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LIE COHOMOLOGY OF REPRESENTATIONS OF NILPOTENT LIE GROUPS AND HOLOMORPHICALLY INDUCED REPRESENTATIONS

BY

RICHARD PENNEY

ABSTRACT. Let U be a locally injective, Moore-Wolf square integrable representation of a nilpotent Lie group N. Let $(\mathcal{H}, \lambda)$ be a complex, maximal subordinate pair corresponding to U and let $\mathcal{H}_0 = \ker \lambda \cap \mathcal{H}$. The space $C^\infty(U)$ of differentiable vectors for U is an $\mathcal{H}_0$ module. In this work we compute the Lie algebra cohomology $H^p(\mathcal{H}_0, C^\infty(U))$ of this Lie module. We show that the cohomology is zero for all but one value of p and that for this specific value the cohomology is one dimensional. These results, when combined with earlier results of ours, yield the existence and irreducibility of holomorphically induced representations for arbitrary (nonpositive), totally complex polarizations.

I. Introduction. Let $\mathcal{H}$ be a connected, simply connected, nilpotent Lie algebra over the complex field. Let M be an $\mathcal{H}$ module over $\mathbf{C}$. By $\Lambda^p(\mathcal{H}, M)$ we shall mean the space of M valued, alternating, *complex* p-linear forms on $\mathcal{H}$. For $f \in \Lambda^p(\mathcal{H}, M)$, $X \in \mathcal{H}$ we define the "canonical action" of X on f by

$$(Xf)(X_1, \ldots, X_p) = X(f(X_1, \ldots, X_p)) - \sum_1^p f(X_1, \ldots, [X, X_i], \ldots, X_p).$$

This action defines an $\mathcal{H}$-module structure on $\Lambda^p(\mathcal{H}, M)$. For $X \in \mathcal{H}$ we also define a mapping $\delta_X\colon \Lambda^p(\mathcal{H}, M) \to \Lambda^{p-1}(\mathcal{H}, M)$ by

$$\delta_X f(X_1, \ldots, X_{p-1}) = f(X, X_1, \ldots, X_{p-1}).$$

We define the usual derivation $\partial\colon \Lambda^p(\mathcal{H}, M) \to \Lambda^{p+1}(\mathcal{H}, M)$ inductively by the equation

$$\delta_X(\partial f) = Xf - \partial(\delta_X f)$$

for all $X \in \mathcal{H}$. (See Hochschild-Serre [3].) It is easily seen that

$$\partial f(X_1, \ldots, X_{p+1}) = \sum (-1)^i X_i(f(X_1, \ldots, \tilde{X_i}, \ldots, X_{p+1})) - \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_1, \ldots, \tilde{X_i}, \ldots, \tilde{X_j}, \ldots, X_{p+1}).$$

The corresponding cohomology groups are denoted $H^p(\mathcal{H}, M)$.

In this paper we are interested in a specific case of the above construction. Let N be a connected, simply connected nilpotent Lie group over $\mathbf{R}$ with Lie algebra $\mathcal{N}$. Let U be an irreducible locally injective representation of N which is square

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integrable modulo its kernel. Let $C^\infty(U)$ be the space of infinitely differentiable vectors for U. (See [9].) For $v \in C^\infty(U)$ and $X \in \mathfrak{N}$, we set

$$Xv = \partial U(X)v = \frac{d}{dt}\Big|_{t=0} U(\exp tX)v.$$

This defines an $\mathfrak{N}$-module structure on $C^\infty(U)$. This $\mathfrak{N}$-module structure will be extended as usual to an action of complexification $\mathfrak{N}_c$ of $\mathfrak{N}$ on $C^\infty(U)$.

Now let $\lambda \in \mathfrak{N}^*$ be a linear functional which corresponds to U under the Kirillov correspondence [4]. We extend λ to a complex linear functional (also denoted λ) on $\mathfrak{N}_c$. A complex subalgebra $\mathcal{H}$ of $\mathfrak{N}_c$ is subordinate to λ if λ is null on $[\mathcal{H}, \mathcal{H}]$. $\mathcal{H}$ is maximal subordinate if $\mathcal{H}$ has maximal dimension among all such subalgebras. Let $\mathcal{H}_0 = \mathcal{H} \cap \ker \lambda$. $\mathcal{H}_0$ is a complex subalgebra of $\mathcal{H}$ which is called the reduced subalgebra. In this work we are interested in computing $H^p(\mathcal{H}_0, C^\infty(U))$ for arbitrary maximal subordinate pairs $(\mathcal{H}, \lambda)$ corresponding to U.

We shall explain our interest in this subject momentarily. First. however, we need some notation. The form $B_\lambda(X, Y) = \lambda([X, Y])$ on $\mathfrak{N}_c \times \mathfrak{N}_c$ is called the antisymmetric form of the pair $(\mathcal{H}, \lambda)$ while the form $\phi_\lambda(X, Y) = -i\lambda([X, \bar{Y}])$ on $\mathcal{H}_0 \times \mathcal{H}_0$ is called the Hermitian form of the pair. The subalgebra $\mathfrak{R}_\lambda$ of elements of $\mathfrak{N}_c$ annihilated by B_λ is called the radical of the pair. A fundamental result of Moore and Wolf [5] says that a locally injective representation U is square integrable iff $\mathfrak{R}_\lambda$ is the center of $\mathfrak{N}_c$. Local injectivity also implies that the center is one dimensional. It is easily seen that ϕ_λ is Hermitian symmetric. It follows that there is a basis $X_1, \ldots, X_d$ of $\mathcal{H}_0$ such that $\phi_\lambda(X_i, X_j) = \varepsilon_{ij}$ where ε_{ij} is zero if $i \neq j$ and $\varepsilon_{ij} \in \{0, -1, +1\}$ for all i. The number p_0 of positive values of ε is called the signature of the pair $(\mathcal{H}, \lambda)$ and the number $d - p_0 = q_0$ is called the deficit of the pair. Our main result is the following

THEOREM 1. *Let U be a locally injective square integrable representation of N and let $(\mathcal{H}, \lambda)$ be a complex, maximal subordinate pair corresponding to U. Then $H^p(\mathcal{H}_0, C^\infty(U)) = 0$ if $p \neq q_0$ where q_0 is the deficit of the pair $(\mathcal{H}, \lambda)$. Furthermore $H^{q_0}(\mathcal{H}_0, C^\infty(U))$ is one dimensional and the image of $\Lambda^{q_0-1}(\mathcal{H}_0, C^\infty(U))$ in $\Lambda^{q_0}(\mathcal{H}_0, C^\infty(U))$ under ∂ is closed in the $C^\infty(U)$ topology.*

The $C^\infty(U)$ topology is defined as the weakest topology making all maps $v \to X_1X_2 \cdots X_n v$ continuous from $C^\infty(U)$ into the representation space of U where $X_i \in \mathfrak{N}$ and $n \in \mathbf{N}$. The corresponding topology on $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ is obtained by identifying $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ with $\Lambda^p((\mathcal{H}_0)^*) \otimes C^\infty(U)$.

