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PERIODIC ORBITS OF CONTINUOUS MAPPINGS OF THE CIRCLE

BY

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ABSTRACT. Let f be a continuous map of the circle into itself and let $P(f)$ denote the set of positive integers n such that f has a periodic point of period n. It is shown that if $1 \in P(f)$ and $n \in P(f)$ for some odd positive integer n then for every integer $m > n$, $m \in P(f)$. Furthermore, if $P(f)$ is finite then there are integers m and n (with $m > 1$ and $n > 0$) such that $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$.

1. Introduction. Let I denote a closed bounded interval on the real line and let $C^0(I, I)$ denote the space of continuous maps of I into itself. For $f \in C^0(I, I)$, let $P(f)$ denote the set of all positive integers n such that f has a periodic point of period n (see §2 for definition). One may ask the following question. If $k \in P(f)$, what other integers must be elements of $P(f)$?

This question is answered by a theorem of Šarkovskii. Consider the following ordering of the positive integers:

$$1, 2, 4, 8, \dots \quad \dots, 7 \cdot 8, 5 \cdot 8, 3 \cdot 8, \dots, 7 \cdot 4, 5 \cdot 4, 3 \cdot 4, \dots, \\ 7 \cdot 2, 5 \cdot 2, 3 \cdot 2, \dots, 7, 5, 3.$$

Šarkovskii's theorem states that if $n \in P(f)$ and m is to the left of n in the above ordering then $m \in P(f)$ (see [3] or [4]). Furthermore, if m is to the right of n in the above ordering, then there is a map $f \in C^0(I, I)$ with $n \in P(f)$ and $m \notin P(f)$.

In this paper we obtain some similar results in $C^0(S^1, S^1)$ (the space of continuous maps of the circle into itself). Since for any positive integer n, there is a map $f \in C^0(S^1, S^1)$ with $P(f) = \{n\}$ (where $P(f)$ is defined as above) one cannot obtain an ordering as in Šarkovskii's theorem. However, we do obtain the following result.

THEOREM A. *Let $f \in C^0(S^1, S^1)$. Suppose $1 \in P(f)$ and $n \in P(f)$ for some odd integer $n > 1$. Then for every integer $m > n$, $m \in P(f)$.*

We remark that if the hypothesis of Theorem A is satisfied, it is possible that for every integer k with $1 < k < n$, $k \notin P(f)$ (see Proposition 12 in §5). Using Theorem A we obtain the following result which characterizes $P(f)$ when $P(f)$ is finite.

THEOREM B. *Let $f \in C^0(S^1, S^1)$ and suppose that $P(f)$ is finite. Then there are integers m and n (with $m \geqslant 1$ and $n \geqslant 0$) such that $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \dots, 2^n \cdot m\}$.*

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It is known that for any integers m and n with $m \geqslant 1$ and $n \geqslant 0$ there is a differentiable map f of the circle with $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \ldots, 2^n \cdot m\}$ (see **[1]**). Also, in **[1]**, Theorem B is proved for a certain subset of $C^0(S^1, S^1)$.

Finally, note that Šarkovskii's theorem implies that for $f \in C^0(I, I)$ if $3 \in P(f)$ then $P(f) = N$, where N denotes the set of positive integers. This result is also proved in **[2]**. It also follows from Šarkovskii's theorem that if $S \subset N$ with the property that for any $f \in C^0(I, I)$, $S \subset P(f) \Rightarrow P(f) = N$, then $3 \in S$. In this paper we obtain the following analogous result for the circle. The first statement, of course, follows immediately from Theorem A.

THEOREM C. *Let $f \in C^0(S^1, S^1)$. If $\{1, 2, 3\} \subset P(f)$ then $P(f) = N$. Conversely, if $S \subset N$ with the property that for any $f \in C^0(S^1, S^1)$, $S \subset P(f) \Rightarrow P(f) = N$, then $\{1, 2, 3\} \subset S$.*

2. Preliminary definitions and results. Let $f \in C^0(S^1, S^1)$. For any $n \in N$, we define f^n inductively by $f^1 = f$ and $f^n = f \circ f^{n-1}$. Let f^0 denote the identity map of S^1.

Let $x \in S^1$. x is said to be a fixed point of f if $f(x) = x$. x is said to be a periodic point of f if $f^n(x) = x$ for some $n \in N$. In this case, the smallest element of $\{n \in N: f^n(x) = x\}$ is called the period of x.

We define the orbit of x to be $\{f^n(x): n = 0, 1, 2, \ldots\}$. If x is a periodic point of period n, we say the orbit of x is a periodic orbit of period n. In this case the orbit of x contains exactly n points, each of which is a periodic point of period n.

We will use the following notation throughout this paper.

NOTATION. *Let $a \in S^1$ and $b \in S^1$ with $a \neq b$. We write $[a, b]$, (a, b), $(a, b]$, or $[a, b)$ to denote the closed, open, or half-open interval from a counterclockwise to b.*

We will also use the following definition.

