

## PERIODIC ORBITS OF CONTINUOUS MAPPINGS OF THE CIRCLE

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**ABSTRACT.** Let  $f$  be a continuous map of the circle into itself and let  $P(f)$  denote the set of positive integers  $n$  such that  $f$  has a periodic point of period  $n$ . It is shown that if  $1 \in P(f)$  and  $n \in P(f)$  for some odd positive integer  $n$  then for every integer  $m > n$ ,  $m \in P(f)$ . Furthermore, if  $P(f)$  is finite then there are integers  $m$  and  $n$  (with  $m > 1$  and  $n > 0$ ) such that  $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$ .

**1. Introduction.** Let  $I$  denote a closed bounded interval on the real line and let  $C^0(I, I)$  denote the space of continuous maps of  $I$  into itself. For  $f \in C^0(I, I)$ , let  $P(f)$  denote the set of all positive integers  $n$  such that  $f$  has a periodic point of period  $n$  (see §2 for definition). One may ask the following question. If  $k \in P(f)$ , what other integers must be elements of  $P(f)$ ?

This question is answered by a theorem of Šarkovskii. Consider the following ordering of the positive integers:

$$1, 2, 4, 8, \dots, \dots, 7 \cdot 8, 5 \cdot 8, 3 \cdot 8, \dots, 7 \cdot 4, 5 \cdot 4, 3 \cdot 4, \dots, \\ 7 \cdot 2, 5 \cdot 2, 3 \cdot 2, \dots, 7, 5, 3.$$

Šarkovskii's theorem states that if  $n \in P(f)$  and  $m$  is to the left of  $n$  in the above ordering then  $m \in P(f)$  (see [3] or [4]). Furthermore, if  $m$  is to the right of  $n$  in the above ordering, then there is a map  $f \in C^0(I, I)$  with  $n \in P(f)$  and  $m \notin P(f)$ .

In this paper we obtain some similar results in  $C^0(S^1, S^1)$  (the space of continuous maps of the circle into itself). Since for any positive integer  $n$ , there is a map  $f \in C^0(S^1, S^1)$  with  $P(f) = \{n\}$  (where  $P(f)$  is defined as above) one cannot obtain an ordering as in Šarkovskii's theorem. However, we do obtain the following result.

**THEOREM A.** *Let  $f \in C^0(S^1, S^1)$ . Suppose  $1 \in P(f)$  and  $n \in P(f)$  for some odd integer  $n > 1$ . Then for every integer  $m > n$ ,  $m \in P(f)$ .*

We remark that if the hypothesis of Theorem A is satisfied, it is possible that for every integer  $k$  with  $1 < k < n$ ,  $k \notin P(f)$  (see Proposition 12 in §5). Using Theorem A we obtain the following result which characterizes  $P(f)$  when  $P(f)$  is finite.

**THEOREM B.** *Let  $f \in C^0(S^1, S^1)$  and suppose that  $P(f)$  is finite. Then there are integers  $m$  and  $n$  (with  $m \geq 1$  and  $n \geq 0$ ) such that  $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$ .*

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It is known that for any integers  $m$  and  $n$  with  $m \geq 1$  and  $n \geq 0$  there is a differentiable map  $f$  of the circle with  $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$  (see [1]). Also, in [1], Theorem B is proved for a certain subset of  $C^0(S^1, S^1)$ .

Finally, note that Šarkovskii's theorem implies that for  $f \in C^0(I, I)$  if  $3 \in P(f)$  then  $P(f) = N$ , where  $N$  denotes the set of positive integers. This result is also proved in [2]. It also follows from Šarkovskii's theorem that if  $S \subset N$  with the property that for any  $f \in C^0(I, I)$ ,  $S \subset P(f) \Rightarrow P(f) = N$ , then  $3 \in S$ . In this paper we obtain the following analogous result for the circle. The first statement, of course, follows immediately from Theorem A.

**THEOREM C.** *Let  $f \in C^0(S^1, S^1)$ . If  $\{1, 2, 3\} \subset P(f)$  then  $P(f) = N$ . Conversely, if  $S \subset N$  with the property that for any  $f \in C^0(S^1, S^1)$ ,  $S \subset P(f) \Rightarrow P(f) = N$ , then  $\{1, 2, 3\} \subset S$ .*

**2. Preliminary definitions and results.** Let  $f \in C^0(S^1, S^1)$ . For any  $n \in N$ , we define  $f^n$  inductively by  $f^1 = f$  and  $f^n = f \circ f^{n-1}$ . Let  $f^0$  denote the identity map of  $S^1$ .

Let  $x \in S^1$ .  $x$  is said to be a fixed point of  $f$  if  $f(x) = x$ .  $x$  is said to be a periodic point of  $f$  if  $f^n(x) = x$  for some  $n \in N$ . In this case, the smallest element of  $\{n \in N: f^n(x) = x\}$  is called the period of  $x$ .

We define the orbit of  $x$  to be  $\{f^n(x): n = 0, 1, 2, \dots\}$ . If  $x$  is a periodic point of period  $n$ , we say the orbit of  $x$  is a periodic orbit of period  $n$ . In this case the orbit of  $x$  contains exactly  $n$  points, each of which is a periodic point of period  $n$ .

We will use the following notation throughout this paper.

**NOTATION.** Let  $a \in S^1$  and  $b \in S^1$  with  $a \neq b$ . We write  $[a, b]$ ,  $(a, b)$ ,  $(a, b]$ , or  $[a, b)$  to denote the closed, open, or half-open interval from  $a$  counterclockwise to  $b$ .

We will also use the following definition.

