf{Z}$ STRUCTURALLY STABLE ACTION

BY

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ABSTRACT. We consider in the product of spheres $S^m \times S^n$ the $\mathbf{Z} \times \mathbf{Z}$-action generated by two simple Morse-Smale diffeomorphisms; if they have some kind of general position, the action is shown to be stable. An application is made to foliations.

1. Introduction. Our aim here is to present a very simple example of a structurally stable $\mathbf{Z} \times \mathbf{Z}$-action. The stability of differentiable Lie group actions on manifolds has been extensively studied when the group is $\mathbf{R}$ or $\mathbf{Z}$. For compact groups, we have the following theorem **[5]**: if G is a compact Lie group, every C^1 G-action is (parametrically) structurally stable. We also mention the treatment in **[7]** of Ω-stability of actions. For noncompact Lie groups other than $\mathbf{Z}$ or $\mathbf{R}$ it is interesting to look for structurally stable examples. This has been done in some cases. In **[2]** we have examples of stable $\mathbf{R}^2$-actions on spheres S^n, of class C^2, and in **[1]** and **[6]** the result that for an open subset $\mathcal{Q} \subseteq \mathrm{Diff}^\infty(M)$, with the C^∞ topology, the pair (f^n, f^m) generates a structurally stable action of $\mathbf{Z} \times \mathbf{Z}$ on M, where $f \in \mathcal{Q}$ and $n, m \in \mathbf{Z}$. This last statement follows from the fact that diffeomorphisms belonging to $\mathcal{Q}$ have discrete centralizers. In our examples we use such diffeomorphisms but in a different manner. Here, the action is not generated by the powers of a single diffeomorphism.

First of all we make some definitions. Let G be a Lie group and M a C^∞ manifold. An *action* ϕ: $G \times M \to M$ is a C^∞ map satisfying

(i) $\phi(1, x) = x$, $\forall x \in M$,

(ii) $\phi(g_1, (g_2, x)) = \phi(g_1 \cdot g_2, x)$, $\forall g_1, g_2 \in G$ and $x \in M$.

Let $\phi(g)$: $M \to M$ be the C^∞ diffeomorphism given by $\phi(g)(x) = \phi(g, x)$.

The actions ϕ and Ψ are (parametrically) *conjugate* if there exists a homeomorphism h: $M \to M$ satisfying $h \cdot \phi(g) = \Psi(g) \cdot h$, $\forall g \in G$.

If M is compact we introduce the C^r metric ($r \geqslant 1$) in the set of actions: let d be the uniform C^r metric in $\mathrm{Diff}^\infty(M)$ and $K \subseteq G$ be a compact generator; then $\bar{d}(\phi, \Psi) = \sup_{g \in K} d(\phi(g), \Psi(g))$.

Now we can say that an action ϕ is (parametrically) *C^r-structurally stable* if there exists a neighborhood $N(\phi)$ of ϕ in the C^r topology such that every $\Psi \in N(\phi)$ is conjugate to ϕ.

We restrict ourselves to the case $G = \mathbf{Z} \times \mathbf{Z}$. If ϕ: $(\mathbf{Z} \times \mathbf{Z}) \times M \to M$ is an

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Received by the editors September 11, 1979 and, in revised form, October 30, 1979.

AMS (MOS) subject classifications (1970). Primary 34C35.

Key words and phrases. Morse-Smale diffeomorphisms, normally hyperbolicity, resonance between eigenvalues, linearizations.

action, the diffeomorphisms $\phi((1, 0)) = F_\phi$ and $\phi((0, 1)) = H_\phi$ are its generators. The definitions given above reduce to (i) ϕ and Ψ are C^r-close if F_ϕ is C^r-close to F_Ψ and H_ϕ is C^r-close to H_Ψ, (ii) ϕ is C^r-structurally stable if for every action Ψ close to ϕ there exists a homeomorphism h: $M \to M$ such that $hF_\phi = F_\Psi h$ and $hH_\phi = H_\Psi h$. Clearly, $F_\phi H_\phi = H_\phi F_\phi$. If the generators of ϕ are powers of the same diffeomorphism we call it an *elementary action*.

THEOREM. *There exist nonelementary C^3-structurally stable $\mathbf{Z} \times \mathbf{Z}$-actions on $S^n \times S^m$.*

These actions have a fairly simple nature. We choose $f \in \text{Diff}^\infty(S^n)$ and $g \in \text{Diff}^\infty(S^m)$ with discrete centralizers (and another condition we will give later on), and take ϕ defined by $F_\phi = (f, \text{Id})$ and $H_\phi = (\text{Id}, g)$. The idea is to show that the product structure of the action persists, in some sense, under perturbations.

I thank C. Camacho for several conversations and suggestions.