DEFINITION. *Let I and J be proper closed intervals on S^1 and let $f \in C^0(S^1, S^1)$. We say I f-covers J if for some closed interval $K \subset I$, $f(K) = J$.*

We conclude this section by proving four lemmas, which use this definition.

LEMMA 1. *Let $I = [a, b]$ be a proper closed interval on S^1 and let $f \in C^0(S^1, S^1)$. Suppose $f(a) = c$ and $f(b) = d$ and $c \neq d$. Then either I f-covers $[c, d]$ or I f-covers $[d, c]$.*

PROOF. Let $A = \{x \in I: f(x) = c\}$. There is a point $v \in A$ such that $(v, b] \cap A = \varnothing$. Let $B = \{x \in [v, b]: f(x) = d\}$. There is a point $w \in B$ such that $[v, w) \cap B = \varnothing$.

We have $f(v) = c$, $f(w) = d$, and if $x \in (v, w)$ then $f(x) \notin \{c, d\}$. Hence, if $K = [v, w]$ then $f(K) = [c, d]$ or $f(K) = [d, c]$. Q.E.D.

LEMMA 2. *Let $f \in C^0(S^1, S^1)$. Let I and J be proper closed intervals on S^1 such that I f-covers J. Suppose L is a closed interval with $L \subset J$. Then I f-covers L.*

PROOF. By hypothesis, there is a closed interval $K \subset I$ with $f(K) = J$. Let $L = [c, d]$. There are points $a \in K$ and $b \in K$ with $f(a) = c$ and $f(b) = d$. Let K_1

be the closed interval with endpoints a and b such that $K_1 \subset K$. By Lemma 1, either K_1 f-covers $[c, d]$ or K_1 f-covers $[d, c]$. Since $K_1 \subset K$ and $f(K) = J$, K_1 cannot f-cover $[d, c]$. Hence K_1 f-covers $[c, d]$. Since $K_1 \subset K \subset I$, I f-covers $[c, d]$. Q.E.D.

LEMMA 3. *Let $f \in C^0(S^1, S^1)$. Suppose N is a proper closed interval on S^1 such that N f-covers N. Then f has a fixed point in N.*

PROOF. By hypothesis, for some closed interval $K \subset N$, $f(K) = N$. There are points $v \in K$ and $w \in K$ such that $f(v)$ and $f(w)$ are the two endpoints of N. Let L be the closed interval with endpoints v and w such that $L \subset K$. By continuity, f has a fixed point in L. Q.E.D.

LEMMA 4. *Suppose $f \in C^0(S^1, S^1)$ and suppose $M_1, M_2, \ldots, M_n$ are proper closed intervals on S^1 such that M_k f-covers M_{k+1} for $k = 1, \ldots, n-1$, and M_n f-covers M_1. Then there is a fixed point z of f^n such that $z \in M_1, f(z) \in M_2, \ldots, f^{n-1}(z) \in M_n$.*

PROOF. Since M_n f-covers M_1, there is an interval $J_n \subset M_n$ such that $f(J_n) = M_1$. Similarly, there are intervals $J_1, \ldots, J_{n-1}$ such that for each $k = 1, \ldots, n-1$, $J_k \subset M_k$ and $f(J_k) = J_{k+1}$. It follows that $f^n(J_1) = M_1$. By the proof of Lemma 3, f^n has a fixed point $z \in J_1$. Clearly $z \in M_1$, $f(z) \in M_2, \ldots, f^{n-1}(z) \in M_n$. Q.E.D.

3. Proof of Theorem A.

DEFINITION. *Let $f \in C^0(S^1, S^1)$ and let $P = \{p_1, \ldots, p_n\}$ be a periodic orbit of f of period n. We say P is labeled in order if for $k = 1, \ldots, n-1$, $P \cap (p_k, p_{k+1}) = \varnothing$, and $P \cap (p_n, p_1) = \varnothing$. In this case we define the intervals determined by P to be the n closed intervals $I_1 = [p_1, p_2]$, $I_2 = [p_2, p_3], \ldots, I_{n-1} = [p_{n-1}, p_n]$, $I_n = [p_n, p_1]$.*

LEMMA 5. *Let $f \in C^0(S^1, S^1)$. Let $P = \{p_1, \ldots, p_n\}$ be a periodic orbit of f of odd period $n \geq 3$. Suppose P is labeled in order and let $I_1, \ldots, I_n$ be the intervals determined by P. Suppose that for some j and k with $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, n\}$, I_i does not f-cover I_j for all $i \in \{1, \ldots, n\}$ with $i \neq j$, and I_i does not f-cover I_k for all $i \in \{1, \ldots, n\}$ with $i \neq k$. Then $j = k$.*

PROOF. Suppose $j \neq k$. Let v be a point in the interior of I_k, and let w be a point in the interior of I_j. Then $v \neq w$. Let $A = P \cap (v, w)$ and $B = P \cap (w, v)$. Then $A \neq \varnothing$, $B \neq \varnothing$, $A \cap B = \varnothing$ and $A \cup B = P$.