**DEFINITION.** Let  $I$  and  $J$  be proper closed intervals on  $S^1$  and let  $f \in C^0(S^1, S^1)$ . We say  $I$   $f$ -covers  $J$  if for some closed interval  $K \subset I$ ,  $f(K) = J$ .

We conclude this section by proving four lemmas, which use this definition.

**LEMMA 1.** *Let  $I = [a, b]$  be a proper closed interval on  $S^1$  and let  $f \in C^0(S^1, S^1)$ . Suppose  $f(a) = c$  and  $f(b) = d$  and  $c \neq d$ . Then either  $I$   $f$ -covers  $[c, d]$  or  $I$   $f$ -covers  $[d, c]$ .*

**PROOF.** Let  $A = \{x \in I: f(x) = c\}$ . There is a point  $v \in A$  such that  $(v, b] \cap A = \emptyset$ . Let  $B = \{x \in [v, b]: f(x) = d\}$ . There is a point  $w \in B$  such that  $[v, w) \cap B = \emptyset$ .

We have  $f(v) = c$ ,  $f(w) = d$ , and if  $x \in (v, w)$  then  $f(x) \notin \{c, d\}$ . Hence, if  $K = [v, w]$  then  $f(K) = [c, d]$  or  $f(K) = [d, c]$ . Q.E.D.

**LEMMA 2.** *Let  $f \in C^0(S^1, S^1)$ . Let  $I$  and  $J$  be proper closed intervals on  $S^1$  such that  $I$   $f$ -covers  $J$ . Suppose  $L$  is a closed interval with  $L \subset J$ . Then  $I$   $f$ -covers  $L$ .*

**PROOF.** By hypothesis, there is a closed interval  $K \subset I$  with  $f(K) = J$ . Let  $L = [c, d]$ . There are points  $a \in K$  and  $b \in K$  with  $f(a) = c$  and  $f(b) = d$ . Let  $K_1$

be the closed interval with endpoints  $a$  and  $b$  such that  $K_1 \subset K$ . By Lemma 1, either  $K_1$   $f$ -covers  $[c, d]$  or  $K_1$   $f$ -covers  $[d, c]$ . Since  $K_1 \subset K$  and  $f(K) = J$ ,  $K_1$  cannot  $f$ -cover  $[d, c]$ . Hence  $K_1$   $f$ -covers  $[c, d]$ . Since  $K_1 \subset K \subset I$ ,  $I$   $f$ -covers  $[c, d]$ . Q.E.D.

LEMMA 3. Let  $f \in C^0(S^1, S^1)$ . Suppose  $N$  is a proper closed interval on  $S^1$  such that  $N$   $f$ -covers  $N$ . Then  $f$  has a fixed point in  $N$ .

PROOF. By hypothesis, for some closed interval  $K \subset N$ ,  $f(K) = N$ . There are points  $v \in K$  and  $w \in K$  such that  $f(v)$  and  $f(w)$  are the two endpoints of  $N$ . Let  $L$  be the closed interval with endpoints  $v$  and  $w$  such that  $L \subset K$ . By continuity,  $f$  has a fixed point in  $L$ . Q.E.D.

LEMMA 4. Suppose  $f \in C^0(S^1, S^1)$  and suppose  $M_1, M_2, \dots, M_n$  are proper closed intervals on  $S^1$  such that  $M_k$   $f$ -covers  $M_{k+1}$  for  $k = 1, \dots, n - 1$ , and  $M_n$   $f$ -covers  $M_1$ . Then there is a fixed point  $z$  of  $f^n$  such that  $z \in M_1, f(z) \in M_2, \dots, f^{n-1}(z) \in M_n$ .

PROOF. Since  $M_n$   $f$ -covers  $M_1$ , there is an interval  $J_n \subset M_n$  such that  $f(J_n) = M_1$ . Similarly, there are intervals  $J_1, \dots, J_{n-1}$  such that for each  $k = 1, \dots, n - 1$ ,  $J_k \subset M_k$  and  $f(J_k) = J_{k+1}$ . It follows that  $f^n(J_1) = M_1$ . By the proof of Lemma 3,  $f^n$  has a fixed point  $z \in J_1$ . Clearly  $z \in M_1, f(z) \in M_2, \dots, f^{n-1}(z) \in M_n$ . Q.E.D.

**3. Proof of Theorem A.**

DEFINITION. Let  $f \in C^0(S^1, S^1)$  and let  $P = \{p_1, \dots, p_n\}$  be a periodic orbit of  $f$  of period  $n$ . We say  $P$  is labeled in order if for  $k = 1, \dots, n - 1, P \cap (p_k, p_{k+1}) = \emptyset$ , and  $P \cap (p_n, p_1) = \emptyset$ . In this case we define the intervals determined by  $P$  to be the  $n$  closed intervals  $I_1 = [p_1, p_2], I_2 = [p_2, p_3], \dots, I_{n-1} = [p_{n-1}, p_n], I_n = [p_n, p_1]$ .

LEMMA 5. Let  $f \in C^0(S^1, S^1)$ . Let  $P = \{p_1, \dots, p_n\}$  be a periodic orbit of  $f$  of odd period  $n \geq 3$ . Suppose  $P$  is labeled in order and let  $I_1, \dots, I_n$  be the intervals determined by  $P$ . Suppose that for some  $j$  and  $k$  with  $j \in \{1, \dots, n\}$  and  $k \in \{1, \dots, n\}, I_i$  does not  $f$ -cover  $I_j$  for all  $i \in \{1, \dots, n\}$  with  $i \neq j$ , and  $I_i$  does not  $f$ -cover  $I_k$  for all  $i \in \{1, \dots, n\}$  with  $i \neq k$ . Then  $j = k$ .