2. Proof of the Theorem. First we give an outline of the proof. Let $f \in \text{Diff}^\infty(S^n)$ and $g \in \text{Diff}^\infty(S^m)$ be Morse-Smale diffeomorphisms with two periodic points (a source and a sink); call A and B the sink and the source of f and C and D the sink and the source of g, respectively. Let ϕ be the $\mathbf{Z} \times \mathbf{Z}$-action generated by $F_\phi = (f, \text{Id})$ and $H_\phi = (\text{Id}, g)$. The set $\Omega_\phi = S^n \times \{C\} \cup S^n \times \{D\} \cup \{A\} \times S^m \cup \{B\} \times S^m$ is invariant under ϕ (the nonwandering set of ϕ). The stable and unstable manifolds of the points in Ω_ϕ form a grid on $S^n \times S^m$ in the following sense. Each $\{P\} \times S^m$ is the H_ϕ-stable manifold of (P, C) and each $S^n \times \{Q\}$ is the F_ϕ-stable manifold of (A, Q). These manifolds coincide with the H_ϕ and F_ϕ-unstable manifolds of (P, D) and (B, Q), respectively. An action Ψ close to ϕ has generators F_Ψ and H_Ψ close to F_ϕ and H_ϕ. We begin by showing that there exists a Ψ-invariant set Ω_Ψ close to Ω_ϕ (and homeomorphic to it). Furthermore, the restrictions of both actions to these sets are conjugate. Let V_Ψ^1, V_Ψ^2, W_Ψ^1 and W_Ψ^2 be the subsets of Ω_Ψ corresponding to $S^n \times \{C\}$, $S^n \times \{D\}$, $\{A\} \times S^m$ and $\{B\} \times S^m$. We show then that the H_Ψ-stable manifolds of the points in V_Ψ^1 are exactly the same as the H_Ψ-unstable manifolds of the points in V_Ψ^2 (the corresponding fact holds for F_Ψ too) if f and g satisfy some conditions which relate them. Therefore we have, as before, a grid for Ψ, and from that we can prove that ϕ and Ψ are conjugate.

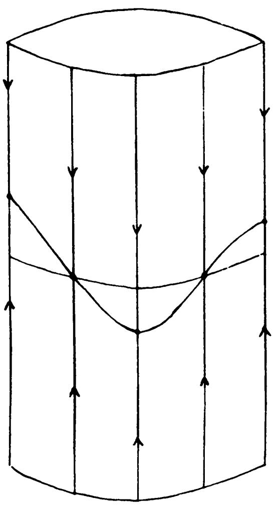
Now we come to the proof.

Step 1. We choose $f \in \text{Diff}^\infty(S^n)$ and $g \in \text{Diff}^\infty(S^m)$, both Morse-Smale diffeomorphisms with two periodic points (a sink and a source) and having C^0-discrete centralizers (we may assume persistence of this property under perturbations in the C^3 topology, see [**1**]). We observe that for existence of such f and g it is necessary to assume at each fixed point that all eigenvalues are distinct and have no resonance relations.

Let us call the condition above C1 (C^0-discrete centralizers and persistence). In Step 4 conditions C2, C3, C4 and C5 are introduced. Let A and B be the sink and the source of f, C and D the sink and the source of g and ϕ the $\mathbf{Z} \times \mathbf{Z}$-action generated by $F_\phi = (f, \text{Id})$ and $H_\phi = (\text{Id}, g)$.

CLAIM. If Ψ is C^3-close to ϕ there exist C^∞ submanifolds of $S^n \times S^m$—V^1_Ψ, V^2_Ψ, W^1_Ψ and W^2_Ψ—close to $S^n \times \{C\}$, $S^n \times \{D\}$, $\{A\} \times S^m$ and $\{B\} \times S^m$ and invariant under Ψ. Furthermore, $\phi|_{S^n \times \{C\}}$ is conjugate to $\Psi|_{V^1_\Psi}$, $\phi|_{S^n \times \{D\}}$ is conjugate to $\Psi|_{V^2_\Psi}$ and so on.

In fact, $S^n \times \{C\}$ is a normally hyperbolic attracting submanifold for $H_\phi = (\mathrm{Id}, g)$ and $H_\phi|_{S^n \times \{C\}} = \mathrm{Id}$. Therefore H_Ψ has an attracting C^∞ submanifold V^1_Ψ, normally hyperbolic and close to $S^n \times \{C\}$ (see [3]). It is easy to see that V^1_Ψ is also invariant by F_Ψ. We "project" $F_\Psi|_{V^1_\Psi}$ along the H_Ψ-stable manifolds of the points in V^1_Ψ in order to get $\tilde{F}$: $S^n \times \{C\} \to S^n \times \{C\}$. The diffeomorphism $\tilde{F}$ is differentiably conjugate to $F_\Psi|_{V^1_\Psi}$ and C^3-close to $F_\phi|_{S^n \times \{C\}}$. Then $\tilde{F}$ is a northpole-southpole diffeomorphism conjugate to $F_\phi|_{S^n \times \{C\}}$, and from that we deduce that $F_\Psi|_{V^1_\Psi}$ and $F_\phi|_{S^n \times \{C\}}$ are conjugate.



Furthermore, "projecting" $H_\Psi|_{V^1_\Psi}$ along the H_Ψ-stable manifolds of the points in V^1_Ψ in order to get $\tilde{H}$: $S^n \times \{C\} \to S^n \times \{C\}$, we have that $\tilde{H}$ and $\tilde{F}$ are C^0-close to the identity. It turns out that $\tilde{H} = \mathrm{Id}$ and from this we get $H_\Psi|_{V^1_\Psi} = \mathrm{Id}$. The same argument holds for the other submanifolds. We note that the four points $V^i_\Psi \cap W^j_\Psi$ are fixed points for the action Ψ.

Step 2. Now we prove the following lemma.