Our interest in this theorem is that it implies the existence and irreducibility of holomorphic induction for not necessarily positive polarizations of nilpotent Lie groups. Specifically, suppose $\mathcal{H}$ satisfies, in addition to the above assumptions,

(i) $\mathcal{H} + \bar{\mathcal{H}} = \mathfrak{N}_c$,

(ii) $\mathcal{H} \cap \bar{\mathcal{H}} = \mathfrak{R}_\lambda = \mathfrak{Z}(\mathfrak{N}_c)$.

Such subalgebras are called totally complex. In our previous work [8], we showed that the existence and irreducibility of holomorphic induction for the pair $(\mathcal{H}, \lambda)$ was equivalent to the fact that $H^p(\mathcal{H}_0, \lambda)$ is one dimensional when p equals the

deficit and zero otherwise (Theorems 10 and 2). We refer the reader to **[8]** for details and applications.

The existence and irreducibility of holomorphic induction has been proven previously only in special cases. Camora **[1]** and Satake **[10]** proved it for the Heisenberg group. Moscovici **[7]** proved it for groups which admit rational forms in the case that $(\mathcal{H}, \lambda)$ satisfies a certain rationality assumption and λ is "sufficiently distant" from 0. In our previous work **[8]**, we established the existence and irreducibility in the case that $\mathcal{H}$ is abelian. Moscovici has also shown the vanishing part of our Theorem 1 above in the case that λ is "sufficiently distant" from zero **[7]**.[1]

Several interesting features of Theorem 1 are worth mentioning. First, the theorem does not assume that the subalgebra $\mathcal{H}$ is totally complex. One can in fact use our theorem to prove irreducibility in the case $\mathcal{H}$ is real. This suggests that there should be a way of defining realizations of U corresponding to arbitrary complex maximal subordinate pairs $(\mathcal{H}, \lambda)$. Actually, the elimination of the totally complex assumption seems to be an essential ingredient of the proof of Theorem 1, as our induction scheme forces us out of the totally complex case.

The second interesting comment concerning Theorem 1 is that its proof is constructive: Given a form f which is homologous to zero we could (given time) explicitly construct a form f_0 such that $\partial f_0 = f$.

II. Proofs. Our main tool in the proof of Theorem 1 is the lemma stated below. Let $C^p, p \geqslant 0$ be a differential complex of vector spaces over $\mathbf{C}$ and let $\partial\colon C^* \to C^*$ and $\delta\colon C^* \to C^*$ be differentials of degree $+1$ and -1 respectively. Let $X = \partial\delta + \delta\partial$. It is easily seen that X commutes with ∂ and δ and X is of degree 0. Let

$$C_1^p = (\delta C^{p+1} + XC^p)/XC^p = \delta C^{p+1}/\delta C^{p+1} \cap XC^p.$$

The space $\delta C^{p+1} + XC^p$ is ∂ invariant since $\partial\delta = X - \delta\partial$ and ∂ commutes with X. It follows that ∂ induces a derivation ∂_X of the complex C_1^p.

There is also a dual concept related to the kernel of X. Let $C_0^p = \ker X \cap \ker \delta$.

On C_0^p, δ and ∂ anticommute so again ∂ gives rise to a derivation ∂_X of C_0^p.

LEMMA 2. *If X is injective on each C^p then δ induces an isomorphism between $H^p(C^*)$ and $H^{p-1}(C_1^*)$ for $p > 0$. Also $H^0(C^*) = 0$.*

If X is surjective on each C^p then the injection of C_0^p in C^p induces an isomorphism between $H^p(C_0^*)$ and $H^p(C^*)$ for all p.

PROOF. We first consider the injective case. The identity $\partial\delta = -\partial\delta + X$ shows that δ anticommutes with ∂ modulo the image of X so δ induces a chain map δ_X of C^p into C_1^{p-1}. Let the corresponding map on cohomology be δ_X^*.

To see that δ_X^* is injective, suppose f is a ∂ closed element of C^p and δf is a boundary in C_1^p. Then $\delta f = \partial g + Xh$ for some $g \in \delta C^p$. We claim that $\partial h = f$ so f is a boundary in C^p. In fact

$$X\partial h = \partial Xh = \partial\delta f = Xf - \delta\partial f = Xf.$$

[1]ADDED IN PROOF. Recently J. Rosenberg has also proven the general result using some of our results below.

Hence $\partial h = f$ as claimed.

To see that δ_X^* is surjective, let $g = \delta g_1 + XC^p$ be a ∂_X closed coset of C_1^p. Then $\partial\delta g_1 = Xh$ in C^{p+1}. We claim that h is closed in C^{p+1} and $\delta h = g \bmod XC^p$. h is closed because $X\partial h = \partial Xh = \partial^2\delta g_1 = 0$. Also

$$X\delta h = \delta Xh = \delta\partial\delta g_1 = (\delta\partial + \partial\delta)\delta g_1 = X\delta g_1.$$

Hence $\delta h = \delta g_1 = g \bmod XC^p$, as claimed.

If $p = 0$, $X = \delta\partial$; so X injective implies $\ker\partial = 0$ as claimed.