PROOF. Suppose  $j \neq k$ . Let  $v$  be a point in the interior of  $I_k$ , and let  $w$  be a point in the interior of  $I_j$ . Then  $v \neq w$ . Let  $A = P \cap (v, w)$  and  $B = P \cap (w, v)$ . Then  $A \neq \emptyset, B \neq \emptyset, A \cap B = \emptyset$  and  $A \cup B = P$ .

If  $f(x) \in A$  for some  $x \in A$  then by hypothesis and Lemma 1,  $f(A) \subset A$ . This is impossible, since  $P$  is a periodic orbit. Thus,  $f(x) \notin A$  for all  $x \in A$ . Hence,  $f(A) \subset B$ . Similarly it follows that  $f(B) \subset A$ .

Since  $f$  maps  $P$  onto  $P$  it follows that  $f(A) = B$  and  $f(B) = A$ . Thus,  $A$  and  $B$  have the same number of elements. Since  $A \cup B = P$  and  $A \cap B = \emptyset$ , this contradicts the fact that  $P$  has an odd number of elements. Q.E.D.

LEMMA 6. Suppose  $f \in C^0(S^1, S^1)$  and  $f$  has a periodic orbit  $P = \{p_1, \dots, p_n\}$  of odd period  $n \geq 3$ . Suppose  $P$  is labeled in order, and let  $I_1, \dots, I_n$  be the intervals determined by  $P$ . Suppose also that  $f$  has a fixed point  $e$ . Then  $f$  has a fixed point  $z$  with the property that if  $I_k$  is the interval determined by  $P$  with  $z \in I_k$ , there is some  $j \in \{1, \dots, n\}$  with  $j \neq k$  such that  $I_j$   $f$ -covers  $I_k$ .

PROOF. Without loss of generality we may assume that  $e \in I_n$ . We may also assume that for each  $j \in \{1, \dots, n-1\}$ ,  $I_j$  does not  $f$ -cover  $I_n$  (or else the conclusion of the lemma holds with  $z = e$ ).

Let  $m$  be the smallest positive integer such that if  $f(p_m) = p_r$ , then  $r < m$ . Note that  $2 \leq m \leq n$ , so  $1 \leq m-1 \leq n-1$ . In particular  $m-1 \neq n$ .

Since  $I_{m-1}$  does not  $f$ -cover  $I_n$ , it follows from Lemmas 1 and 2 that  $I_{m-1}$   $f$ -covers  $I_{m-1}$ . By Lemma 3,  $f$  has a fixed point  $z \in I_{m-1}$ . Since  $I_j$  does not  $f$ -cover  $I_n$  for all  $j \in \{1, \dots, n-1\}$ , it follows from Lemma 5 that for some  $j \in \{1, \dots, n\}$  with  $j \neq m-1$ ,  $I_j$   $f$ -covers  $I_{m-1}$ . Q.E.D.

LEMMA 7. Let  $f \in C^0(S^1, S^1)$  and let  $P$  be a periodic orbit of  $f$  of period  $m$  where  $m \geq 3$ . Suppose that  $\{M_1, \dots, M_k\}$  is a collection of closed intervals with  $2 \leq k \leq m$  such that

- (1) for each  $j \in \{1, \dots, k\}$ , there are no elements of  $P$  in the interior of  $M_j$ ,
- (2) if  $i \neq j$ ,  $M_i$  and  $M_j$  have disjoint interiors,
- (3) if  $j \in \{2, \dots, k\}$  the endpoints of  $M_j$  are in  $P$ ,
- (4) if  $b$  is an endpoint of  $M_1$ , either  $b \in P$  or  $b$  is a fixed point of  $f$ ,
- (5) for each  $j \in \{1, \dots, k-1\}$ ,  $M_j$   $f$ -covers  $M_{j+1}$ ,
- (6)  $M_1$   $f$ -covers  $M_1$  and  $M_k$   $f$ -covers  $M_1$ .

Then for any positive integer  $n > k$ ,  $f$  has a periodic point of period  $n$ .

PROOF. Let  $n > k$ . We may assume  $n \neq m$  since, by hypothesis,  $f$  has a periodic point of period  $m$ .

Let  $L_1 = M_1, L_2 = M_1, \dots, L_{n-k} = M_1, L_{n-k+1} = M_1, L_{n-k+2} = M_2, L_{n-k+3} = M_3, \dots, L_{n-k+k} = L_n = M_k$ . By Lemma 4 (applied to  $L_1, L_2, \dots, L_n$ ), there is a fixed point  $z$  of  $f^n$  such that  $z \in L_1, f(z) \in L_2, \dots, f^{n-1}(z) \in L_n$ . Since  $z \in M_1$  and  $f^{n-k+1}(z) \in M_2$ , it follows from (2) and (3) of the hypothesis that  $z$  is not a fixed point of  $f$ .

We claim that  $z \notin P$ . To prove this, first suppose that  $n \geq k+2$ . Then  $L_1 = L_2 = L_3 = M_1$ . Hence  $z \in M_1, f(z) \in M_1$  and  $f^2(z) \in M_1$ . Since  $P$  is a periodic orbit of period  $m \geq 3$ , it follows from (1) of the hypothesis that  $z \notin P$ . Now, suppose that  $n < k+2$ . Then  $n < m+2$ . Since  $n \neq m$  and  $m \geq 3$ ,  $n$  is not a multiple of  $m$ . Since  $f^n(z) = z$ , it follows that  $z \notin P$ .

