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ALGEBRAS OF FOURIER TRANSFORMS WITH CLOSED RESTRICTIONS

BY

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ABSTRACT. Let G denote a compact abelian group and let B denote a Banach subalgebra of A, the algebra of complex-valued functions on G whose Fourier series is absolutely convergent. If B contains the constant functions, separates the points of G, and if the restriction algebra, $B(E)$, is closed in $A(E)$ for every closed subset E of G, then $B = A$.

1. Introduction. Let G denote a compact abelian group and A the algebra under the pointwise operations of complex-valued functions on G having absolutely convergent Fourier series. Thus, a function f in A has a representation as $\Sigma\, a_n(\gamma_n, g)$ where $\Sigma\, |a_n| < \infty$, the latter quantity serving as the norm of f.

Let B denote a subalgebra continuously embedded in A, containing the constant functions, and separating the points of G. For E, a closed subset of G, $B(E)$ and $A(E)$ denote the restriction algebras to the set E and are furnished with the quotient norm.

THEOREM. *Suppose that for every closed set $E \subset G$, $B(E)$ is a closed linear subspace of $A(E)$. Then $B = A$.*

This result is the exact analogue of a result [**1**] in function algebras with the role of the continuous complex-valued functions played by A. In his thesis [**4**] Sungwoo Suh proved the theorem for locally compact totally disconnected groups.

To illustrate the content of the Theorem, consider the case of G equal to T, the circle group, and let B be the closure of the linear span of A-functions taking the values 1 and 0 on a closed totally disconnected set $F \subset T$. It is known (see [**2**, p. 40], and [**3**]) that there are certain F for which B is a proper subalgebra of A. From our Theorem it follows that $B(E)$ fails to be closed in $A(E)$ for some closed E that must intersect F in a proper subset. A qualitative statement of this fact is the following. There are functions locally constant on E that are the restrictions of A-functions having small A-norm, but such that any attempt to extend them to remain locally constant on F will force their A-norms to become large.

We now indicate some notations and tools to be used in the proof of the Theorem. The norm of a function $f \in B(E)$ is given by

$$\|f\|_{B(E)} = \inf\{\|f + h\|_B : h \in B \text{ and } h = 0 \text{ on } E\}.$$

If $f \in B(E)$, a *representative* of f is a function f_0 belonging to B such that $f_0|E = f$.

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The dual space of A as a Banach space is PM, the space of pseudomeasures on G. The action of a pseudomeasure S on an A-function f will be denoted by (S, f). The symbol Sf will denote that pseudomeasure such that $(Sf, g) = (S, fg)$ for all $g \in A$. A pseudomeasure S is said to be supported by E, and we write $S \in PM(E)$, provided $(S, f) = 0$ whenever f vanishes in a neighborhood of E. The dual space of the restriction algebra $A(E)$ consists of all those pseudomeasures belonging to $PM(E)$ that annihilate every A-function vanishing on E. Finally, $B^{\perp}$ will denote all of those pseudomeasures that annihilate every function in B.

2. Preliminary lemmas. We begin by showing that under the assumptions of the Theorem the restriction algebra $B(E)$ must contain all of the idempotents of $A(E)$. In the function algebra case (cf. [**1**]) this step is almost immediate. In this section G need not be a group, only a compact topological space. The algebra A need only be a commutative semisimple regular Banach algebra.

Let A be such an algebra under the pointwise operations on its maximal ideal space G, and let B denote a closed subalgebra of A containing 1 and separating the points of G.

LEMMA 1. *Assume that for every closed subset $E \subset G$ the inclusion mapping $B(E) \to A(E)$ is closed. Then, for every closed subset $E \subset G$, $B(E)$ contains all the idempotent functions on E belonging to $A(E)$.*

PROOF. Let E be an arbitrary closed subset of G. We will show that the maximal ideal space of $B(E)$ is E. Thus, appealing to the Shilov idempotent theorem, we have that if f belongs to $A(E)$ and is idempotent on E, then it must also belong to $B(E)$.

Let h denote a complex homomorphism of $B(E)$, and suppose that h is not evaluation at some point of E. Then for each $x \in E$, there exists a function g belonging to $B(E)$ such that $h(g) = 1$ and such that $g(x) = 0$. Let W_x be a closed neighborhood of x such that $|g(y)| \leqslant 1/4$ for all $y \in W_x$. Of course, the maximal ideal space of $B(E \cap W_x)$, Δ, is a closed subset of the maximal ideal space of $B(E)$. We shall now show that h does not belong to Δ.