Now suppose X is surjective. Let i: $C_0^p \to C^p$ be the injection. i is a chain map. Let i^* be the induced map on cohomology. We first show that i^* is surjective. Let $f \in C^p$, $\partial f = 0$. Let $h \in C^p$ be such that $Xh = f$. Then we claim that $f - \partial\delta h = g$ belongs to C_0^p. Clearly i^* maps the class of g onto that of f. To see that g belongs to C_0^p, we compute

$$\delta g = \delta f - \delta\partial\delta g = \delta f - \delta(X - \delta\partial)g = \delta f - \delta f = 0,$$
$$Xg = Xf - \partial\delta Xh = \partial\delta f - \partial\delta f = 0.$$

To see that i^* is injective suppose $g \in C_0^p$ and $g = \partial f$ for some $f \in C^{p-1}$. We claim $g = \partial h$ for some h in C_0^{p-1}. To see this, let h_0 be such that $Xh_0 = f$ and set $h = f - \partial\delta h_0$. Then $g = \partial h$. To see that $h \in C_0^{p-1}$, we again compute

$$\delta h = \delta f - \delta\partial\delta h_0 = \delta f - \delta Xh_0 = \delta f - \delta f = 0,$$
$$Xh = Xf - \delta\partial Xh_0 = Xf - \delta\partial f = \delta\partial f - \delta\partial f = 0.$$

This proves g is null in $H^p(C_0^*)$ and hence our lemma is proven.

We shall also require a topological version of the above proposition. Suppose that the C^p are locally convex topological vector spaces and that δ and ∂ are both continuous. We endow $H^p(C^*)$, C_i^p and $H^p(C_i^*)$, $i = 0, 1$, with their respective quotient (subspace) topologies (which may be non-Hausdorff). The mappings δ_X^* and i^* are continuous, bijective mappings of vector spaces.

LEMMA 3. *If X has a continuous inverse W defined on the closure of the image of X, then δ_X^* is a topological isomorphism. If X is open, i^* is a topological isomorphism.*

PROOF. From the proof of the above proposition, the inverse of δ_X^* is given by mapping cosets of the form $g + XC^p$ into the cohomology class of $W\partial g$. This is clearly continuous.

To prove the statement about i^*, let X be an open mapping. We shall show that i^* is open. Let $\mathfrak{U}_0$ be an open subset of $H^p(C_0^*)$. By definition of the quotient topology, $\mathfrak{U}_0$ is a projection to $H^p(C_0^*)$ of a set of the form $\mathfrak{U} \cap C_0^p \cap \ker\partial$ where $\mathfrak{U}$ is open in C^p. $i^*\mathfrak{U}_0$ will be open iff $\mathfrak{U} \cap C_0^p \cap \ker\partial + \partial C^{p-1}$ is open in $\ker\partial$. From the proof of Lemma 2, if h is any element of C^p and $\partial Xh = 0$, then $Xh - \partial\delta h$ belongs to $C_0^p \cap \ker\partial$. Let $\tilde{\mathfrak{U}} = (X - \delta\partial)^{-1}\mathfrak{U}$. Then $\tilde{\mathfrak{U}}$ is open so $X\tilde{\mathfrak{U}}$ is open in C^p. Thus $X\tilde{\mathfrak{U}} \cap \ker\partial$ is open in $\ker\partial$. It $Xh \in X\tilde{\mathfrak{U}} \cap \ker\partial$ for some $h \in \tilde{\mathfrak{U}}$, then $Xh - \partial\delta h \in \mathfrak{U} \cap \ker\partial \cap C_0^p$ so

$$X\tilde{\mathfrak{U}} \cap \ker\partial + \partial C^{p-1} \subset \mathfrak{U} \cap \ker\partial \cap C_0^p + \partial C^{p-1}.$$

In fact it is not difficult to see that the "$\subset$" above is really an equality. This proves the lemma.

In our applications of the above lemma, ∂ will be the derivation of $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ and δ will be the map $\delta_X f(X_1 \cdots X_{p-1}) = f(X, X_1, \ldots, X_{p-1})$ for some fixed $X \in \mathcal{H}_0$. The inductive definition of ∂ shows that $\partial\delta_X + \delta_X\partial$ is simply the usual action of X on $\Lambda^p(\mathcal{H}_0, C^\infty(U))$. Hence $\partial\delta_X + \delta_X\partial = X$. The crux of our proof of Theorem 1 will be to find appropriate X's and to describe C_0^p or C_1^p as the case may be.

We now proceed to the proof of Theorem 1.

PROOF. Let $\mathcal{Z}(\mathcal{N})$ denote the center of N. $\mathcal{Z}(\mathcal{N})$ is one dimensional by virtue of the local injectivity of U. There is an element $Y \in \mathcal{N}$ such that $Y \notin \mathcal{Z}(\mathcal{N})$ but $[\mathcal{N}, Y] \subset \mathcal{Z}(\mathcal{N})$. We may pick Y so that $\lambda(Y) = 0$. Let $\mathcal{N}_1$ be the centralizer of Y in $\mathcal{N}$. $\mathcal{N}_1$ is the kernel of ad Y. $\mathcal{N}_1$ has codimension one in $\mathcal{N}$ as ad Y has a one-dimensional image. Hence $\mathcal{N}_1$ is an ideal in $\mathcal{N}$. Let N_1 be the corresponding subgroup of N. Let $\lambda_1 = \lambda|\mathcal{N}_1$ and let U_1 be the irreducible representation corresponding to λ_1 under the Kirillov correspondence. Let U_1 be realized in the Hilbert space $\mathcal{H}_1$. It is a well-known consequence of Kirillov theory that U is equivalent with the representation induced from N_1 by U_1. Let $X_0 \in \mathcal{N}$ be complementary to $\mathcal{N}_1$. X_0 may be chosen so $\lambda([X_0, Y]) = 1$. Let $Z = [X_0, Y]$. We use X_0 as usual to realize U in $L^2(\mathbf{R}, \mathcal{H}_1)$ by restriction of functions to the one-parameter subgroup defined by X_0. The action of U on $L^2(\mathbf{R}, \mathcal{H}_1)$ is described by the equations

$$(U(\exp tX_0)f)(s) = f(s + t), \qquad t \in \mathbf{R},$$
$$(U(x)f)(s) = U_1(x^s)(f(s)), \qquad x \in N_1,$$

where $x^s = (\exp - sX_0)x(\exp sX_0)$.

We shall require a description of the space $C^\infty(U)$ relative to the above realization.