If $f(x) \in A$ for some $x \in A$ then by hypothesis and Lemma 1, $f(A) \subset A$. This is impossible, since P is a periodic orbit. Thus, $f(x) \notin A$ for all $x \in A$. Hence, $f(A) \subset B$. Similarly it follows that $f(B) \subset A$.

Since f maps P onto P it follows that $f(A) = B$ and $f(B) = A$. Thus, A and B have the same number of elements. Since $A \cup B = P$ and $A \cap B = \varnothing$, this contradicts the fact that P has an odd number of elements. Q.E.D.

LEMMA 6. *Suppose $f \in C^0(S^1, S^1)$ and f has a periodic orbit $P = \{p_1, \ldots, p_n\}$ of odd period $n \geq 3$. Suppose P is labeled in order, and let $I_1, \ldots, I_n$ be the intervals determined by P. Suppose also that f has a fixed point e. Then f has a fixed point z with the property that if I_k is the interval determined by P with $z \in I_k$, there is some $j \in \{1, \ldots, n\}$ with $j \neq k$ such that I_j f-covers I_k.*

PROOF. Without loss of generality we may assume that $e \in I_n$. We may also assume that for each $j \in \{1, \ldots, n - 1\}$, I_j does not f-cover I_n (or else the conclusion of the lemma holds with $z = e$).

Let m be the smallest positive integer such that if $f(p_m) = p_r$ then $r < m$. Note that $2 \leq m \leq n$, so $1 \leq m - 1 \leq n - 1$. In particular $m - 1 \neq n$.

Since I_{m-1} does not f-cover I_n, it follows from Lemmas 1 and 2 that I_{m-1} f-covers I_{m-1}. By Lemma 3, f has a fixed point $z \in I_{m-1}$. Since I_j does not f-cover I_n for all $j \in \{1, \ldots, n - 1\}$, it follows from Lemma 5 that for some $j \in \{1, \ldots, n\}$ with $j \neq m - 1$, I_j f-covers I_{m-1}. Q.E.D.

LEMMA 7. *Let $f \in C^0(S^1, S^1)$ and let P be a periodic orbit of f of period m where $m \geq 3$. Suppose that $\{M_1, \ldots, M_k\}$ is a collection of closed intervals with $2 \leq k \leq m$ such that*

(1) *for each $j \in \{1, \ldots, k\}$, there are no elements of P in the interior of M_j,*

(2) *if $i \neq j$, M_i and M_j have disjoint interiors,*

(3) *if $j \in \{2, \ldots, k\}$ the endpoints of M_j are in P,*

(4) *if b is an endpoint of M_1, either $b \in P$ or b is a fixed point of f,*

(5) *for each $j \in \{1, \ldots, k - 1\}$, M_j f-covers M_{j+1},*

(6) *M_1 f-covers M_1 and M_k f-covers M_1.*

Then for any positive integer $n > k$, f has a periodic point of period n.

PROOF. Let $n > k$. We may assume $n \neq m$ since, by hypothesis, f has a periodic point of period m.

Let $L_1 = M_1$, $L_2 = M_1, \ldots, L_{n-k} = M_1$, $L_{n-k+1} = M_1$, $L_{n-k+2} = M_2$, $L_{n-k+3} = M_3, \ldots, L_{n-k+k} = L_n = M_k$. By Lemma 4 (applied to $L_1, L_2, \ldots, L_n$), there is a fixed point z of f^n such that $z \in L_1, f(z) \in L_2, \ldots, f^{n-1}(z) \in L_n$. Since $z \in M_1$ and $f^{n-k+1}(z) \in M_2$, it follows from (2) and (3) of the hypothesis that z is not a fixed point of f.

We claim that $z \notin P$. To prove this, first suppose that $n \geq k + 2$. Then $L_1 = L_2 = L_3 = M_1$. Hence $z \in M_1, f(z) \in M_1$ and $f^2(z) \in M_1$. Since P is a periodic orbit of period $m \geq 3$, it follows from (1) of the hypothesis that $z \notin P$. Now, suppose that $n < k + 2$. Then $n < m + 2$. Since $n \neq m$ and $m \geq 3$, n is not a multiple of m. Since $f^n(z) = z$, it follows that $z \notin P$.

Since z is not a fixed point of f and $z \notin P$ it follows from (4) of the hypothesis that z is in the interior of M_1. Also, since $f^n(z) = z \notin P$, for any positive integer $r < n, f^r(z) \notin P$ (and $f^r(z)$ is not a fixed point of f). Thus, by (3) and (4), for any positive integer $r < n, f^r(z)$ is not an endpoint of any of the intervals $M_1, \ldots, M_k$. It follows from this, and (2), and the fact that $z \in M_1$, $f(z) \in M_1$, $f^2(z) \in M_1, \ldots, f^{n-k}(z) \in M_1$, $f^{n-k+1}(z) \in M_2, \ldots, f^{n-1}(z) \in M_k$, that z is a periodic point of f of period n. Q.E.D.