Suppose on the contrary that $h \in \Delta$, and define $a = \sup_{\phi \in \Delta}|\phi(g)|$. Then $a \geqslant 1$, and for every positive integer n

$$\|(g/a)^n\|_{B(W_x \cap E)} \geqslant 1. \tag{1}$$

Since $|g(y)| \leqslant 1/4$ for all $y \in W_x$, and the maximal ideal space of $A(W_x \cap E)$ is just $W_x \cap E$, it follows that

$$\lim\sup\|(g/a)^n\|_{A(W_x \cap E)^{1/n}} \leqslant 1/4.$$

Thus, for n large we have

$$\|(g/a)^n\|_{A(W_x \cap E)} \leqslant (1/2)^n. \tag{2}$$

Now, (1) and (2) together contradict the closedness of the mapping $B(W_x \cap E) \to A(W_x \cap E)$. This establishes that h does not belong to Δ.

Therefore, there is a function $g \in B(E)$ that vanishes on W_x such that $h(g) = 1$. We may repeat the above argument for each point of E, and by the compactness of

E choose a finite set of functions g_i, $i = 1, \ldots, n$, so that $h(g_i) = 1$ and so that the product g_0 vanishes on E. Of course, $h(g_0) = 1$. This contradicts the fact that h was chosen in the maximal ideal space of $B(E)$. The proof of the lemma is complete.

DEFINITION. Let N_E denote the norm of the inverse of the inclusion mapping $B(E) \to A(E)$. Following **[1]**, we say that B is *bounded on a subset* V of G if there is a positive constant C_V for which, whenever F is closed and contained in V, $N_F \leqslant C_V$. We say that B is bounded at $x \in G$ if B is bounded on some neighborhood V of x.

LEMMA 2. *Let V_1 and V_2 be open subsets of G and let B be bounded on each V_i, $i = 1, 2$. Then B is bounded on every closed subset of $V_1 \cup V_2$.*

PROOF. Choose a closed subset F of $V_1 \cup V_2$. Since $F \setminus V_2$ is a closed subset of V_1, we can find open $W_1 \supset F \setminus V_2$ such that $\overline{W}_1 \subset V_1$. Since $F \setminus W_1 \subset V_2$, there is an open $W_2 \supset F \setminus W_1$ such that $\overline{W}_2 \subset V_2$. Now $F \subset W_1 \cup W_2$. Furthermore, we may choose open L satisfying $F \setminus W_2 \subset L \subset \overline{L} \subset W_1$, and by Lemma 1 a function $h \in B$ that is 1 on $F \setminus W_2$ and 0 on L^c. For $f \in B(F)$ there are $f_i \in B$, $i = 1, 2$, such that $f_i | \overline{W}_i = f | \overline{W}_i$ and $\|f_i\|_B < C_{V_i}$. But now it is easy to check that $f = f_1 \cdot h + (1 - h) f_2$ on F. We have, therefore

$$\|f\|_{B(F)} \leqslant \|f_1\|_B \|h\|_B + \|1 - h\|_B \|f_2\|_B < \|h\|_B C_{V_1} + \|1 - h\|_B C_{V_2}. \qquad (3)$$

The right-hand side of (3) may be taken to be C_F. The proof of the lemma is now complete.

LEMMA 3. *If F is closed in G, and B is bounded at each $x \in F$, there is an open V containing F on which B is bounded.*

PROOF. The proof is an obvious consequence of compactness of F and Lemma 2.

LEMMA 4. *There are at most finitely many $x \in G$ at which B is not bounded.*

PROOF. Assume on the contrary that B is not bounded at each member of the sequence $\{y_n\}$. By dropping to a subsequence, if necessary, we may suppose that no element of the sequence $\{y_n\}$ is a limit point of the sequence. Choose open neighborhoods U_{y_1}, W_{y_1} of y_1 that are disjoint from the set of limit points of the sequence $\{y_n\}$, such that $\overline{W}_{y_1} \subset U_{y_1}$. By Lemma 1 we may choose a function $k_1 \in B$ that is 1 on $\overline{W}_{y_1}$ and 0 on $U_{y_1}^c$. Since B is unbounded in every neighborhood of y_1, we may pick closed $F_1 \subset W_{y_1}$ so that there is a $g_1 \in B(F_1)$ with $\|g_1\|_{B(F_1)} > 1$ while having $\|g_1\|_{A(F_1)} < \|k_1\|_A^{-1}$. Set $f_1 = h_1 k_1$ where $h_1 | F_1 = g_1$ and h_1 is a representative of g_1 whose A-norm is less than $\|k_1\|_A^{-1}$. Then $\|f_1\|_A \leqslant \|h_1\|_A \|k_1\|_A < 1$, but, of course, $\|f_1\|_{B(F_1)} > 1$ since $f_1 = g_1$ on F_1. Note that f_1 itself may not belong to B, since h_1 need not.