Since  $z$  is not a fixed point of  $f$  and  $z \notin P$  it follows from (4) of the hypothesis that  $z$  is in the interior of  $M_1$ . Also, since  $f^n(z) = z \notin P$ , for any positive integer  $r < n$ ,  $f^r(z) \notin P$  (and  $f^r(z)$  is not a fixed point of  $f$ ). Thus, by (3) and (4), for any positive integer  $r < n$ ,  $f^r(z)$  is not an endpoint of any of the intervals  $M_1, \dots, M_k$ . It follows from this, and (2), and the fact that  $z \in M_1, f(z) \in M_1, f^2(z) \in M_1, \dots, f^{n-k}(z) \in M_1, f^{n-k+1}(z) \in M_2, \dots, f^{n-1}(z) \in M_k$ , that  $z$  is a periodic point of  $f$  of period  $n$ . Q.E.D.

**THEOREM A.** *Suppose  $f \in C^0(S^1, S^1)$ . Suppose  $1 \in P(f)$  and  $n \in P(f)$  for some odd integer  $n > 1$ . Then for every integer  $m > n$ ,  $m \in P(f)$ .*

**PROOF.** By hypothesis  $f$  has a periodic orbit  $P = \{p_1, \dots, p_n\}$  of period  $n$ . Without loss of generality we may assume that  $P$  is labeled in order. Let  $I_1, \dots, I_n$  be the intervals determined by  $P$ .

Also, by hypothesis,  $f$  has a fixed point  $e$ . We may assume without loss of generality that  $e \in I_n$ . By Lemma 6, we may also assume that for some  $j \in \{1, \dots, n - 1\}$ ,  $I_j f$ -covers  $I_n$ .

Let  $f(p_1) = p_s$  and  $f(p_n) = p_t$ . We have two cases.

*Case 1.* Either  $[e, p_1] f$ -covers  $[e, p_s]$  or  $[p_n, e] f$ -covers  $[p_t, e]$ .

Since these are analogous we may assume  $[e, p_1] f$ -covers  $[e, p_s]$ . Thus, by Lemma 2,  $[e, p_1] f$ -covers  $[e, p_1]$ , and  $[e, p_1] f$ -covers each interval  $I_j$  with  $j \in \{1, \dots, s - 1\}$ .

Suppose that for some  $j \in \{1, \dots, s - 1\}$ ,  $I_j f$ -covers  $I_n$ . Then the hypothesis of Lemma 7 is satisfied with  $k = 2$ ,  $M_1 = [e, p_1]$ , and  $M_2 = I_j$ . Hence, by Lemma 7, the conclusion of the theorem holds.

Thus, we may assume that for all  $j \in \{1, \dots, s - 1\}$ ,  $I_j$  does not  $f$ -cover  $I_n$ . Since  $I_j f$ -covers  $I_n$  for some  $j \in \{1, \dots, n - 1\}$ , this implies that  $s - 1 < n - 1$ . Hence,  $s < n$ .

Since  $s < n$ , for some integer  $r$  with  $2 \leq r \leq s$ ,  $f(p_r) \notin \{p_1, \dots, p_s\}$ . We may assume, by choosing  $r$  smaller if necessary that  $f(p_{r-1}) \in \{p_1, \dots, p_s\}$ . Let  $f(p_r) = p_q$ . Since  $I_{r-1}$  does not  $f$ -cover  $I_n$ , by Lemmas 1 and 2,  $I_{r-1} f$ -covers  $[f(p_{r-1}), p_q]$ . Hence, for every positive integer  $j$  with  $s \leq j \leq q - 1$ ,  $I_{r-1} f$ -covers  $I_j$ .

Note that by choice of  $p_r$  and  $p_q$ ,  $s \leq q - 1$ . Suppose that for some positive integer  $j$  with  $s \leq j \leq q - 1$ ,  $I_j f$ -covers  $I_n$ . Then the conclusion of the theorem holds by Lemma 7 (with  $k = 3$ ,  $M_1 = [e, p_1]$ ,  $M_2 = I_{r-1}$  and  $M_3 = I_j$ ).

By repeating the argument of the preceding three paragraphs at most  $n$  times, using the fact that for some  $j \in \{1, \dots, n - 1\}$ ,  $I_j f$ -covers  $I_n$ , eventually we obtain a collection of closed intervals  $\{M_1, M_2, \dots, M_k\}$  with  $k \leq n$ , such that the hypothesis of Lemma 7 is satisfied. Thus, the conclusion of the theorem follows from Lemma 7.

*Case 2.*  $[e, p_1]$  does not  $f$ -cover  $[e, p_s]$  and  $[p_n, e]$  does not  $f$ -cover  $[p_t, e]$ .

By Lemma 1,  $[e, p_1] f$ -covers  $[p_s, e]$  and  $[p_n, e] f$ -covers  $[e, p_t]$ . We claim that  $I_n = [p_n, p_1] f$ -covers  $I_n$ . To prove this, note that since  $[e, p_1] f$ -covers  $[p_s, e]$ , there is a point  $a \in (e, p_1]$  such that  $f(a) = p_n$  but  $f(x) \neq p_n$  for all  $x \in (e, a)$ . Since  $f(e) = e$  and  $f(a) = p_n$ ,  $[e, a] f$ -covers  $[e, p_n]$  or  $[e, a] f$ -covers  $[p_n, e]$ . Since  $[e, p_1]$  does not  $f$ -cover  $[e, p_s]$ ,  $[e, a]$  does not  $f$ -cover  $[e, p_s]$ . By Lemma 2,  $[e, a]$  does not  $f$ -cover  $[e, p_n]$ . Hence,  $[e, a] f$ -covers  $[p_n, e]$ . In particular  $f([e, a]) \supset [p_n, e]$ .