LEMMA 4. *A function f in $L^2(\mathbf{R}, \mathcal{H}_1)$ is in $C^\infty(U)$ iff f is a $C^\infty(U_1)$-valued Schwartz map of $\mathbf{R}$. That is to say, f is C^∞ as a mapping of $\mathbf{R}$ into $C^\infty(U_1)$ and for any polynomial function p and integer n, the function $p(d/dt)^n f$ is bounded as a map of $\mathbf{R}$ into $C^\infty(U_1)$.*

PROOF. This is not difficult to prove from a few basic facts about C^∞ vectors. However, it is somewhat quicker to use results of Corwin-Greenleaf-Penney [2]. Specifically, there is a connected subgroup H_1 of N_1 and a character χ of H_1 such that U_1 is the representation of N_1 induced by χ and U is the representation of N induced by χ. Let $\mathcal{T}_1$ be a vector complement to $\log H_1$ in $\mathcal{N}_1$ and let $T_1 = \exp \mathcal{T}_1$. We use T_1 to define a realization of U_1 in $L^2(\mathcal{T}_1)$ by restriction of functions to the cross-section T_1 and composition with the exponential map. Similarly we define a realization of U in $L^2(\mathcal{T}_1 \times \mathbf{R})$ by means of the mapping $(t, s) \to \exp t \exp sX_0$ for $t \in \mathcal{T}_1$, $s \in \mathbf{R}$. The results of [2] imply that in these realizations $C^\infty(U_1)$ is the Schwartz space $\mathcal{S}(\mathcal{T}_1)$ of rapidly decreasing C^∞ functions on $\mathcal{T}_1$ while $C^\infty(U)$ is the Schwartz space on $\mathcal{T}_1 \times \mathbf{R}$. Our lemma follows easily from the fact that $\mathcal{S}(\mathcal{T}_1 \times \mathbf{R})$ can be identified with the space of Schwartz maps of $\mathbf{R}$ into $\mathcal{S}(\mathcal{T}_1)$. Q.E.D.

Our argument now splits into three cases depending on whether $Y \in \mathcal{H}$ or $Y \in \mathcal{H} + \overline{\mathcal{H}}$ or $Y \notin \mathcal{H} + \overline{\mathcal{H}}$. First, however, let us make some preliminary observations concerning U_1. Y is central in $\mathfrak{N}_1$ so U_1 is scalar on Y and the value of U_1 on $\exp sY$ is $e^{2\pi i\lambda_1(sY)}I = I$. Hence $\exp sY$ belongs to the kernel of U_1. We shall let $/Y$ be a generic symbol for concepts on $\mathfrak{N}_1$ or N_1 reduced modulo Y. For example U_1/Y is the representation of $N_1/\tilde{Y}$ where $\tilde{Y}$ is the subgroup $\{\exp sY | s \in \mathbf{R}\}$. We note that U_1/Y is a locally injective square integrable representation which corresponds to λ_1/Y. In fact $\mathfrak{N}_1$ is the orthogonal space of $Y + \mathfrak{Z}(\mathfrak{N})$ relative to B_{λ_1} so $\operatorname{span}_{\mathbf{C}} Y + \mathfrak{Z}(\mathfrak{N})$ is the radical of B_{λ_1}. Hence $\mathfrak{Z}(\mathfrak{N})/Y$ is the radical of $B_{\lambda_1/Y}$. The radical always contains the center so $\mathfrak{Z}(\mathfrak{N})/Y$ is the center of $\mathfrak{N}_1/Y$. Hence U_1/Y is square integrable modulo its kernel by the Moore-Wolf Theorem [5]. U_1/Y is nontrivial on $\mathfrak{Z}(\mathfrak{N})/Y$ so U_1/Y is locally injective. We are now ready to deal with Case I.

Case I. $Y \in \mathcal{H}$. In this case $\lambda([Y, \mathcal{H}]) = 0$ so $\mathcal{H} \subset (\mathfrak{N}_1)_C$, Case I will follow by induction from the following lemma.

LEMMA 4′. *In Case* I, $H^0(\mathcal{H}_0, C^\infty(U)) = 0$ *and* $H^p(\mathcal{H}_0, C^\infty(U))$ *is topologically isomorphic with* $H^{p-1}(\mathcal{H}_0/Y, C^\infty(U_1/Y))$.

PROOF. It is easily computed that $\partial U(Y)$ is the mapping of $C^\infty(U)$ defined by $\partial U(Y)f(s) = isf(s)$. Since Y centralizes $\mathcal{H}_0$, the canonical action of Y on forms f in $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ is just $f \to \partial U(Y)f$. It follows that Lemma 2 applies with $\delta = \delta_Y$ and $X = Y$. The image of Y in $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ is the set of forms $s \to f(s)$ which are zero at $s = 0$. It follows that the mapping $f \to f(0)$ defines an isomorphism of $\Lambda^p(\mathcal{H}_0, C^\infty(U))/Y\Lambda^p(\mathcal{H}_0, C^\infty(U))$ with $\Lambda^p(\mathcal{H}_0, C^\infty(U_1))$. The image in $\Lambda^{p-1}(\mathcal{H}_0, C^\infty(U))$ of $\Lambda^p(\mathcal{H}_0 C^\infty(U))$ under δ_Y is the space of forms which are zero whenever any argument belongs to the span over $\mathbf{C}$ of Y. This in turn is isomorphic with $\Lambda^{p-1}(\mathcal{H}_0/Y, C^\infty(U))$. It follows that the space C_1^{p-1} of Lemma 2 is

$$\delta C^p/XC^{p-1} \cap \delta C^p = \Lambda^{p-1}(\mathcal{H}_0/Y, C^\infty(U_1)).$$

Lemma 2 implies the algebraic isomorphism of the spaces in question. The image of $\partial U(Y)$ is clearly closed and multiplication by $(is)^{-1}$ is continuous on the image of $\partial U(Y)$ by the closed graph theorem. Thus Lemma 3 also applies, proving Lemma 4.

The signature of $\phi_{\lambda_1/Y}$ is the same as that of ϕ_{λ_1} while the dimension of $\mathcal{H}_0/Y$ decreases by one. Hence the deficit decreases by one. Also the deficit of $(\mathcal{H}_0, \lambda)$ cannot be zero because Y is in the radical of ϕ_λ. Therefore the validity of Theorem 1 for $(\mathcal{H}_0/Y, \lambda_1/Y)$ implies Theorem 1 for $(\mathcal{H}_0, \lambda_1)$ and Case I follows by induction.