THEOREM A. *Suppose* $f \in C^0(S^1, S^1)$. *Suppose* $1 \in P(f)$ *and* $n \in P(f)$ *for some odd integer* $n > 1$. *Then for every integer* $m > n$, $m \in P(f)$.

PROOF. By hypothesis f has a periodic orbit $P = \{p_1, \ldots, p_n\}$ of period n. Without loss of generality we may assume that P is labeled in order. Let $I_1, \ldots, I_n$ be the intervals determined by P.

Also, by hypothesis, f has a fixed point e. We may assume without loss of generality that $e \in I_n$. By Lemma 6, we may also assume that for some $j \in \{1, \ldots, n-1\}$, I_j f-covers I_n.

Let $f(p_1) = p_s$ and $f(p_n) = p_t$. We have two cases.

Case 1. Either $[e, p_1]$ f-covers $[e, p_s]$ or $[p_n, e]$ f-covers $[p_t, e]$.

Since these are analogous we may assume $[e, p_1]$ f-covers $[e, p_s]$. Thus, by Lemma 2, $[e, p_1]$ f-covers $[e, p_1]$, and $[e, p_1]$ f-covers each interval I_j with $j \in \{1, \ldots, s-1\}$.

Suppose that for some $j \in \{1, \ldots, s-1\}$, I_j f-covers I_n. Then the hypothesis of Lemma 7 is satisfied with $k = 2$, $M_1 = [e, p_1]$, and $M_2 = I_j$. Hence, by Lemma 7, the conclusion of the theorem holds.

Thus, we may assume that for all $j \in \{1, \ldots, s-1\}$, I_j does not f-cover I_n. Since I_j f-covers I_n for some $j \in \{1, \ldots, n-1\}$, this implies that $s - 1 < n - 1$. Hence, $s < n$.

Since $s < n$, for some integer r with $2 \leqslant r \leqslant s$, $f(p_r) \notin \{p_1, \ldots, p_s\}$. We may assume, by choosing r smaller if necessary that $f(p_{r-1}) \in \{p_1, \ldots, p_s\}$. Let $f(p_r) = p_q$. Since I_{r-1} does not f-cover I_n, by Lemmas 1 and 2, I_{r-1} f-covers $[f(p_{r-1}), p_q]$. Hence, for every positive integer j with $s \leqslant j \leqslant q-1$, I_{r-1} f-covers I_j.

Note that by choice of p_r and p_q, $s \leqslant q - 1$. Suppose that for some positive integer j with $s \leqslant j \leqslant q-1$, I_j f-covers I_n. Then the conclusion of the theorem holds by Lemma 7 (with $k = 3$, $M_1 = [e, p_1]$, $M_2 = I_{r-1}$ and $M_3 = I_j$).

By repeating the argument of the preceding three paragraphs at most n times, using the fact that for some $j \in \{1, \ldots, n-1\}$, I_j f-covers I_n, eventually we obtain a collection of closed intervals $\{M_1, M_2, \ldots, M_k\}$ with $k \leqslant n$, such that the hypothesis of Lemma 7 is satisfied. Thus, the conclusion of the theorem follows from Lemma 7.

Case 2. $[e, p_1]$ does not f-cover $[e, p_s]$ and $[p_n, e]$ does not f-cover $[p_t, e]$.

By Lemma 1, $[e, p_1]$ f-covers $[p_s, e]$ and $[p_n, e]$ f-covers $[e, p_t]$. We claim that $I_n = [p_n, p_1]$ f-covers I_n. To prove this, note that since $[e, p_1]$ f-covers $[p_s, e]$, there is a point $a \in (e, p_1]$ such that $f(a) = p_n$ but $f(x) \neq p_n$ for all $x \in (e, a)$. Since $f(e) = e$ and $f(a) = p_n$, $[e, a]$ f-covers $[e, p_n]$ or $[e, a]$ f-covers $[p_n, e]$. Since $[e, p_1]$ does not f-cover $[e, p_s]$, $[e, a]$ does not f-cover $[e, p_s]$. By Lemma 2, $[e, a]$ does not f-cover $[e, p_n]$. Hence, $[e, a]$ f-covers $[p_n, e]$. In particular $f([e, a]) \supset [p_n, e]$.