Suppose that for some  $z \in (e, a)$ ,  $f(z) \notin [p_n, p_1]$ . Since  $f(z) \notin [p_n, p_1]$  and  $f(e) \in [p_n, p_1]$ , by continuity, for some  $q \in (e, a)$ ,  $f(q) = p_1$  or  $f(q) = p_n$ . Since  $q \in (e, a)$  it follows from the choice of  $a$  that  $f(q) \neq p_n$ . Hence  $f(q) = p_1$ . Now, it follows from the choice of  $a$ , that  $f([e, a])$  is a proper closed interval on  $S^1$  and  $p_n$  is an endpoint of  $f([e, a])$ . Also,  $e \in f([e, a])$  and  $p_1 \in f([e, a])$ . Hence, either  $[p_n, p_1] \subset f([e, a])$  or  $[e, p_n] \subset f([e, a])$ . If  $[e, p_n] \subset f([e, a])$ , it follows as in the proof of

Lemma 2, using the fact that  $f([e, a]) \neq S^1$ , that  $[e, a]$   $f$ -covers  $[e, p_n]$ . This implies, by Lemma 2, that  $[e, a]$   $f$ -covers  $[e, p_s]$ . Thus,  $[e, p_1]$   $f$ -covers  $[e, p_s]$ , a contradiction. Hence  $[p_n, p_1] \subset f([e, a])$ . Since  $f([e, a]) \neq S^1$ , this implies that  $[e, a]$   $f$ -covers  $[p_n, p_1]$ . Thus,  $[p_n, p_1]$   $f$ -covers  $[p_n, p_1]$ .

We have shown that our claim holds if  $f(z) \notin [p_n, p_1]$  for some  $z \in (e, a)$ . Hence, we may assume that  $f([e, a]) \subset [p_n, p_1]$ .

Since  $[p_n, e]$   $f$ -covers  $[e, p_t]$ , there is a point  $b \in [p_n, e]$  such that  $f(b) = p_1$  but  $f(x) \neq p_1$  for all  $x \in (b, e)$ . It follows that  $f([b, e]) \supset [e, p_1]$  (by the same argument used to show  $f([e, a]) \supset [p_n, e]$ ). Also, we may assume that  $f([b, e]) \subset [p_n, p_1]$  (by the same argument used to show that we may assume that  $f([e, a]) \subset [p_n, p_1]$ ). Thus,  $f([b, a]) = [p_n, p_1]$ . Since  $[b, a] \subset [p_n, p_1]$ , this establishes our claim that  $[p_n, p_1]$   $f$ -covers  $[p_n, p_1]$ .

Now, since  $[e, p_1]$   $f$ -covers  $[p_s, e]$ ,  $[p_n, p_1]$   $f$ -covers  $[p_s, e]$ . Also, since  $[p_n, e]$   $f$ -covers  $[e, p_t]$ ,  $[p_n, p_1]$   $f$ -covers  $[e, p_t]$ . Hence, by Lemma 2, for any integer  $j$  with  $1 \leq j \leq t - 1$  or  $s \leq j \leq n - 1$ ,  $[p_n, p_1]$   $f$ -covers  $I_j$ .

Suppose that for some integer  $j$  with  $1 \leq j \leq t - 1$  or  $s \leq j \leq n - 1$ ,  $I_j$   $f$ -covers  $I_n$ . Then the conclusion of the theorem holds by Lemma 7 (with  $k = 2$ ,  $M_1 = I_n$  and  $M_2 = I_j$ ). Hence, we may assume that for every integer  $j$  with  $1 \leq j \leq t - 1$  or  $s \leq j \leq n - 1$ ,  $I_j$  does not  $f$ -cover  $I_n$ . Since  $I_j$   $f$ -covers  $I_n$  for some integer  $j$  with  $1 \leq j \leq n - 1$ , this implies that  $t < s$ .

Note that we cannot have both  $f(\{p_1, \dots, p_t\}) \subset \{p_s, \dots, p_n\}$  and  $f(\{p_s, \dots, p_n\}) \subset \{p_1, \dots, p_t\}$ . This follows from the fact that  $\{p_1, \dots, p_n\}$  is a periodic orbit and  $t < s$ , and uses the fact that  $n$  is odd in the case  $t = s - 1$ . Without loss of generality we may assume that  $f(\{p_1, \dots, p_t\})$  is not a subset of  $\{p_s, \dots, p_n\}$ .

Let  $w$  be the smallest positive integer such that  $f(p_w) \notin \{p_s, \dots, p_n\}$ . Then  $2 \leq w \leq t$  and  $I_{w-1}$   $f$ -covers the interval  $[f(p_w), f(p_{w-1})]$  (since  $I_{w-1}$  does not  $f$ -cover  $I_n$ ). Hence  $I_{w-1}$   $f$ -covers the interval  $[f(p_w), p_s]$ . Let  $f(p_w) = p_v$ . Then  $v \leq s - 1$  and  $I_{w-1}$   $f$ -covers each interval  $I_j$  with  $v \leq j \leq s - 1$ .

Suppose for some integer  $j$  with  $v \leq j \leq s - 1$ ,  $I_j$   $f$ -covers  $I_n$ . Then the conclusion of the theorem holds by Lemma 7 (with  $k = 3$ ,  $M_1 = I_n$ ,  $M_2 = I_{w-1}$  and  $M_3 = I_j$ ).

By repeating the argument of the preceding four paragraphs at most  $n$  times, using the fact that for some  $j \in \{1, \dots, n - 1\}$   $I_j$   $f$ -covers  $I_n$ , we eventually obtain a collection of closed intervals  $\{M_1, M_2, \dots, M_k\}$  with  $k \leq n$  such that the hypothesis of Lemma 7 is satisfied. Thus, the conclusion of the theorem follows from Lemma 7. Q.E.D.