Case II. $Y \in \mathcal{H} + \overline{\mathcal{H}}$ and $Y \notin \mathcal{H}$. In this case $\mathcal{H} \not\subset (\mathfrak{N}_1)_c$ since Y cannot centralize $\mathcal{H}$ by the maximality of $\mathcal{H}$. Let $\mathcal{H}_1 = \mathcal{H}_0 \cap (\mathfrak{N}_1)_c$ and let $\mathcal{H}_2 = \operatorname{span}_{\mathbf{C}} Y + \mathcal{H}_1$.

Case II splits into two subcases.

Subcase II.A. $Y \in \mathcal{H}_1 + \overline{\mathcal{H}}_1$. In this case there is a real $V \in \mathfrak{N}_1$ such that $Y + iV \in \mathcal{H}_1$.

LEMMA 5. *$\partial U(Y + iV)$ is injective on $C^\infty(U)$.*

PROOF. Suppose $\partial U(Y + iV)f = 0$. Then $i(s + \partial U_1(V^s))f(s) = 0$ for all s. In particular $f(s)$ is an eigenvector for $\partial U_1(V^s)$ of eigenvalue $-s$. However $\partial U_1(V^s)$ is skew-symmetric so $f(s) = 0$ unless $s = 0$. The continuity of f then implies $f \equiv 0$. This proves the lemma.

We cannot conclude directly from Lemma 5 that Lemma 2 is applicable as $W = Y + iV$ need not centralize $\mathcal{H}_0$. However, for $f \in \Lambda^p(\mathcal{H}, C^\infty(U))$ and $W \in \mathcal{H}$ let

$$W^*f(X_1, \ldots, X_p) = \sum_i f(X_1, \ldots, [W, X_i], \ldots, X_p)$$

so $Wf = \partial U(W)f - W^*f$. It is clear that $\partial U(W)$ and the mapping $W^*: f \to W^*f$ commute. Furthermore W^* is nilpotent on $\Lambda^p(\mathcal{H}_0, C^\infty(U))$.

LEMMA 6. *If $\partial U(W)$ is injective on $C^\infty(U)$ then $f \to Wf$ is injective on $\Lambda^P(\mathcal{H}_0, C^\infty(U))$.*

PROOF. If $Wf = 0$, then $\partial U(W)f = W^*f$. By iteration $\partial U(W)^n f = (W^*)^n f = 0$ if n is sufficiently large. Hence $f = 0$, proving the lemma.

Thus Lemma 2 is applicable with $\delta = \delta_W$ and X given by the action of $W = Y + iV$ on $\Lambda^p(\mathcal{H}_0, C^\infty(U))$. The main difficulty in our treatment of this subcase will be the identification of the image of W.

To aid in this identification we introduce a new $\mathfrak{N}_c$ module. As we shall also use this construction later, we shall be slightly more general than our current needs dictate. Let $X_1 \in \mathfrak{N}_c \sim (\mathfrak{N}_1)_c$. We define a mapping D_{X_1} of $C^\infty(U)$ into the product space $\Pi_{n=1}^\infty C^\infty(U_1)$ by $D_{X_1}f = (f_0, f_1, \ldots)$ where $f_i = \partial U(X_1)^n f(0)/n!$.

The image of D_{X_1} is denoted M_{X_1} and is called the evaluation module relative to X_1. We define an action of $\mathfrak{N}_c$ on M_{X_1} so that D_{X_1} is a Lie module homomorphism as follows. For $K \in (\mathfrak{N}_1)_c$, let $K_n = (\operatorname{ad} X_1)^n(K)/n!$. For $f = (f_0, f_1, \ldots) \in M_{X_1}$ we set $Kf = (g_0, g_1, \ldots)$ where $g_n = \Sigma_{i+j=n} \partial U(K_i)f_j$. We also define $X_1 f = (f_1, 2f_2, 3f_3, \ldots)$.

The action is extended to the rest of $\mathfrak{N}_c$ by linearity.

LEMMA 7. *M_{X_1} is an $\mathfrak{N}_c$ module and D_{X_1} is an $\mathfrak{N}_c$ module homomorphism. The isomorphism class of M_{X_1} is independent of X_1, and as a vector space $M_{X_1} = \Pi_{n=0}^\infty C^\infty(U_1)$.*

PROOF. The following identity in the complex enveloping algebra $\mathfrak{U}_c(\mathfrak{N})$ of $\mathfrak{N}$ is well known.

$$X_1^n K = \sum_{i+j=n} \binom{n}{j} (\operatorname{ad} X_1)^j(K) X_1^i.$$

This implies that

$$(X_1^n/n!)K = \sum_{i+j=n} K_j(X_1^i/i!).$$

Hence $D_{X_1}Kf = KD_{X_1}f$. The property $X_1D_{X_1} = D_{X_1}X_1$ is clear. The Lie-module property of M_{X_1} follows easily from the surjectivity of D_{X_1} and the above comments.

To prove the isomorphism of the various modules, assume that X_2 is another element of $\mathfrak{N}_c \sim (\mathfrak{N}_1)_c$. From the Poincaré-Birkhoff-Whitt Theorem every element X of $\mathfrak{U}(\mathfrak{N})$ is expressible *uniquely* in the form

$$X = \sum_{i=0}^{k} \left(C_i(X)X_1^i\right)/i!$$

where k is some integer depending on X and $C_i(X)$ belongs to $\mathfrak{U}_c(\mathfrak{N}_1)$. In particular we may write

$$X_2^n/n! = \sum_{i=0}^{k(n)} \left(C_{n,i}X_1^i\right)/i!.$$

Let C: $M_{X_1} \to M_{X_2}$ be the mapping C: $(f_0, f_1, \ldots) \to (g_0, g_1, \ldots)$ where

$$g_n = \sum_{i=0}^{k(n)} \partial U_1(C_{n,i})f_i.$$

(Here we have extended ∂U_1 to $\mathfrak{U}_c(\mathfrak{N}_1)$ as usual.)

SUBLEMMA. $CD_{X_1} = D_{X_2}C$.