Next choose $y_2' \in \{y_n\}_{n=2}^{\infty}$ and open neighborhoods $U_{y_2'}$, $W_{y_2'}$, of y_2' that are disjoint from the set of limit points of the sequence $\{y_n\}$, such that $\overline{W}_{y_2'} \subset U_{y_2'}$ and $U_{y_1} \cap U_{y_2'} = \varnothing$. Choose a function $k_2 \in B$ such that k_2 is 1 on $\overline{W}_{y_2'}$ and is 0 on $U_{y_2'}^c$. Since B is not bounded in every neighborhood of y_2', it follows that there is a closed set $F_2 \subset W_{y_2'}$ and a function $g_2 \in B(F_2)$ with $\|g_2\|_{B(F_2)} > 2$, but $\|g_2\|_{A(F_2)} < \|k_2\|_A^{-1}$.

Set $f_2 = h_2 \cdot k_2$ where $h_2|F_2 = g_2$ and h_2 is a representative of g_2 whose A-norm is less than $\|k_2\|_A^{-1}$. Then $\|f_2\|_A < 1$, but, of course, $\|f_2\|_{B(F_2)} > 2$. Continuing in this way, we arrive at a disjoint sequence of closed sets $\{F_n\}$ and a sequence of functions $\{f_n\}$ such that

$$\|f_n\|_A < 1, \qquad \|f_n\|_{B(F_n)} > n, \qquad n = 1, 2, \ldots . \tag{4}$$

Denote by F the closed set $\bigcup_{n=1}^{\infty} F_n \cup F_0$, where F_0 is the set of limit points of $\bigcup_{n=1}^{\infty} F_n$. For each m, $f_m|F_m \in B(F_m)$ and $f_m = 0$ on the closed set $\bigcup_{n \neq m} F_n \cup F_0$. The latter set is disjoint from F_m. It follows from Lemma 1, by multiplying any representative of $f_m|F_m$ in B by a function in B equal to 1 on F_m and 0 on $\bigcup_{n \neq m} F_n \cup F_0$, that $f_m \in B(F)$. Therefore, the closedness of the inclusion mapping $B(F) \to A(F)$ is contradicted by (4). The proof of the lemma is complete.

3. Proof of the Theorem for the circle T. Assume on the contrary that $B \neq A$ and that H is the finite set of points at which B is not bounded. Our first claim is that there is a closed set F such that F^c contains H and such that $B(F) \neq A(F)$. If not, then for every such F, $B(F) = A(F)$. Let S denote a pseudomeasure in $B^{\perp}$, and suppose that f belongs to A and is 0 in a neighborhood of H. It follows from Lemma 1 that B is a normal algebra of functions on T. Therefore, since f belongs locally to B at each point of T, f belongs to B. Since S is orthogonal to B, $(S, f) = 0$. Thus the support of S is the finite set H, and S is in fact a measure. Since B separates the points of T, it follows that $S\{p\} = 0$ for every $p \in H$. This establishes that $B(F) \neq A(F)$ for some closed set F such that F^c contains H. Henceforth we shall assume that F^c is a finite union of disjoint open intervals centered at the points of H. Then, by Lemma 4, B is bounded, say with constant N_W, on some open set W containing F and missing H.

Now, at least one of the functions $e^{\pm ix}$, say e^{ix}, when restricted to F, fails to belong to $B(F)$. By the Hahn-Banach Theorem there is a nonzero element S of $A(F)^*$ that annihilates $B(F)$ and such that $(S, e^{ix}) = 1$. We shall regard S as a pseudomeasure supported on F.

Let m be a fixed positive integer satisfying

$$\exp(20/m) < 1 + \left(8e^{20}\|S\|N_W^2 C\right)^{-1}, \tag{5}$$

where C denotes the norm of the mapping $B \to A$.

Let σ denote the trapezoidal function equal to 1 on the interval $(-\pi/4, \pi/4)$, zero outside the interval $(-\pi/2, \pi/2)$ and linear on intermediate intervals. Set $V_1(x) = \sigma(mx)$ and $V_2 = 1 - V_1$. From standard estimates it follows that $\|V_j\|_A \leqslant 4$ for $j = 1, 2$. By Lemma 1 and the boundedness of B on W, there is a B-function v_1 with its B-norm bounded by $4N_W$, and such that $v_1 = V_1$ on the intersection of F with the intervals where V_1 is constant. Set $v_2 = 1 - v_1$.