#### 4. Proof of Theorem B.

**COROLLARY 8.** *Let  $f \in C^0(S^1, S^1)$ . Suppose  $m \in P(f)$  and  $n \in P(f)$  where  $m$  and  $n$  are odd integers with  $m \neq n$ . Then  $P(f)$  is infinite.*

**PROOF.** Without loss of generality, we may assume that  $m < n$ . Then  $1 \in P(f^m)$  and for some odd integer  $k > 1$ ,  $k \in P(f^m)$ . By Theorem A,  $P(f^m)$  is infinite. Hence,  $P(f)$  is infinite. Q.E.D.

**COROLLARY 9.** *Suppose  $f \in C^0(S^1, S^1)$  and  $P(f)$  is finite. Then for some integers  $m$  and  $n$  with  $m \geq 1$  and  $n \geq 0$ ,  $P(f) \subset \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$ .*

**PROOF.** Let  $m$  be the smallest element of  $P(f)$ . Since  $P(f)$  is finite, it suffices to prove that if  $k \in P(f)$  then  $k = 2^i \cdot m$  for some nonnegative integer  $i$ .

Let  $k \in P(f)$ . Let  $m = 2^r \cdot s$  where  $s$  is odd,  $s \geq 1$  and  $r \geq 0$ , and let  $k = 2^v \cdot w$  where  $w$  is odd,  $w \geq 1$  and  $v \geq 0$ . Let  $j$  be the largest element of  $\{2^r, 2^v\}$ . Then  $P(f^j)$  is finite,  $s \in P(f^j)$ ,  $w \in P(f^j)$  and  $s$  and  $w$  are odd. By Corollary 8,  $s = w$ .

Since  $m$  is the smallest element of  $P(f)$  and  $s = w$ , we have  $v \geq r$ . Let  $i = v - r$ . Then  $i \geq 0$  and  $k = 2^v \cdot w = 2^v \cdot s = 2^{v-r} \cdot 2^r \cdot s = 2^i \cdot m$ . Q.E.D.

**LEMMA 10.** *Let  $f \in C^0(S^1, S^1)$  and suppose that  $\{p_1, p_2, p_3, p_4\}$  is a periodic orbit of  $f$  of period 4, labeled in order. Suppose that one of the following holds.*

- (i)  $f(p_1) = p_2, f(p_2) = p_3, f(p_3) = p_4, f(p_4) = p_1$ .
- (ii)  $f(p_1) = p_4, f(p_4) = p_3, f(p_3) = p_2, f(p_2) = p_1$ .

*Suppose also that  $1 \in P(f)$ . Then  $5 \in P(f)$ .*

**PROOF.** Since (i) and (ii) are analogous, we may assume that (i) holds. Let  $I_1, I_2, I_3$  and  $I_4$  be the intervals determined by  $\{p_1, p_2, p_3, p_4\}$ .

By Lemma 1, either  $I_1$   $f$ -covers  $I_2$  or  $I_1$   $f$ -covers  $I_1 \cup I_3 \cup I_4$ . Suppose  $I_1$   $f$ -covers  $I_1 \cup I_3 \cup I_4$ . If  $I_4$   $f$ -covers  $I_1$ , then it follows from Lemma 7 (with  $k = 2, M_1 = I_1$  and  $M_2 = I_4$ ) that  $5 \in P(f)$ . Also, if  $I_3$   $f$ -covers  $I_1$ , then it follows from Lemma 7 (with  $k = 2, M_1 = I_1$  and  $M_2 = I_3$ ) that  $5 \in P(f)$ . Hence we may assume that  $I_4$  does not  $f$ -cover  $I_1$  and  $I_3$  does not  $f$ -cover  $I_1$ . This implies (by Lemma 1) that  $I_4$   $f$ -covers  $I_2 \cup I_3 \cup I_4$  and  $I_3$   $f$ -covers  $I_4$ . Hence, by Lemma 7 (with  $k = 2, M_1 = I_4$  and  $M_2 = I_3$ ),  $5 \in P(f)$ .

Thus we may assume that  $I_1$   $f$ -covers  $I_2$ . Similarly, we may assume that  $I_2$   $f$ -covers  $I_3, I_3$   $f$ -covers  $I_4$  and  $I_4$   $f$ -covers  $I_1$ .

By hypothesis  $f$  has a fixed point  $e$ . Without loss of generality we may assume that  $e \in I_4$ . Let  $I_{4A} = [p_4, e]$  and let  $I_{4B} = [e, p_1]$ .

By Lemma 1, either  $I_{4B}$   $f$ -covers  $I_{4B} \cup I_1$  or  $I_{4B}$   $f$ -covers  $I_2 \cup I_3 \cup I_{4A}$ . If  $I_{4B}$   $f$ -covers  $I_{4B} \cup I_1$ , then it follows from Lemma 7 (with  $k = 4, M_1 = I_{4B}, M_2 = I_1, M_3 = I_2$  and  $M_4 = I_3$ ) that  $5 \in P(f)$ . Hence, we may assume that  $I_{4B}$   $f$ -covers  $I_2 \cup I_3 \cup I_{4A}$ .

Also, by Lemma 1, either  $I_{4A}$   $f$ -covers  $I_{4B}$  or  $I_{4A}$   $f$ -covers  $I_1 \cup I_2 \cup I_3 \cup I_{4A}$ . If  $I_{4A}$   $f$ -covers  $I_1 \cup I_2 \cup I_3 \cup I_{4A}$ , then it follows from Lemma 7 (with  $k = 2, M_1 = I_{4A}$  and  $M_2 = I_3$ ) that  $5 \in P(f)$ . Hence, we may assume that  $I_{4A}$   $f$ -covers  $I_{4B}$ .