PROOF. Let $f \in C^\infty(U)$ and let $f_i = (\partial U(X_1)^i f)(0)/i!$. Then

$$\left(\partial U\left(C_{n,i}X_1^i\right)f\right)(0) = \partial U_1(C_{n,i})\left(\partial U(X_1)^i f(0)\right).$$

Hence, if g_n is as above,

$$g_n = \sum_{i=0}^{k(n)} \left(\partial U\left(C_{n,i}X_1^i\right)f\right)(0)/i! = \partial U(X_2^n)f(0)/n!.$$

This proves the sublemma.

We may also define an inverse $\tilde{C}$ to C, for, by the same reasoning,

$$X_1^n/n! = \sum_{i=0}^{l(n)} \tilde{C}_{n,i}X_2^i/i!$$

with $\tilde{C}_{n,i} \in \mathfrak{U}_c(\mathfrak{N}_1)$. Hence $\tilde{C}_{n,i}$ gives rise to a mapping $\tilde{C}$ of M_{X_2} into M_{X_1}. The mappings C and $\tilde{C}$ are inverse to each other. In fact

$$X_2^n/n! = \sum_{i=0}^{k(n)} C_{n,i}X_1^n/i! = \sum_{i=0}^{k(n)} \sum_{j=0}^{l(i)} C_{n,i}\tilde{C}_{i,j}X_2^j/j!.$$

Thus

$$\sum_{i=0}^{k(n)} C_{n,i}\tilde{C}_{i,j} = \delta_{j,n}$$

showing the inverse property. This proves the isomorphism of the modules.

To prove that D_{X_1} maps $C^\infty(U)$ onto $\prod_{n=0}^{\infty} C^\infty(U_1)$ for all X_1, we claim that it suffices to prove this for one single X_1. In fact, the isomorphism C between M_{X_1} and M_{X_2} is definable on all of $\prod_{n=0}^{\infty} C^\infty(U_1)$ by the same formula. The mapping $\tilde{C}$

is still inverse to C so if D_{X_1} maps onto the product space for a single X_1, then it will map onto for all X_1. We choose $X_1 = X_0$, the (real) element of $\mathfrak{N}$ complementary to $\mathfrak{N}_1$ chosen above.

To prove our lemma, let $f = (f_0, f_1, \dots)$ belong to M_{X_0}. We need to show that there is a function $g \in C^\infty(U_1)$ such that

$$f_i = \partial U(X_0)^n g(0)/n! = (d/dt)^n g(0)/n!.$$

Let ψ: $\mathbf{R} \to \mathbf{R}$ be a C^∞ function which is supported in the interval $[-1, 1]$ and which is one on a neighborhood of 0 in $\mathbf{R}$. Let h_n be the function $h_n(t) = t^n\psi(t)f_n$. Let λ_n be a sequence of positive numbers and consider the functions

$$g_n(t) = \lambda_n^{-n}h_n(\lambda_n t) = t^n\psi(\lambda_n t)f_n.$$

Let $\{X_0, Y_1, Y_2, \dots, Y_k\}$ be a basis for $\mathfrak{N}$ with $Y_i \in \mathfrak{N}_1$ and X_0 as above. It is possible to choose λ_n large enough so that

$$\|\partial U(X_0^{n_0}Y_1^{n_1}Y_2^{n_2} \cdots Y_k^{n_k})g_n\| < 2^{-n}$$

whenever $\sum n_i < n - 2$. This is because

$$X_0^m g_n = (d/dt)^m g_n = \lambda_n^{m-n}(d^m/dt^m)h_n(\lambda_n t).$$

Then $g = \sum_{n=0}^\infty g_n$ converges in $C^\infty(U)$. It is easily seen that

$$\partial U(X_0)^n g(0)/n! = f_n.$$

This proves the lemma.

The relevance of the above construction to our current problem is the following lemma. Let $X_1 \in \mathfrak{N}_c \sim (\mathfrak{N}_1)_c$.

LEMMA 8. *Let $f \in C^\infty(U)$; then f is in the image of $\partial U(W)$ iff $D_{X_1}f$ is in the image of W in M_{X_1}. Furthermore W is injective on M_{X_1}.*

PROOF. From Lemma 7 it suffices to assume $X_1 = X_0$. It is clear that if f is in the image of $\partial U(W)$ then $D_{X_0}f$ is in the image of W.

Conversely, suppose $D_{X_0}f = W\tilde{g}$. We need to show that f is in the image of W. From Lemma 7, there is a $g \in C^\infty(U)$ such that $\tilde{g} = D_{X_0}g$. Then $D_{X_0}(f - Wg) = 0$ so it suffices to consider the case $D_{X_0}f = 0$.

Now, recall that $W = Y + iV$.

$$\partial U(Y)f(s) = \partial U_1(Y^s)f(s) = isf(s).$$

Hence

$$\partial U(W)f(s) = i(s + \partial U_1(V^s))f(s).$$

The spectrum of $\partial U_1(V^s)$ is purely imaginary so $s + \partial U_1(V^s)$ is invertible for $s \neq 0$. The inverse is given by

$$\begin{aligned}(s + \partial U_1(V^s))^{-1} &= \int_0^\infty \exp - (st + \partial U_1(tV^s))\, dt \\ &= \int_0^\infty e^{-st}U_1(\exp - tV^s)\, dt \qquad (*)\end{aligned}$$

for $s > 0$ and

$$(s + \partial U_1(V^s))^{-1} = -\int_0^\infty e^{st} U_1(\exp tV^s)\, dt \tag{$*$}$$

for $s < 0$.

These integrals converge in the strong operator topology on the representation space of U_1. Let g be the mapping of $\mathbf{R}$ into the representation space of U_1 given by

$$g(s) = -i(s + \partial U_1(V^s))^{-1} f(s), \quad s \neq 0, \qquad g(0) = 0.$$

SUBLEMMA. *$g \in C^\infty(U)$ and $\partial U(W)g = f$.*

PROOF. From Lemma 4, we need to show that g is a C^∞ mapping of $\mathbf{R}$ into $C^\infty(U_1)$ and all derivatives of g remain bounded in the $C^\infty(U_1)$ topology when multiplied by an arbitrary polynomial.