Since $(S, e^{ix}) = 1$, it follows that for at least one j, $|(Sv_j, e^{ix})| \geqslant 1/2$. There will be no loss of generality in assuming that this occurs for $j = 1$. The ensuing argument is similar in the other case. Let K denote the support of the function v_1. Of course, since S is orthogonal to B, so also is the pseudomeasure $L = Sv_1$. The latter's support is contained in the finite union of closed intervals $F \cap K$, and its pseudomeasure norm satisfies

$$\|L\| \leqslant \|S\| \, \|v_1\|_A \leqslant 4\|S\| N_W C. \tag{6}$$

Let $\zeta(x)$ denote the continuous piecewise linear function such that $\zeta(-\pi/2) = 0$, $\zeta(\pi/2) = \pi$ and ζ has slope -1 on the intervals complementary to $(-\pi/2, \pi/2)$. Set $Z(x) = \zeta(mx)$. By standard estimates the A-norm of Z is no greater than 20. The important property for present purposes of the function Z is that the restriction of the function $e^{iZ/m}$ to the support of v_1 is just $\exp\{i(x + \lambda_j)\}$ for constants λ_j and x belonging to the jth interval of support of $v_1, j = 1, 2, \ldots, m$.

The A-norm of the function $e^{i(x-Z/m)}$ is no greater than e^{20}. On the set $F \cap K$ it is equal to the constant $e^{-i\lambda_j}$ for x belonging to the jth interval of K. In particular, it is locally constant on $F \cap K$ and so belongs to $B(F \cap K)$. Its $B(F \cap K)$-norm is no greater than $N_W e^{20}$. Let $\alpha(x)$ be a representative in B of the restriction of $e^{i(x-Z/m)}$ to the set $F \cap K$ satisfying

$$\|\alpha\|_B \leqslant N_W e^{20}. \tag{7}$$

Likewise, let $\alpha_1(x)$ be a representative in B of $e^{-i(x-Z/m)}$ restricted to $F \cap K$. Thus, we have that

$$1/2 \leqslant |(L\alpha\alpha_1, e^{ix})| = |(L\alpha, \alpha_1 e^{ix})| = |(L\alpha, e^{iZ/m})|. \tag{8}$$

Since $\alpha \in B$, the pseudomeasure $L\alpha$ is still orthogonal to B.

We have

$$\|e^{iZ/m} - 1\|_{A(F\cap K)} \leqslant \exp\{\|Z\|_A/m\} - 1. \tag{9}$$

From the triangle inequality we see that

$$|(L\alpha, e^{iZ/m})| \leqslant |(L\alpha, e^{iZ/m} - 1)| + |(L\alpha, 1)|. \tag{10}$$

Since $L\alpha$ is orthogonal to B, the second term on the right-hand side of (10) is 0. Since the set $F \cap K$ is a finite union of closed intervals, $L\alpha$ may be regarded as an element of the dual space of $A(F \cap K)$. Therefore, the first term on the right-hand side of (10) is at most $\|L\| \, \|\alpha\|_A \|e^{iZ/m} - 1\|_{A(F\cap K)}$. By our choice of m in (5) and by inequalities (6), (7) and (9), it follows that the right-hand side of (10) is less than 1/2. This contradicts (8) and completes the proof of the Theorem for the case of the circle.

4. Proof of the Theorem for an arbitrary compact abelian group. First we consider a product of circles $\prod_\alpha T_\alpha$. Let f denote the element in the dual group constant in every coordinate except the βth coordinate, and of the form e^{ix} in the βth coordinate. Consider the products $E = \prod_{\alpha\neq\beta} T_\alpha \times E_\beta$, where the projection onto the βth coordinate is a closed set E_β.

Since $B(F)$ is closed in $A(F)$ for all F closed and contained in $\prod_\alpha T_\alpha$, it follows from Lemma 1 that $B(E)$ contains the span of the idempotents of $A(E)$. By Lemma 4 B is bounded on the complement of any neighborhood of a certain finite set of points. To deny that f belongs to B leads to a contradiction as before. Therefore, all characters having the form of f belong to B, and since B is closed in A it must equal A.

Finally, let G be an arbitrary compact abelian group. Then every neighborhood of the identity contains a closed subgroup H such that G/H is isomorphic to a (finite) product of circles with a finite group. By the above remarks and Lemma 1, B will contain those characters of the dual group G/H. If $B(G/H)$ is interpreted to mean those B-functions constant on the cosets of H, it follows that $B(G/H) = A(G/H)$. In particular, B contains all trigonometric polynomials, and by the closedness of B in A, we again conclude that $B = A$.

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