Suppose that for some $z \in (e, a)$, $f(z) \notin [p_n, p_1]$. Since $f(z) \notin [p_n, p_1]$ and $f(e) \in [p_n, p_1]$, by continuity, for some $q \in (e, a)$, $f(q) = p_1$ or $f(q) = p_n$. Since $q \in (e, a)$ it follows from the choice of a that $f(q) \neq p_n$. Hence $f(q) = p_1$. Now, it follows from the choice of a, that $f([e, a])$ is a proper closed interval on S^1 and p_n is an endpoint of $f([e, a])$. Also, $e \in f([e, a])$ and $p_1 \in f([e, a])$. Hence, either $[p_n, p_1] \subset f([e, a])$ or $[e, p_n] \subset f([e, a])$. If $[e, p_n] \subset f([e, a])$, it follows as in the proof of

Lemma 2, using the fact that $f([e, a]) \neq S^1$, that $[e, a]$ f-covers $[e, p_n]$. This implies, by Lemma 2, that $[e, a]$ f-covers $[e, p_s]$. Thus, $[e, p_1]$ f-covers $[e, p_s]$, a contradiction. Hence $[p_n, p_1] \subset f([e, a])$. Since $f([e, a]) \neq S^1$, this implies that $[e, a]$ f-covers $[p_n, p_1]$. Thus, $[p_n, p_1]$ f-covers $[p_n, p_1]$.

We have shown that our claim holds if $f(z) \notin [p_n, p_1]$ for some $z \in (e, a)$. Hence, we may assume that $f([e, a]) \subset [p_n, p_1]$.

Since $[p_n, e]$ f-covers $[e, p_t]$, there is a point $b \in [p_n, e)$ such that $f(b) = p_1$ but $f(x) \neq p_1$ for all $x \in (b, e)$. It follows that $f([b, e]) \supset [e, p_1]$ (by the same argument used to show $f([e, a]) \supset [p_n, e]$). Also, we may assume that $f([b, e]) \subset [p_n, p_1]$ (by the same argument used to show that we may assume that $f([e, a]) \subset [p_n, p_1]$). Thus, $f([b, a]) = [p_n, p_1]$. Since $[b, a] \subset [p_n, p_1]$, this establishes our claim that $[p_n, p_1]$ f-covers $[p_n, p_1]$.

Now, since $[e, p_1]$ f-covers $[p_s, e]$, $[p_n, p_1]$ f-covers $[p_s, e]$. Also, since $[p_n, e]$ f-covers $[e, p_t]$, $[p_n, p_1]$ f-covers $[e, p_t]$. Hence, by Lemma 2, for any integer j with $1 \leqslant j \leqslant t - 1$ or $s \leqslant j \leqslant n - 1$, $[p_n, p_1]$ f-covers I_j.

Suppose that for some integer j with $1 \leqslant j \leqslant t - 1$ or $s \leqslant j \leqslant n - 1$, I_j f-covers I_n. Then the conclusion of the theorem holds by Lemma 7 (with $k = 2$, $M_1 = I_n$ and $M_2 = I_j$). Hence, we may assume that for every integer j with $1 \leqslant j \leqslant t - 1$ or $s \leqslant j \leqslant n - 1$, I_j does not f-cover I_n. Since I_j f-covers I_n for some integer j with $1 \leqslant j \leqslant n - 1$, this implies that $t < s$.

Note that we cannot have both $f(\{p_1, \ldots, p_t\}) \subset \{p_s, \ldots, p_n\}$ and $f(\{p_s, \ldots, p_n\}) \subset \{p_1, \ldots, p_t\}$. This follows from the fact that $\{p_1, \ldots, p_n\}$ is a periodic orbit and $t < s$, and uses the fact that n is odd in the case $t = s - 1$. Without loss of generality we may assume that $f(\{p_1, \ldots, p_t\})$ is not a subset of $\{p_s, \ldots, p_n\}$.

Let w be the smallest positive integer such that $f(p_w) \notin \{p_s, \ldots, p_n\}$. Then $2 \leqslant w \leqslant t$ and I_{w-1} f-covers the interval $[f(p_w), f(p_{w-1})]$ (since I_{w-1} does not f-cover I_n). Hence I_{w-1} f-covers the interval $[f(p_w), p_s]$. Let $f(p_w) = p_v$. Then $v \leqslant s - 1$ and I_{w-1} f-covers each interval I_j with $v \leqslant j \leqslant s - 1$.

Suppose for some integer j with $v \leqslant j \leqslant s - 1$, I_j f-covers I_n. Then the conclusion of the theorem holds by Lemma 7 (with $k = 3$, $M_1 = I_n$, $M_2 = I_{w-1}$ and $M_3 = I_j$).

By repeating the argument of the preceding four paragraphs at most n times, using the fact that for some $j \in \{1, \ldots, n - 1\}$ I_j f-covers I_n, we eventually obtain a collection of closed intervals $\{M_1, M_2, \ldots, M_k\}$ with $k \leqslant n$ such that the hypothesis of Lemma 7 is satisfied. Thus, the conclusion of the theorem follows from Lemma 7. Q.E.D.

4. Proof of Theorem B.

COROLLARY 8. *Let $f \in C^0(S^1, S^1)$. Suppose $m \in P(f)$ and $n \in P(f)$ where m and n are odd integers with $m \neq n$. Then $P(f)$ is infinite.*

PROOF. Without loss of generality, we may assume that $m < n$. Then $1 \in P(f^m)$ and for some odd integer $k > 1$, $k \in P(f^m)$. By Theorem A, $P(f^m)$ is infinite. Hence, $P(f)$ is infinite. Q.E.D.