We begin by showing that g is a polynomially bounded mapping of $\mathbf{R}$ into $C^\infty(U_1)$. To see this it suffices to show that $g(s)$ belongs to $C^\infty(U_1)$ and $\|\partial U_1(A)g(s)\|$ is polynomially bounded in s for all A in the enveloping algebra $\mathfrak{U}(\mathfrak{N}_1)$. For each $s \in \mathbf{R}$ let $A(s, t)$ denote $e^{t \operatorname{ad} V^s}(A)$ where ad V^s is extended as usual to the enveloping algebra. Formally, we have the following identity for $s > 0$.

$$\begin{aligned}\partial U_1(A)g(s) &= \int_0^\infty \partial U_1(A)\, U_1(\exp - tV^s) f(s) e^{-st}\, dt \\ &= \int_0^\infty U_1(\exp - tV^s)\partial U_1(A(s, t)) f(s) e^{-st}\, dt.\end{aligned}$$

The analytic validity of this identity is easily verfied by differentiation under the integral. Note that this implies $g(s) \in C^\infty(U_1)$. Now there are fixed elements $A_1 \cdots A_k \in \mathfrak{U}(\mathfrak{N}_1)$ and polynomials $p_i(s, t)$ such that

$$A(s, t) = \sum_{i=1}^k p_i(s, t) A_i.$$

If $p(s, t)$ is any polynomial, then $\int_0^\infty p(s, t) e^{-st}\, dt$ increases at most polynomially in s as $s \to \infty$. It follows that there are polynomials $q_i(s)$ such that

$$\|\partial U(A_1)g(s)\| \leqslant \sum_{i=1}^n q_i(s) \|\partial U_1(A_i) f(s)\|.$$

The polynomial boundedness of f now is seen to imply the polynomial boundedness of $g(s)$ for large positive s. The case of large negative s is proven similarly. The boundedness for all s will follow once we have shown that g is C^∞ as a mapping of $\mathbf{R}$ into $C^\infty(U_1)$.

The fact that g is C^∞ at zero follows from the fact that $\lim_{s\to 0} f(s)/s^n = 0$ for all n since $(d/ds)^n f(0) = 0$. In fact

$$\lim_{s\to 0^+} g(s)/s^n = \lim_{s\to 0^+} \int_0^\infty s^2 e^{-st} U_1(\exp - tV^s)(f(s)/s^{n+2})\, dt.$$

As $s \to 0^+$, $s^2 e^{-st} \to 0$ in $L^1([0, \infty))$ and the rest of the integrand remains bounded so $\lim_{s\to 0^+} g(s)/s^n = 0$. A similar computation for $s < 0$ shows $(d/ds)^n g(0) = 0$.

To show that g is C^∞ at nonzero s we adopt another point of view. Again let

$s > 0$. The function $s \to U_1(\exp - tV^s)f(s)$ is merely

$$(U(\exp - tV)f)(s) = (U(\exp sX_0)U(\exp - tV)f)(0).$$

Let δ: $C^\infty(U) \to C^\infty(U_1)$ be the map $\delta(f) = f(0)$. For $s > 0$ let T_s be the map

$$T_s f = \int_0^\infty e^{-st}U(\exp - tV)f\,dt.$$

Then

$$g(s) = \delta(U(\exp sX_0)T_s f).$$

If we can show that $s \to U(\exp sX_0)T_s f$ is a C^∞ mapping of $\mathbf{R}$ into $C^\infty(U)$, our claim will follow from the continuity of δ. The fact that T_s maps $C^\infty(U)$ into $C^\infty(U)$ is shown in much the same way that we showed that $g(s) \in C^\infty(U_1)$. In fact, if $B \in \mathfrak{U}(\mathfrak{N})$, then

$$\begin{aligned}\partial U(B)T_s f &= \int_0^\infty e^{-st}U(\exp - tV)\partial U(B(t))f \\ &= \sum_{i=1}^k \int_0^\infty e^{-st}U(\exp - tV)r_i(t)\partial U(B_i)f\,dt\end{aligned}$$

where $B(t) = e^{t\,\mathrm{ad}\,V}B$ and r_i are polynomials. Differentiation with respect to s under the integral is justified if $s > 0$ and shows the C^∞ nature of $T_s f$.

It now follows from the product rule for differentiation that g is C^∞ for $s > 0$ and

$$\left(\frac{d}{ds}\right)^n g(s) = \delta\left(U(\exp sX_0)\left(\sum_{j=0}^n \binom{n}{j}\partial U(X_0)^{n-j}T_s^j f\right)\right)$$

where

$$T_s^j f = \left(\frac{d}{ds}\right)^j T_s f = \int_0^\infty (-1)^j t^j e^{-st}U(\exp - tV)f\,dt.$$

Bringing the $\partial U(X_0)^{n-j}$ past the integral defining T_s^j as above we see that

$$\left(\frac{d}{ds}\right)^n g(s) = \sum_{j=1}^k \left(\int \sigma_j(t)e^{-st}U(\exp - tV)f^j\,dt\right)(s)$$

where the σ_j are certain polynomials and the f_j's are elements of $C^\infty(U)$. The proof given above for the polynomial boundedness of g applies to the terms in the above sum as well and shows the polynomial boundedness for $(d/ds)^n g(s)$ for s large, positive. Similar arguments apply for negative s and our sublemma is proven.

To finish the lemma, we need only show the injectivity of W on M_{X_0}. We begin by showing that $\partial U_1(W) = i\partial U_1(V)$ is injective on $C^\infty(U_1)$. (Note that $\exp Y \in \ker U_1$.) Let χ_0 be the character of the center $\mathfrak{Z}(N_1)$ of N_1 defined by

$$\chi_0(x) = \exp i\lambda_1(\log x).$$

Let U^{χ_0} be the representation of N_1 induced by χ_0. By results of Moore and Wolf, U_1 can be realized as a subrepresentation of U^{χ_0}. The operator $\partial U_1(V)$ is a left invariant differential operator so $\partial U_1(V)f = 0$ implies that f is constant on right cosets of $\{\exp tV | t \in \mathbf{R}\}$. As $V \notin \mathfrak{Z}(\mathfrak{N}_1) = \operatorname{span} Y + \mathfrak{Z}(N)$, the constancy

of f along V-cosets is inconsistent with the square integrability of $\hat{f}$ unless $f \equiv 0$.