LEMMA 1. *Let* L: $\mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n \times \mathbf{R}^m$ *be a linear map*, $L(\nu, 0) = (\nu, 0)$ *for* $\nu \in \mathbf{R}^n$ *and*

$$L|_{\{0\} \times \mathbf{R}^m} = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{bmatrix},$$

with $0 < |\mu_j| < 1$ *distinct and* $\mu_j \in \mathbf{R}$, $1 \leqslant j \leqslant m$. *Consider a* C^∞ *map* $\xi\colon V \times D \to \mathbf{R}^m$, *where* $V \subseteq \mathbf{R}^n$ *is compact*, $D \subseteq \mathbf{R}^m$ *is an open neighborhood of* $0 \in \mathbf{R}^n$ *and* $\xi_1(T, O) = \cdots = \xi_m(T, O) = 0$, *and the m-parameter family of submanifolds*

$$S_x = \left\{(T, \xi_1(T, x), \ldots, \xi_m(T, x)),\ T \in V \text{ and } x = (x_1, \ldots, x_j, \ldots, x_m) \in D\right\}.$$

If this family is L-invariant, that is, if $S_{L(x)} = LS_x$, *and if* $\mu_j \neq \prod_{i=1}^m \mu_i^{n_i}$, $\forall n_i > 0$, $\sum_{i=1}^m n_i \geqslant 2$, *then there exist* C^∞ *maps* $A_j(T)$ *such that* $\xi_j(T, x) = x_j A_j(T)$.

PROOF. Invariance means

$$\xi_j\left(T, \mu_1^k x_1, \ldots, \mu_j^k x_j, \ldots, \mu_m^k x_m\right) = \mu_j^k \xi_j(T, x_1, \ldots, x_j, \ldots, x_m)$$

or $\xi_j(T, Ux) = \mu_j \xi_j(T, x)$.

We have $\xi_j(T, x) = \sum_{|\sigma|<k} A_\sigma^j(T) x^\sigma + R(T, x)$ where $R(T, x)/|x|^k \to 0$ as $|x| \to 0$, and

$$x^\sigma = x_1^{\sigma_1} \cdots x_m^{\sigma_m}, \qquad |\sigma| = \sigma_1 + \cdots + \sigma_m.$$

We then have

$$\begin{aligned}\xi_j(T, U^l x) &= \sum_{|\sigma|<k} A_\sigma^j(T)(Ux)^\sigma + R(T, U^l x)\\ &= \sum_{|\sigma|<k} A_\sigma^j(T)\mu^\sigma x^\sigma + R(T, U^l x) = \mu_j^l \xi_j(T, x)\\ &= \mu_j^l \sum_{|\sigma|<k} A_\sigma^j(T) x^\sigma + \mu_j^l R(T, x).\end{aligned}$$

Given $\varepsilon > 0$, for $|x|$ small enough we may write $|R(T, x)| < \varepsilon |x|^k$. Since U is a contraction, we have $|R(T, U^l x)| < \varepsilon |U^l x|^k \leqslant \varepsilon |U|^{lk} |x|^k$, $\forall l > N_0$.

Then $|\mu_j^l|\,|R(T, x)| < \varepsilon |U|^{lk} |x|^k$. From this, it follows that $|R(T, x)| < \varepsilon(|U|^{lk}/|\mu_j|^l)|x|^k$.

We choose k in order to have $|\mu_i^k| < |\mu_j|$, $\forall i \neq j$; as $l \to \infty$ we get $R(T, x) = 0$. Then,

$$\xi_j(T, x) = \sum_{|\sigma|<k} A_\sigma^j(T) x_\sigma.$$

Now, $\mu^\sigma A_\sigma^j(T) = \mu_j^l A_\sigma^j(T)$. From the absence of resonances and $\mu_i \neq \mu_j$ for $i \neq j$ we obtain $A_\sigma^j(T) = 0$ whenever $|\sigma| \geqslant 2$ or $\sigma \neq j$ if $|\sigma| = 1$.

REMARKS. (1) Lemma 1 is essentially a theorem of [**4**, p. 167].

(2) The following is implied by the lemma: if $(T, \xi_1(T, a), \ldots, \xi_m(T, a)) \in S_a$ and $(T, \xi_1(T, b), \ldots, \xi_m(T, b)) \in S_b$ then

$$\xi_i(T, a)/a_i = \xi_i(T, b)/b_i \quad \text{or} \quad \xi_i(T, b) = (b_i/a_i)\xi_i(T, a).$$

This means that knowledge of the submanifold for some value of the parameter gives a knowledge of all the submanifolds of the family.

(3) It is easy to extend the lemma to the case where L has complex eigenvalues.

We consider

(i)

$$L = \begin{bmatrix} \mu_1 & & & & & \\ & \ddots & & & 0 & \\ & & \mu_m & & & \\ & & & A_1 & & \\ & 0 & & & \ddots & \\ & & & & & A_s \end{bmatrix}$$

where $0 < |\mu_j| < 1$, $\mu_j \in \mathbf{R}$ $(1 \leqslant j \leqslant m)$,

$$A_j = e^{\lambda_i}\begin{bmatrix} \cos\theta_i & -\text{sen}\,\theta_i \\ \text{sen}\,\theta_i & \cos\theta_i \end{bmatrix}$$

and $\lambda_i < 0$ $(1 \leqslant i \leqslant s)$,

(ii) a differentiable map $\phi\colon V \times D \to R^{m+2s}$, $\phi = (\xi_1, \ldots, \xi_m, \eta_1, \zeta_1, \ldots, \eta_s, \zeta_s)$, (where $V \subseteq \mathbf{R}^n$ is an open set and $D \subseteq \mathbf{R}^{m+2s}$ is a neighborhood of $0 \in \mathbf{R}^{m+2s}$), and the $(m + 2s)$-parameter family of submanifolds

$$\begin{aligned} S_{(x,z)} = \{(T, \xi_1(T, x, z), \ldots, \xi_m(T, x, z), \eta_1(T, x, z), \\ \zeta_1(T, x, z), \ldots, \eta_s(T, x, z), \zeta_s(T, x, z)), \\ T \in V, (x, z) = (x_1, \ldots, x_m, z_1, w_1, \ldots, z_s, w_s) \in D\}. \end{aligned}$$