COROLLARY 9. *Suppose $f \in C^0(S^1, S^1)$ and $P(f)$ is finite. Then for some integers m and n with $m \geq 1$ and $n \geq 0$, $P(f) \subset \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \ldots, 2^n \cdot m\}$.*

PROOF. Let m be the smallest element of $P(f)$. Since $P(f)$ is finite, it suffices to prove that if $k \in P(f)$ then $k = 2^i \cdot m$ for some nonnegative integer i.

Let $k \in P(f)$. Let $m = 2^r \cdot s$ where s is odd, $s \geq 1$ and $r \geq 0$, and let $k = 2^v \cdot w$ where w is odd, $w \geq 1$ and $v \geq 0$. Let j be the largest element of $\{2^r, 2^v\}$. Then $P(f^j)$ is finite, $s \in P(f^j)$, $w \in P(f^j)$ and s and w are odd. By Corollary 8, $s = w$.

Since m is the smallest element of $P(f)$ and $s = w$, we have $v \geq r$. Let $i = v - r$. Then $i \geq 0$ and $k = 2^v \cdot w = 2^v \cdot s = 2^{v-r} \cdot 2^r \cdot s = 2^i \cdot m$. Q.E.D.

LEMMA 10. *Let $f \in C^0(S^1, S^1)$ and suppose that $\{p_1, p_2, p_3, p_4\}$ is a periodic orbit of f of period 4, labeled in order. Suppose that one of the following holds.*

(i) $f(p_1) = p_2, f(p_2) = p_3, f(p_3) = p_4, f(p_4) = p_1$.

(ii) $f(p_1) = p_4, f(p_4) = p_3, f(p_3) = p_2, f(p_2) = p_1$.

Suppose also that $1 \in P(f)$. Then $5 \in P(f)$.

PROOF. Since (i) and (ii) are analogous, we may assume that (i) holds. Let I_1, I_2, I_3 and I_4 be the intervals determined by $\{p_1, p_2, p_3, p_4\}$.

By Lemma 1, either I_1 f-covers I_2 or I_1 f-covers $I_1 \cup I_3 \cup I_4$. Suppose I_1 f-covers $I_1 \cup I_3 \cup I_4$. If I_4 f-covers I_1, then it follows from Lemma 7 (with $k = 2$, $M_1 = I_1$ and $M_2 = I_4$) that $5 \in P(f)$. Also, if I_3 f-covers I_1, then it follows from Lemma 7 (with $k = 2$, $M_1 = I_1$ and $M_2 = I_3$) that $5 \in P(f)$. Hence we may assume that I_4 does not f-cover I_1 and I_3 does not f-cover I_1. This implies (by Lemma 1) that I_4 f-covers $I_2 \cup I_3 \cup I_4$ and I_3 f-covers I_4. Hence, by Lemma 7 (with $k = 2$, $M_1 = I_4$ and $M_2 = I_3$), $5 \in P(f)$.

Thus we may assume that I_1 f-covers I_2. Similarly, we may assume that I_2 f-covers I_3, I_3 f-covers I_4 and I_4 f-covers I_1.

By hypothesis f has a fixed point e. Without loss of generality we may assume that $e \in I_4$. Let $I_{4A} = [p_4, e]$ and let $I_{4B} = [e, p_1]$.

By Lemma 1, either I_{4B} f-covers $I_{4B} \cup I_1$ or I_{4B} f-covers $I_2 \cup I_3 \cup I_{4A}$. If I_{4B} f-covers $I_{4B} \cup I_1$, then it follows from Lemma 7 (with $k = 4$, $M_1 = I_{4B}$, $M_2 = I_1$, $M_3 = I_2$ and $M_4 = I_3$) that $5 \in P(f)$. Hence, we may assume that I_{4B} f-covers $I_2 \cup I_3 \cup I_{4A}$.

Also, by Lemma 1, either I_{4A} f-covers I_{4B} or I_{4A} f-covers $I_1 \cup I_2 \cup I_3 \cup I_{4A}$. If I_{4A} f-covers $I_1 \cup I_2 \cup I_3 \cup I_{4A}$, then it follows from Lemma 7 (with $k = 2$, $M_1 = I_{4A}$ and $M_2 = I_3$) that $5 \in P(f)$. Hence, we may assume that I_{4A} f-covers I_{4B}.

Now, we have that I_{4A} f-covers I_{4B}, I_{4B} f-covers I_3 and I_3 f-covers I_{4B}. By Lemma 4, there is a fixed point z of f^3 such that $z \in I_{4A}$, $f(z) \in I_{4B}$ and $f^2(z) \in I_3$. Since $I_{4A} \cap I_{4B} \cap I_3 = \emptyset$, z is not a fixed point of f. Hence, $3 \in P(f)$. Thus, by Theorem A, $5 \in P(f)$. Q.E.D.