Now, we have that  $I_{4A}$   $f$ -covers  $I_{4B}, I_{4B}$   $f$ -covers  $I_3$  and  $I_3$   $f$ -covers  $I_{4B}$ . By Lemma 4, there is a fixed point  $z$  of  $f^3$  such that  $z \in I_{4A}, f(z) \in I_{4B}$  and  $f^2(z) \in I_3$ . Since  $I_{4A} \cap I_{4B} \cap I_3 = \emptyset$ ,  $z$  is not a fixed point of  $f$ . Hence,  $3 \in P(f)$ . Thus, by Theorem A,  $5 \in P(f)$ . Q.E.D.

**LEMMA 11.** *Let  $f \in C^0(S^1, S^1)$ . Suppose  $1 \in P(f), 4 \in P(f)$  and  $5 \notin P(f)$ . Then  $2 \in P(f)$ .*

PROOF. Let  $\{p_1, p_2, p_3, p_4\}$  be a periodic orbit of  $f$  of period 4, labeled in order. By Lemma 10, one of the following must hold.

- (i)  $f(p_1) = p_3, f(p_3) = p_2, f(p_2) = p_4$  and  $f(p_4) = p_1$ .
- (ii)  $f(p_1) = p_3, f(p_3) = p_4, f(p_4) = p_2$  and  $f(p_2) = p_1$ .
- (iii)  $f(p_1) = p_4, f(p_4) = p_2, f(p_2) = p_3$  and  $f(p_3) = p_1$ .
- (iv)  $f(p_1) = p_2, f(p_2) = p_4, f(p_4) = p_3$  and  $f(p_3) = p_1$ .

Note that (ii) is analogous to (i), because if (ii) holds and we let  $q_1 = p_4, q_2 = p_1, q_3 = p_2$  and  $q_4 = p_3$  then  $f(q_1) = q_3, f(q_3) = q_2, f(q_2) = q_4$  and  $f(q_4) = q_1$ . Also, (iii) is analogous to (i) because if (iii) holds and we let  $q_1 = p_4, q_2 = p_3, q_3 = p_2$  and  $q_4 = p_1$  then  $f(q_1) = q_3, f(q_3) = q_2, f(q_2) = q_4$  and  $f(q_4) = q_1$ . Finally, (iv) is analogous to (i), because if (iv) holds and we let  $q_1 = p_2, q_2 = p_3, q_3 = p_4$  and  $q_4 = p_1$  then  $f(q_1) = q_3, f(q_3) = q_2, f(q_2) = q_4$  and  $f(q_4) = q_1$ . Hence, we may assume that (i) holds.

We claim that  $I_1$   $f$ -covers  $I_3$ . Suppose that  $I_1$  does not  $f$ -cover  $I_3$ . By Lemma 1,  $I_1$   $f$ -covers  $I_4 \cup I_1 \cup I_2$ . If  $I_2$   $f$ -covers  $I_4 \cup I_1$  then we obtain a contradiction (to the fact that  $5 \notin P(f)$ ) by Lemma 7 (with  $k = 2, M_1 = I_1$  and  $M_2 = I_2$ ). Hence, by Lemma 1,  $I_2$   $f$ -covers  $I_2 \cup I_3$ . Also, if  $I_3$   $f$ -covers  $I_1$  then we obtain a contradiction by Lemma 7 (with  $k = 3, M_1 = I_1, M_2 = I_2, M_3 = I_3$ ). Hence, by Lemma 1,  $I_3$   $f$ -covers  $I_2 \cup I_3 \cup I_4$ . Again, by Lemma 7 (with  $k = 2, M_1 = I_2, M_2 = I_3$ ) we obtain a contradiction. This establishes our claim that  $I_1$   $f$ -covers  $I_3$ .

We claim also that  $I_3$   $f$ -covers  $I_1$ . Suppose that  $I_3$  does not  $f$ -cover  $I_1$ . Then  $I_3$   $f$ -covers  $I_2 \cup I_3 \cup I_4$  by Lemma 1. If  $I_2$   $f$ -covers  $I_2 \cup I_3$  then we obtain a contradiction by Lemma 7. Hence, by Lemma 1,  $I_2$   $f$ -covers  $I_4 \cup I_1$ . Since  $I_1$   $f$ -covers  $I_3$ , we again obtain a contradiction by Lemma 7 (with  $k = 3, M_1 = I_3, M_2 = I_2$  and  $M_3 = I_1$ ). This establishes our claim that  $I_3$   $f$ -covers  $I_1$ .

We have shown that  $I_1$   $f$ -covers  $I_3$  and  $I_3$   $f$ -covers  $I_1$ . Since  $I_1 \cap I_3 = \emptyset$ , it follows from Lemma 4 that  $2 \in P(f)$ . Q.E.D.

**THEOREM B.** *Let  $f \in C^0(S^1, S^1)$  and suppose that  $P(f)$  is finite. Then there are integers  $m$  and  $n$  (with  $m \geq 1$  and  $n \geq 0$ ) such that  $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$ .*

PROOF. Let  $m$  be the smallest element of  $P(f)$  and let  $k$  be the largest element of  $P(f)$ . By Corollary 9,  $k = 2^n \cdot m$  for some nonnegative integer  $n$  and  $P(f) \subset \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$ .

If  $n = 0$  or  $n = 1$  the conclusion of the theorem follows immediately, so we may assume that  $n \geq 2$ . Let  $r = 2^{n-2}$ . Then  $1 \in P(f^{r \cdot m})$  and  $4 \in P(f^{r \cdot m})$ . Also, since  $P(f)$  is finite,  $P(f^{r \cdot m})$  is finite. Hence, by Theorem A,  $5 \notin P(f^{r \cdot m})$ . By Lemma 11,  $2 \in P(f^{r \cdot m})$ . It follows from this, and the fact that  $P(f) \subset \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$ , that  $2^{n-1} \cdot m \in P(f)$ .