If this family is L-invariant, there exist differentiable maps $A_j(T)$, $B_i(T)$, $C_i(T)$ such that

$$\xi_j(T, x, z) = x_j A_j(T), \qquad \eta_i(T, x, z) = z_i B_i(T) - w_i C_i(T),$$
$$\zeta_i(T, x, z) = z_i C_i(T) + w_i B_i(T).$$

Step 3. The diffeomorphisms $f \in \text{Diff}^\infty(S^n)$ and $g \in \text{Diff}^\infty(S^m)$ chosen at Step 1 satisfy the condition of no resonances between the eigenvalues of sinks (and sources) (see **[1]**). This condition implies that we can linearize them near those critical points, and that the linearizations vary "continuously" with the diffeomorphisms.

We then have the following lemma (notation as before).

LEMMA 2. *Let $D \subseteq S^n \times S^m$ and $D_1 \times D_2 \subseteq \mathbf{R}^n \times \mathbf{R}^m$ be disks such that $(A, C) \in D$ and $(0, 0) \in (D_1 \times D_2)$. Then there exist a neighborhood N of the action ϕ (in the C^3-topology) and a C^0 map $R\colon N \to \text{Emb}^2(D, D_1 \times D_2)$ such that, for $\Psi \in N$, we have*

(i) *$R(\Psi)H_\Psi R(\Psi)^{-1}(\nu_1, \nu_2) = (\nu_1, L\nu_2)$ where $L \in GL(\mathbf{R}^m)$,*

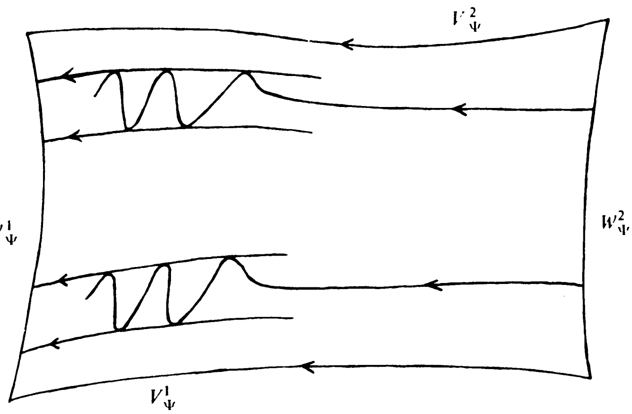
(ii) *$R(\Psi)(V_\Psi^1 \cap D) \subseteq D_1$, $R(\Psi)(W_\Psi^1 \cap D) \subseteq D_2$, $R(\Psi)(W_{H_\Psi}^s(a) \cap D) = \{R(\Psi)(a)\} \times D_2$ and $R(\Psi)(W_{F_\Psi}^s(b) \cap D) = D_1 \times \{R(\Psi)(b)\}$ for $a \in V^1$ and $b \in W^1$ (here $W_{H_\Psi}^s(a)$ denotes the H_Ψ-stable manifold of a).*

PROOF. The embedding $R(\phi)$ is defined as follows. Take $\overline{R}(\phi)$, from a neighborhood of (A, C) (in W_ϕ^1) to D_2, that linearizes H_ϕ, and $\tilde{R}(\phi)$ an embedding of a neighborhood of (A, C) (in V_ϕ^1) into V_ϕ^1. Given $z \in S^n \times S^m$ belonging to a neighborhood of (A, C), there exist $a \in V_\phi^1$ and $b \in W_\phi^1$ such that $\{z\} = W_{H_\phi}^s(A) \cap W_{F_\phi}^s(b)$. Then we define $R(\phi)(z)$ as $(\tilde{R}(\phi)(a), \overline{R}(\phi(b)))$. The Lemma from [1, p. 145] implies that this construction may be done continuously for actions close to ϕ.

REMARKS. (1) We may suppose that $R(\Psi)H_\Psi R(\Psi)^{-1}|D_2$ is in Jordan normal form.

(2) Lemma 2 is true for the other fixed points.

We relate Lemma 1 and Lemma 2 as follows. By Step 1 every action Ψ close to ϕ has invariant submanifolds V_Ψ^i, W_Ψ^j, $1 \leqslant i, j \leqslant 2$, such that the points $V_\Psi^1 \cap W_\Psi^j$ are fixed for Ψ. The family of F_Ψ-unstable manifolds of points in W_Ψ^2 close to $V_\Psi^2 \cap W_\Psi^2$ hits points close to $V_\Psi^2 \cap W_\Psi^1$. From there it is taken by H_Ψ to a neighborhood of $V_\Psi^1 \cap W_\Psi^1$ so that it coincides with the family of F_Ψ-unstable manifolds that comes from points in W_Ψ^2 close to $V_\Psi^1 \cap W_\Psi^2$ (we observe that H_Ψ preserves respectively the stable and unstable manifolds of the points in W_Ψ^2 and W_Ψ^1). After linearizing H_Ψ near $V_\Psi^i \cap W_\Psi^1$, $i = 1, 2$, we may apply Lemma 1.