LEMMA 11. *Let $f \in C^0(S^1, S^1)$. Suppose $1 \in P(f)$, $4 \in P(f)$ and $5 \notin P(f)$. Then $2 \in P(f)$.*

PROOF. Let $\{p_1, p_2, p_3, p_4\}$ be a periodic orbit of f of period 4, labeled in order. By Lemma 10, one of the following must hold.

(i) $f(p_1) = p_3, f(p_3) = p_2, f(p_2) = p_4$ and $f(p_4) = p_1$.

(ii) $f(p_1) = p_3, f(p_3) = p_4, f(p_4) = p_2$ and $f(p_2) = p_1$.

(iii) $f(p_1) = p_4, f(p_4) = p_2, f(p_2) = p_3$ and $f(p_3) = p_1$.

(iv) $f(p_1) = p_2, f(p_2) = p_4, f(p_4) = p_3$ and $f(p_3) = p_1$.

Note that (ii) is analogous to (i), because if (ii) holds and we let $q_1 = p_4$, $q_2 = p_1$, $q_3 = p_2$ and $q_4 = p_3$ then $f(q_1) = q_3, f(q_3) = q_2, f(q_2) = q_4$ and $f(q_4) = q_1$. Also, (iii) is analogous to (i) because if (iii) holds and we let $q_1 = p_4$, $q_2 = p_3$, $q_3 = p_2$ and $q_4 = p_1$ then $f(q_1) = q_3$, $f(q_3) = q_2$, $f(q_2) = q_4$ and $f(q_4) = q_1$. Finally, (iv) is analogous to (i), because if (iv) holds and we let $q_1 = p_2$, $q_2 = p_3$, $q_3 = p_4$ and $q_4 = p_1$ then $f(q_1) = q_3$, $f(q_3) = q_2$, $f(q_2) = q_4$ and $f(q_4) = q_1$. Hence, we may assume that (i) holds.

We claim that I_1 f-covers I_3. Suppose that I_1 does not f-cover I_3. By Lemma 1, I_1 f-covers $I_4 \cup I_1 \cup I_2$. If I_2 f-covers $I_4 \cup I_1$ then we obtain a contradiction (to the fact that $5 \notin P(f)$) by Lemma 7 (with $k = 2$, $M_1 = I_1$ and $M_2 = I_2$). Hence, by Lemma 1, I_2 f-covers $I_2 \cup I_3$. Also, if I_3 f-covers I_1 then we obtain a contradiction by Lemma 7 (with $k = 3$, $M_1 = I_1$, $M_2 = I_2$, $M_3 = I_3$). Hence, by Lemma 1, I_3 f-covers $I_2 \cup I_3 \cup I_4$. Again, by Lemma 7 (with $k = 2$, $M_1 = I_2$, $M_2 = I_3$) we obtain a contradiction. This establishes our claim that I_1 f-covers I_3.

We claim also that I_3 f-covers I_1. Suppose that I_3 does not f-cover I_1. Then I_3 f-covers $I_2 \cup I_3 \cup I_4$ by Lemma 1. If I_2 f-covers $I_2 \cup I_3$ then we obtain a contradiction by Lemma 7. Hence, by Lemma 1, I_2 f-covers $I_4 \cup I_1$. Since I_1 f-covers I_3, we again obtain a contradiction by Lemma 7 (with $k = 3$, $M_1 = I_3$, $M_2 = I_2$ and $M_3 = I_1$). This establishes our claim that I_3 f-covers I_1.

We have shown that I_1 f-covers I_3 and I_3 f-covers I_1. Since $I_1 \cap I_3 = \varnothing$, it follows from Lemma 4 that $2 \in P(f)$. Q.E.D.

THEOREM B. *Let $f \in C^0(S^1, S^1)$ and suppose that $P(f)$ is finite. Then there are integers m and n (with $m \geqslant 1$ and $n \geqslant 0$) such that $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \ldots, 2^n \cdot m\}$.*

PROOF. Let m be the smallest element of $P(f)$ and let k be the largest element of $P(f)$. By Corollary 9, $k = 2^n \cdot m$ for some nonnegative integer n and $P(f) \subset \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \ldots, 2^n \cdot m\}$.

If $n = 0$ or $n = 1$ the conclusion of the theorem follows immediately, so we may assume that $n \geqslant 2$. Let $r = 2^{n-2}$. Then $1 \in P(f^{r \cdot m})$ and $4 \in P(f^{r \cdot m})$. Also, since $P(f)$ is finite, $P(f^{r \cdot m})$ is finite. Hence, by Theorem A, $5 \notin P(f^{r \cdot m})$. By Lemma 11, $2 \in P(f^{r \cdot m})$. It follows from this, and the fact that $P(f) \subset \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \ldots, 2^n \cdot m\}$, that $2^{n-1} \cdot m \in P(f)$.