Now, if  $n = 2$  the conclusion of the theorem follows immediately. If  $n > 2$ , it follows by the argument of the preceding paragraph (with  $r = 2^{n-3}$  instead of  $r = 2^{n-2}$ ) that  $2^{n-2} \cdot m \in P(f)$ . Thus, it follows by using this argument inductively, that  $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$ . Q.E.D.



**5. Proof of Theorem C.**

**PROPOSITION 12.** *Let  $n$  be an integer with  $n \geq 3$ . There is a map  $f \in C^0(S^1, S^1)$  such that  $1 \in P(f)$  and  $n \in P(f)$ , but for every integer  $k$  with  $1 < k < n$ ,  $k \notin P(f)$ .*

**PROOF.** Let  $f \in C^0(S^1, S^1)$  with the following properties.

- (1)  $f$  has a periodic orbit  $\{p_1, \dots, p_n\}$  of period  $n$ , labeled in order, with  $f(p_i) = p_{i+1}$  for  $i = 1, \dots, n - 1$ , and  $f(p_n) = p_1$ .
- (2)  $f(I_j) = I_{j+1}$  for  $j \in \{1, \dots, n - 1\}$ , where  $I_1, \dots, I_n$  are the intervals determined by  $\{p_1, \dots, p_n\}$ .
- (3)  $f$  has a fixed point  $e \in I_n$ .
- (4)  $f([p_n, e]) = [e, p_1]$  and  $f([e, p_1]) = [e, p_2]$ .
- (5) For any  $x \in (e, p_1)$ ,  $f(x) \neq x$  and  $(x, f(x)) \subset (e, p_1)$ .

By construction  $1 \in P(f)$  and  $n \in P(f)$ . Also, by construction, if  $x \in S^1$  such that  $e$  is not in the orbit of  $x$ , then for any  $j \in \{1, \dots, n\}$ , there is a point in the orbit of  $x$  in  $I_j$ . Thus, for every integer  $k$  with  $1 < k < n$ ,  $k \notin P(f)$ . Q.E.D.

**LEMMA 13.** *There is a map  $f \in C^0(S^1, S^1)$  such that for every integer  $n > 1$ ,  $n \in P(f)$ , but  $1 \notin P(f)$ .*

**PROOF.** Let  $f \in C^0(S^1, S^1)$  with the following properties:

- (1)  $f$  has a periodic orbit  $\{p_1, p_2, p_3\}$ , labeled in order, with  $f(p_1) = p_2$ ,  $f(p_2) = p_3$  and  $f(p_3) = p_1$ .
- (2) There are points  $c_1 \in (p_1, p_2)$ ,  $c_2 \in (p_2, p_3)$  and  $c_3 \in (p_3, p_1)$ , such that  $f(c_1) = p_1$ ,  $f(c_2) = p_2$  and  $f(c_3) = p_3$ .
- (3)  $f([p_1, c_1]) = [p_2, p_1]$ ,  $f([c_1, p_2]) = [p_3, p_1]$ ,  $f([p_2, c_2]) = [p_3, p_2]$ ,  $f([c_2, p_3]) = [p_1, p_2]$ ,  $f([p_3, c_3]) = [p_1, p_3]$ ,  $f([c_3, p_1]) = [p_2, p_3]$ .

Note that by construction,  $1 \notin P(f)$ . Let  $n$  be any integer with  $n > 1$ . Define  $n$  closed intervals,  $M_1, \dots, M_n$ , by  $M_1 = [p_1, c_1]$ ,  $M_k = [p_2, c_2]$  if  $k$  is even and  $2 \leq k \leq n$ , and  $M_k = [p_3, c_3]$  if  $k$  is odd and  $2 \leq k \leq n$ . By Lemma 4, there is a fixed point  $z$  of  $f^n$  such that  $z \in M_1$ ,  $f(z) \in M_2, \dots, f^{n-1}(z) \in M_n$ . Since  $[p_1, c_1] \cap [p_2, c_2] = \emptyset$  and  $[p_1, c_1] \cap [p_3, c_3] = \emptyset$ ,  $z$  is a periodic point of  $f$  of period  $n$ . Thus,  $n \in P(f)$ . Q.E.D.

**THEOREM C.** *Let  $f \in C^0(S^1, S^1)$ . If  $\{1, 2, 3\} \subset P(f)$  then  $P(f) = N$ . Conversely, if  $S \subset N$  with the property that for any  $f \in C^0(S^1, S^1)$ ,  $S \subset P(f) \Rightarrow P(f) = N$ , then  $\{1, 2, 3\} \subset S$ .*

**PROOF.** By Theorem A, if  $\{1, 2, 3\} \subset P(f)$  then  $P(f) = N$ .

Suppose  $S \subset N$  with the property that for any  $f \in C^0(S^1, S^1)$ ,  $S \subset P(f) \Rightarrow P(f) = N$ . Since there are mappings  $g$  of the interval into itself such that  $3 \notin P(g)$  but  $k \in P(g)$  for every positive integer  $k \neq 3$  (see [3] or [4]), there are mappings  $f \in C^0(S^1, S^1)$  such that  $3 \notin P(f)$ , but  $k \in P(f)$  for every positive integer  $k \neq 3$ . Thus  $3 \in S$ . By Proposition 12 (with  $n = 3$ ), there is a map  $f \in C^0(S^1, S^1)$  such that  $1 \in P(f)$  and  $3 \in P(f)$ , but  $2 \notin P(f)$ . By Theorem A, for this map  $f$ ,  $P(f)$  consists of all positive integers except 2. Thus,  $2 \in S$ . Finally, it follows from Lemma 13, that  $1 \in S$ . Hence  $\{1, 2, 3\} \subset S$ . Q.E.D.

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