Step 4. What conditions must $g \in \mathrm{Diff}^\infty(S^m)$ satisfy in order that the F_Ψ-unstable manifolds of the points of W_Ψ^2 coincide with the F_Ψ-stable manifolds of the points of W_Ψ^1 for Ψ close to ϕ? We will answer this question now.

We know that some power H_Ψ^p takes a fundamental domain $\overline{\Sigma}_\Psi$ for H_Ψ, close to $V_\Psi^2 \cap W_\Psi^1$, to a fundamental domain Σ_Ψ for H_Ψ, close to $V_\Psi^1 \cap W_\Psi^1$. This map has the following properties.

(i) $H_\Psi^p(W_\Psi^1 \cap \overline{\Sigma}_\Psi) = W_\Psi^1 \cap \Sigma_\Psi$.

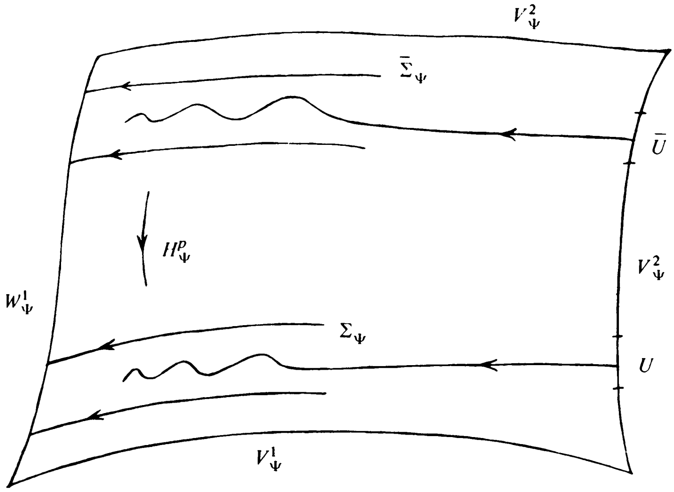
(ii) $H_\Phi^p = g^p$.

(iii) If $(\cdot) \in W_\Psi^1$ then $H_\Psi^p(W^s(\cdot)) = W^s(H_\Psi^p(\cdot))$, where $W_\Psi^s(\cdot)$ is the F_Ψ-stable manifold of the point $(\cdot)$.

(iv) If $(\cdot) \in W_\Psi^2$, then $H_\Psi^p(W^u(\cdot)) = W_\Psi^u(H_\Psi^p(\cdot))$, where $W_\Psi^u(\cdot)$ is the F_Ψ-unstable manifold of the point $(\cdot)$.

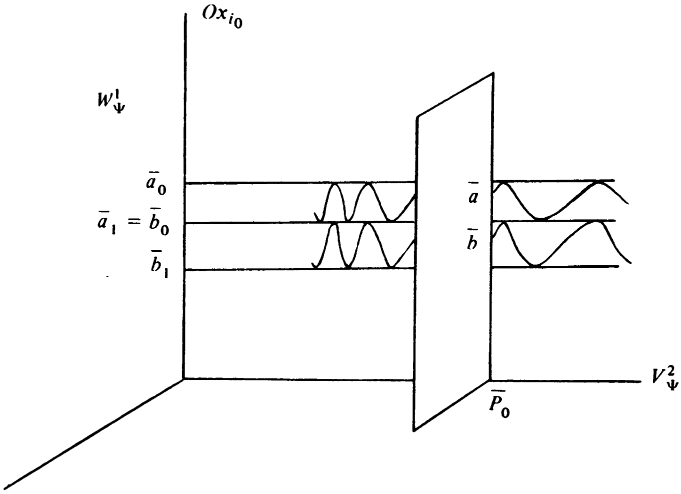
We point out that there exist open subsets $U, \overline{U}$ contained in W_Ψ^2, close to

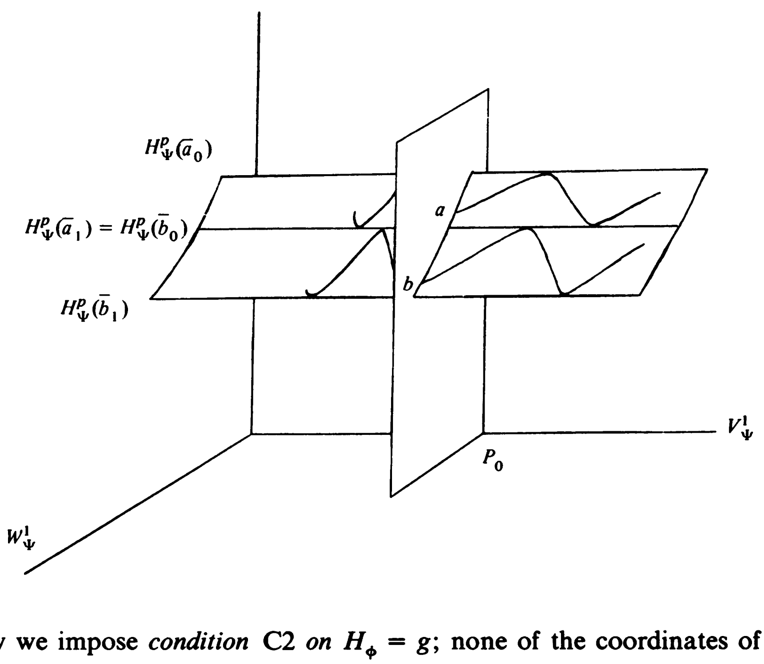
$V_\Psi^2 \cap W_\Psi^2$ and $V_\Psi^1 \cap W_\Psi^2$ such that if $(\cdot) \in U$ ($(\cdot) \in \bar{U}$) then $W_\Psi^u(\cdot) \cap D \subseteq \Sigma_\Psi$ ($W_\Psi^u(\cdot) \subseteq \bar{D} \cap \bar{\Sigma}_\Psi$). ($D$ and $\bar{D}$ are disks around $V_\Psi^1 \cap W_\Psi^1$ and $V_\Psi^2 \cap W_\Psi^1$ where Lemma 2 holds.) After linearizing H_Ψ in D and $\bar{D}$ we get two local actions defined in a neighborhood of $(0, 0) \in \mathbf{R}^n \times \mathbf{R}^m$ (we will maintain the notation after linearizations have been carried out). The $\mathbf{Z} \times \mathbf{Z}$ local action on D is generated by F_Ψ and H_Ψ, $F_\Psi|_{W_\Psi^1} = F_\Psi|_{\{0\}\times\mathbf{R}^m} = \mathrm{Id}$. The F_Ψ-stable manifold of $(0, P) \in W_\Psi^1$ is $\mathbf{R}^n \times \{P\}$ and $H_\Psi|_{\{0\}\times\mathbf{R}^m}$ is diagonalizable in the canonical basis (the semisimple case is analogous). For the $\mathbf{Z} \times \mathbf{Z}$ local action defined on $\bar{D}$ we have corresponding statements.