Now, if $n = 2$ the conclusion of the theorem follows immediately. If $n > 2$, it follows by the argument of the preceding paragraph (with $r = 2^{n-3}$ instead of $r = 2^{n-2}$) that $2^{n-2} \cdot m \in P(f)$. Thus, it follows by using this argument inductively, that $P(f) = \{m, 2 \cdot m, 4 \cdot m, 8 \cdot m, \ldots, 2^n \cdot m\}$. Q.E.D.

5. Proof of Theorem C.

PROPOSITION 12. *Let n be an integer with $n \geqslant 3$. There is a map $f \in C^0(S^1, S^1)$ such that $1 \in P(f)$ and $n \in P(f)$, but for every integer k with $1 < k < n$, $k \notin P(f)$.*

PROOF. Let $f \in C^0(S^1, S^1)$ with the following properties.

(1) f has a periodic orbit $\{p_1, \ldots, p_n\}$ of period n, labeled in order, with $f(p_i) = p_{i+1}$ for $i = 1, \ldots, n-1$, and $f(p_n) = p_1$.

(2) $f(I_j) = I_{j+1}$ for $j \in \{1, \ldots, n-1\}$, where $I_1, \ldots, I_n$ are the intervals determined by $\{p_1, \ldots, p_n\}$.

(3) f has a fixed point $e \in I_n$.

(4) $f([p_n, e]) = [e, p_1]$ and $f([e, p_1]) = [e, p_2]$.

(5) For any $x \in (e, p_1)$, $f(x) \neq x$ and $(x, f(x)) \subset (e, p_1)$.

By construction $1 \in P(f)$ and $n \in P(f)$. Also, by construction, if $x \in S^1$ such that e is not in the orbit of x, then for any $j \in \{1, \ldots, n\}$, there is a point in the orbit of x in I_j. Thus, for every integer k with $1 < k < n$, $k \notin P(f)$. Q.E.D.

LEMMA 13. *There is a map $f \in C^0(S^1, S^1)$ such that for every integer $n > 1$, $n \in P(f)$, but $1 \notin P(f)$.*

PROOF. Let $f \in C^0(S^1, S^1)$ with the following properties:

(1) f has a periodic orbit $\{p_1, p_2, p_3\}$, labeled in order, with $f(p_1) = p_2$, $f(p_2) = p_3$ and $f(p_3) = p_1$.

(2) There are points $c_1 \in (p_1, p_2)$, $c_2 \in (p_2, p_3)$ and $c_3 \in (p_3, p_1)$, such that $f(c_1) = p_1$, $f(c_2) = p_2$ and $f(c_3) = p_3$.

(3) $f([p_1, c_1]) = [p_2, p_1]$, $f([c_1, p_2]) = [p_3, p_1]$, $f([p_2, c_2]) = [p_3, p_2]$, $f([c_2, p_3]) = [p_1, p_2]$, $f([p_3, c_3]) = [p_1, p_3]$, $f([c_3, p_1]) = [p_2, p_3]$.

Note that by construction, $1 \notin P(f)$. Let n be any integer with $n > 1$. Define n closed intervals, $M_1, \ldots, M_n$, by $M_1 = [p_1, c_1]$, $M_k = [p_2, c_2]$ if k is even and $2 \leqslant k \leqslant n$, and $M_k = [p_3, c_3]$ if k is odd and $2 \leqslant k \leqslant n$. By Lemma 4, there is a fixed point z of f^n such that $z \in M_1$, $f(z) \in M_2, \ldots, f^{n-1}(z) \in M_n$. Since $[p_1, c_1] \cap [p_2, c_2] = \varnothing$ and $[p_1, c_1] \cap [p_3, c_3] = \varnothing$, z is a periodic point of f of period n. Thus, $n \in P(f)$. Q.E.D.

THEOREM C. *Let $f \in C^0(S^1, S^1)$. If $\{1, 2, 3\} \subset P(f)$ then $P(f) = N$. Conversely, if $S \subset N$ with the property that for any $f \in C^0(S^1, S^1)$, $S \subset P(f) \Rightarrow P(f) = N$, then $\{1, 2, 3\} \subset S$.*

PROOF. By Theorem A, if $\{1, 2, 3\} \subset P(f)$ then $P(f) = N$.

Suppose $S \subset N$ with the property that for any $f \in C^0(S^1, S^1)$, $S \subset P(f) \Rightarrow P(f) = N$. Since there are mappings g of the interval into itself such that $3 \notin P(g)$ but $k \in P(g)$ for every positive integer $k \neq 3$ (see **[3]** or **[4]**), there are mappings $f \in C^0(S^1, S^1)$ such that $3 \notin P(f)$, but $k \in P(f)$ for every positive integer $k \neq 3$. Thus $3 \in S$. By Proposition 12 (with $n = 3$), there is a map $f \in C^0(S^1, S^1)$ such that $1 \in P(f)$ and $3 \in P(f)$, but $2 \notin P(f)$. By Theorem A, for this map f, $P(f)$ consists of all positive integers except 2. Thus, $2 \in S$. Finally, it follows from Lemma 13, that $1 \in S$. Hence $\{1, 2, 3\} \subset S$. Q.E.D.

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