The family $\mathfrak{F}$ ($\bar{\mathfrak{F}}$) in D ($\bar{D}$) of the F_Ψ-unstable manifolds of the points in W_Ψ^2 close to $V_\Psi^1 \cap W_\Psi^2$ ($V_\Psi^2 \cap W_\Psi^2$) is a differentiable m-parameter family. We take the parameter as the single point where each unstable manifold crosses $\{P_0\} \times \mathbf{R}^n$ for some P_0 ($\{\bar{P}_0\} \times \mathbf{R}^n$). This is an $H_\Psi(H_\Psi^{-1})$ invariant family in the sense of Lemma 1, so it can be described as $\mathfrak{F} = \{S_x\}_{x\in\mathbf{R}^m}$, ($\bar{\mathfrak{F}} = \{\bar{S}_x\}_{x\in\mathbf{R}^m}$) with $S_x = \{(T, \xi_1(T, x), \ldots, \xi_m(T, x))\}$, where T belongs to a fundamental domain of F_Ψ in V_Ψ^1 containing P_0 ($\bar{S}_x = \{(T, \bar{\xi}_1(T, x), \ldots, \bar{\xi}_m(T, x))\}$, T belonging to a fundamental domain of F_Ψ in V_Ψ^2 containing $\bar{P}_0$).

Consider $\bar{\mathfrak{F}}$ when its parameter belongs to a coordinate axis, say the i_0th coordinate axis Ox_{i_0}. By Lemma 1 we see that the submanifold $V_\Psi^2 \times Ox_{i_0} - \{0\} \times Ox_{i_0}$ is satured by $\bar{\mathfrak{F}}$. Fix $\bar{a} \in Ox_{i_0}$, ($\bar{a} \neq 0$). There exists an interval $[\bar{a}_1, \bar{a}_0]$ in Ox_{i_0} such that $\bar{S}_{(0,\ldots,\bar{a},\ldots,0)} \subseteq \mathbf{R}^n \times \{(0, \ldots, \bar{x}, \ldots, 0), \bar{x} \in [\bar{a}_1, \bar{a}_0]\}$ which is minimal for this property. By Lemma 1, if $\bar{b} \in Ox_{i_0}$ is such that $\bar{a}/\bar{b} = \bar{a}_0/\bar{a}_1$, the interval $[\bar{b}_1, \bar{b}_0]$, which is minimal for the property $\bar{S}_{(0,\ldots,\bar{b},\ldots,0)} \subseteq \mathbf{R}^n \times \{(0, \ldots, \bar{x}, \ldots, 0), \bar{x} \in [\bar{b}_1, \bar{b}_0]\}$, satisfies $\bar{b}_0 = \bar{a}_1$ and $\bar{a}/\bar{b} = \bar{b}_0/\bar{b}_1$. It turns out that $\bar{b}_0^2 = \bar{a}_0\bar{b}_1$. We note that if there is no coincidence between the F_Ψ-unstable manifolds of points in W_Ψ^2 and the F_Ψ-stable ones of points in W_Ψ^1, then necessarily $a_0 \neq a_1$ and $b_0 \neq b_1$. Let us fix $\bar{a}_0$ (we do not change it for Ψ close to ϕ); clearly the points $\bar{b}_0 = \bar{a}_1$ and $\bar{b}_1$ depend on Ψ: $\bar{b}_0 = \bar{b}_0(\Psi)$ and $\bar{b}_1 = \bar{b}_1(\Psi)$.





Now we impose *condition* C2 *on* $H_\Phi = g$; none of the coordinates of $H^p_\Phi(\bar a_0) \in \{0\} \times \mathbf{R}^m$ are zero. This assumption still holds for $H^p_\Psi(\bar a_0)$, for Ψ close to ϕ. Applying H^p_Ψ to $\mathbf{R}^n \times \{(0, \dots, \bar x, \dots, 0), \bar b_1 \leqslant \bar x \leqslant \bar a_0\}$ we get a cylinder over the curve whose endpoints are $H^p_\Psi(\bar a_0)$ and $H^p_\Psi(\bar b_1)$ (this curve contains $H^p_\Psi(\bar a_1)$). For some $(a, b) \in Ox_{i_0} \times Ox_{i_0}$ we have

$$S_{(0, \dots, a, \dots, 0)} = H^p_\Psi\big(\bar S_{(0, \dots, \bar a, \dots, 0)}\big) \subseteq \mathbf{R}^n \times \big\{z \in \overparen{H^p_\Psi(\bar a_1), H^p_\Psi(\bar a_0)}\big\}$$

and

$$S_{(0, \dots, b, \dots, 0)} = H^p_\Psi\big(\bar S_{(0, \dots, \bar b, \dots, 0)}\big) \subseteq \mathbf{R}^n \times \big\{z \in \overparen{H^p_\Psi(\bar b_1), H^p(\bar b_0)}\big\};$$

clearly $H_{\Psi}^{\varrho}(\bar{a}_1) = H_{\Psi}^{\varrho}(\bar{b}_0)$. Lemma 1 again implies $(H_{\Psi}^{\varrho}(\bar{a}_0))_j/(H_{\Psi}^{\varrho}(\bar{a}_1))_j = a_j/b_j$ and $(H_{\Psi}^{\varrho}(\bar{b}_0))_j/(H_{\Psi}^{\varrho}(\bar{b}_1))_j = a_j/b_j, j = 1, \ldots, m$ ($(\cdot)_j$ stands for the jth coordinate of $(\cdot) \in \mathbf{R}^m$) and from that

$$(H_{\Psi}^{\varrho}(\bar{a}_0))_j/\left(H_{\Psi}^{\varrho}(\bar{b}_0)\right)_j = \left(H_{\Psi}^{\varrho}(\bar{b}_0)\right)_j/\left(H_{\Psi}^{\varrho}(\bar{b}_1)\right)_j$$

or

$$\left(H_{\Psi}^{\varrho}(\bar{b}_0)\right)_j^2 = (H_{\Psi}^{\varrho}(\bar{a}_0))_j \cdot \left(H_{\Psi}^{\varrho}(\bar{b}_1)\right)_j, \qquad j = 1, \ldots, m.$$

We have obtained that for some interval I around $\bar{a}_0$ the map $H_{\Psi}^{\varrho}: I \to \mathbf{R}^m$ has the following property. If there is no coincidence between the F_{Ψ}-stable manifolds of points in W_{Ψ}^1 and the F_{Ψ}-unstable ones of points in W_{Ψ}^2, then there exists a point $\bar{b}_0$ (depending on Ψ) close to $\bar{a}_0$ but $\bar{b}_0 \neq \bar{a}_0$ such that

$$\left(H_{\Psi}^{\varrho}(\bar{b}_0)\right)_j^2 = (H_{\Psi}^{\varrho}(\bar{a}_0))_j \cdot \left(H_{\Psi}^{\varrho}(\bar{b}_0^2/\bar{a}_0)\right)_j, \qquad j = 1, \ldots, m. \tag{$*$}$$

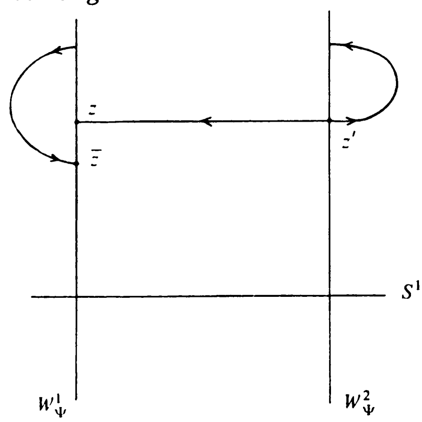
For the action ϕ, $(*)$ holds when $\bar{b}_0 = \bar{a}_0$. Now we give a condition on ϕ to ensure that $(*)$ holds only if $\bar{a}_0 = \bar{b}_0$ and not only for ϕ but even for Ψ close enough to ϕ. Consider $\alpha_{\Psi}: I \to \mathbf{R}^m$.

$$\alpha_{\Psi}(x) = \big((H_{\Psi}^{\varrho}(x))_1^2 - (H_{\Psi}^{\varrho}(\bar{a}_0))_1 \cdot (H_{\Psi}^{\varrho}(x^2/\bar{a}_0))_1, \ldots,$$
$$(H_{\Psi}^{\varrho}(x))_m^2 - (H_{\Psi}^{\varrho}(\bar{a}_0))_m \cdot (H_{\Psi}^{\varrho}(x^2/\bar{a}_0))_m\big).$$

Clearly $\alpha_{\Psi}(\bar{a}_0) = \alpha'_{\Psi}(\bar{a}_0) = 0$. Now we perturb slightly the generator $H_{\phi} = g$ (same notation as before) in the C^3-topology to get *condition* C3, $\alpha''_{\phi}(\bar{a}_0) \neq 0$. For the new action, we have the same properties as already obtained, but $\alpha_{\phi}(x) = 0$ for $x \in I$ close to $\bar{a}_0$ if and only if $x = \bar{a}_0$.

If Ψ is C^2-close to ϕ, we still may say that $\alpha''_{\Psi}(\bar{a}_0) \neq 0$. This implies that the equality $\alpha_{\Psi}(x) = 0$ for x close to $\bar{a}_0$ holds only if $x = \bar{a}_0$. Therefore we have $\bar{b}_0 = \bar{a}_0$ in $(*)$, that is, the submanifold $S_{(0, \ldots, a, \ldots, 0)} \subseteq W_{F_{\Psi}}^s(H_{\Psi}^{\varrho}(\bar{a}_0))$. Now Lemma 1 guarantees that the F_{Ψ}-stable manifolds of the points in W_{Ψ}^2 coincide with the F_{Ψ}-stable ones of the points in W_{Ψ}^1.

Proceeding as before we change $F_{\phi} = f$ in order to get coincidence between the H_{Ψ}-stable manifolds of the points in V_{Ψ}^1 and the H_{Ψ}-unstable ones of the points in V_{Ψ}^2, for Ψ close enough to ϕ in the C^3-topology. We impose on f conditions C4 and C5 analogous to C2 and C3 for g.



REMARK. Connectedness of fundamental domains of $f|_{V^1_\phi}$ and $g||_{W^1_\phi}$ is needed in the proof above. In the case $n = 1$ or $m = 1$ the proof ends as follows (assume $n = 1$). Given $z \in W^1_\Psi$, one of the connected components of $W^2_{F_\Psi}(z) - \{z\}$ coincides with one of the components of $W^u_{F_\Psi}(z') - \{z'\}$ for some $z' \in W^2_\Psi$. The other component of $W^u_{F_\Psi}(z') - \{z'\}$ is equal to one of the components of $W^s_{F_\Psi}(\bar{z}) - \{\bar{z}\}$ for some $\bar{z} \in W^1_\Psi$. The map $z \to \bar{z}$ is a C^∞ diffeomorphism close to Id (if Ψ is close to ϕ) and belongs to the centralizer of H_Ψ: $W^1_\Psi \to W^1_\Psi$. Therefore $\bar{z} = z$ by the claim of Step 1.

Step 5. Now we construct the conjugacy between actions ϕ (described before) and Ψ C^3-close to it. We know that there exist homeomorphisms h_V: $V^1_\phi \to V^1_\phi$ and h_W: $W^1_\phi \to W^1_\phi$ such that $h_V \cdot (F_\phi)|_{V^1_\phi} = (F_\Psi)|_{V^1_\Psi} \cdot h_V$ and $h_W \cdot (H_\phi)|_{W^1_\Psi} \cdot h_W$ (this was proved in Step 1). Given $z \in S^n \times S^m$, we have $\{z\} = W^s_{H_\phi}(z_1) \cap W^s_{F_\phi}(z_2)$ for $z_1 \in V^1_\phi$ and $z^2 \in W^1_\phi$. Define h: $S^n \times S^m \to S^n \times S^m$ by $h(z) = W^s_{H_\Psi}(h_V(z_1)) \cap W^s_{F_\Psi}(h_W(z_2))$; it is easy to see that h is a homeomorphism and $hF_\phi = F_\Psi h$ and $hH_\phi = H_\Psi h$.

REMARK. ϕ *is not* locally structurally stable at its fixed points. The reason is the following. If Ψ is close to ϕ but is defined only on a neighborhood V of a fixed point of ϕ we can not guarantee that Ψ is the identity on some Ψ-invariant submanifold in V.

3. Stable foliations. Let $T^2 = S^1 \times S^1$ and ϕ: $\Pi_1(T^2) \to \text{Diff}^\infty(S^n \times S^m)$ be the action of the Theorem. The *suspension of* ϕ is the foliation defined as follows. Take in $\mathbf{R}^2 \times S^n \times S^m$ the trivial foliation $\mathcal{F}$ by leaves $\mathbf{R}^2 \times (x, y)$ and the equivalence relation

$$(u, v, x, y) \sim (u', v', x', y') \Leftrightarrow \begin{cases} (u - u', v - v') \in \mathbf{Z} \times \mathbf{Z}, \\ f^{(u-u')}(x) = x', \\ g^{(v-v')}(y) = y', \end{cases}$$

where $f = \phi(1, 0)$ and $g = \phi(0, 1)$.

Let Π: $\mathbf{R}^2 \times S^n \times S^m \to \mathbf{R}^2 \times S^n \times S^m/\sim$ be the quotient map; define $\mathcal{F}(\phi)$ as $\Pi_*(\mathcal{F})$. It is not difficult to show that $\mathcal{F}(\phi)$ is structurally stable (as a foliation) if and only if ϕ is structurally stable (as an action). It follows from our theorem that $\mathcal{F}(\phi)$ is C^3 structurally stable.

This kind of construction was done in **[6]** for representations ρ: $\Pi_1(N) \to \text{Diff}^\infty(M)$ (M and N are differentiable manifolds and $\Pi_1(N)$ is finitely generated) satisfying $\rho(g_1) = f$, $\rho(g_i) = \text{Id}$, $i = 2, \ldots, k$, where $\{g_1, \ldots, g_k\}$ are generators of $\Pi_1(N)$ and $f \in \text{Diff}^\infty(M)$ is a structurally stable diffeomorphism with discrete centralizer. Our example shows that this is not the only possible way of getting stable representations. See **[6]** for further information.

Now we should like to pose some questions. (1) Is it possible to prove the theorem using general Morse-Smale diffeomorphisms? (2) Does our construction extend to $\mathbf{Z} \times \mathbf{Z} \times \cdots \times \mathbf{Z}$-stable actions? (3) Does every manifold have a nonelementary $\mathbf{Z} \times \mathbf{Z}$ structurally stable action?

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