Now suppose $f = (f_0, f_1, \ldots) \in M_{X_0}$ is annihilated by V. Then $(Vf)_0 = \partial U_1(V) f_0$ so $f_0 = 0$. But $f_0 = 0$ implies $(Vf)_1 = \partial U_1(V) f_1$ so $f_1 = 0$. By induction it follows that all $f_i = 0$ and V is injective, proving the lemma.

It now follows from the proof of Lemma 6 that Lemma 2 applies to the complex $C^p = \Lambda^p(\mathcal{H}_0, M)$ with $\delta = \delta_W$ and $X = W$. Let $X_1 \in \mathcal{H}_0 \sim \mathcal{N}_1$.

LEMMA 9. *$H^p(\mathcal{H}_0, M_{X_1})$ and $H^p(\mathcal{H}_0, C^\infty(U))$ are isomorphic. The isomorphism is the map $D^*_{X_1}$ induced by D_{X_1} on homology.*

PROOF. Let $C^p = \Lambda^p(\mathcal{H}_0, C^\infty(U))$ and $\tilde{C}^p = \Lambda^p(\mathcal{H}_0, M_{X_1})$. Let δ be the derivation δ_W on C^p and $\tilde{\delta}$ be δ_W on $\tilde{C}^p$. The previous lemma shows that D_{X_1} defines a chain map from C^p to $\tilde{C}^p$ such that f is in the image of $W = X$ in C^p iff f is in the image of W in $\tilde{C}^p$. It follows that D_{X_1} induces a chain isomorphism of the complexes C^p_1 and $\tilde{C}^p_1$. Lemma 2, applied to C^p_1 and $\tilde{C}^p_1$, now shows that $H^p(\mathcal{H}_0, C^\infty(U))$ and $H^p(\mathcal{H}_0, M_{X_1})$ are both isomorphic to $H^{p-1}(C^*_1)$ for $p > 0$ and to zero for $p = 0$.

We are not yet finished as we still must show that the isomorphism can be induced directly by D_{X_1} without the intercession of C^p_1. However, this follows easily from the commutativity of the diagram below and the fact that the isomorphism of $H^p(C^*)$ with $H^p(C^*_1)$ is given by δ_X.

$$\begin{array}{ccc} C^p & \xrightarrow{D_{X_1}} & \tilde{C}^p \\ \delta_W \downarrow & & \downarrow \delta_W \\ C^{p-1} & \xrightarrow{D_{X_1}} & \tilde{C}^{p-1} \end{array}$$

This finishes the lemma.

REMARK. In the above lemma we have not made any claim that the isomorphisms are topological. This is because we have not shown that W has a bounded inverse on either $C^\infty(U)$ or M_{X_1}. Notice, however, that Lemma 9 does not involve W at all except in its proof. We shall obtain the topological properties of the isomorphism later without the aid of Lemma 2. In fact we have not even defined a topology on M_{X_1} as yet. This fact we remedy immediately. We give M_{X_1} the product topology. Notice that the action of X_1 on M_{X_1} is essentially a left shift so X_1 acts surjectively and X_1 is an open mapping. The following is a counterpart to Lemma 6.

LEMMA 10. *The action of X_1 on $\Lambda^p(\mathcal{H}_0, M_{X_1})$ is surjective and open.*

PROOF. The action of X_1 on $f \in \Lambda^p(\mathcal{H}_0, M_{X_1})$ is given by

$$X_1 \cdot f = (X_1 - X_1^*) f$$

where X_1^* is as defined previously (Lemma 6). As a formal power-series

$$(X_1 - X_1^*)^{-1} = \sum_{n=0}^{\infty} X_1^{-(n+1)} (X_1^*)^n.$$

We let f_n be such that

(i) $X_1^{(n+1)} f_n = (X_1^*)^n f$,

(ii) $f_n = 0$ if $(X_1^*)^n f = 0$.

f_n is then a finite sequence and it is easily seen that

$$(X_1 - X_1^*)\left(\sum f_n\right) = f.$$

This shows surjectivity.

To see openness, let $\mathfrak{U}$ be a neighborhood of zero in $\Lambda^p(\mathcal{H}_0, M_{X_1})$ and consider $\mathcal{V} = (X_1 - X_1^*)\mathfrak{U}$. It suffices to show that $\mathcal{V}$ is a neighborhood of zero. Let n be such that $(X_1^*)^n = 0$ and let $\mathcal{V}_0 = \bigcap_{k=0}^{n+1} (X_1)^k \mathfrak{U}$. Let $\mathcal{V}_1 = \bigcap_{k=0}^{n} (X_1^*)^{-k} \mathcal{V}_0$. $\mathcal{V}_1$ is a neighborhood of zero. We claim $\mathcal{V}_1 \subset \mathcal{V}$. In fact, for $f \in \mathcal{V}_1$, the f_n defined above can be chosen in $\mathfrak{U}$ so $f \in (X_1 - X_1^*)\mathfrak{U}$. This proves the lemma.

Now we apply the second portion of Lemmas 2 and 3 to $\Lambda^p(\mathcal{H}_0, M_{X_1})$ with $\delta = \delta_{X_1}$. We come up with the following fascinating conclusion.

LEMMA 11. *$H^p(\mathcal{H}_0, C^\infty(U))$ is isomorphic with $H^p(\mathcal{H}_1, C^\infty(U_1))$.*

PROOF. $H^p(\mathcal{H}_0, C^\infty(U))$ is isomorphic with $H^p(\mathcal{H}_0, M_{X_1})$ by the previous lemma. From Lemma 2, $H^p(\mathcal{H}_0, M_{X_1})$ in turn is isomorphic with $H^p(\mathcal{H}_0, C_0^*)$ where C_0^* is the intersection of the kernel of δ_{X_1} in $\Lambda^p(\mathcal{H}_0, M_{X_1})$ with the kernel of the canonical action of X_1. A form f is in the kernel of δ_{X_1} iff f is constant on additive cosets of $\operatorname{span}_{\mathbf{C}} X_1$ in $\mathcal{H}_0$ in each variable. It follows that restriction of such forms to $\mathcal{H}_1$ is a vector space isomorphism onto $\Lambda^p(\mathcal{H}_1, M_{X_1})$.

SUBLEMMA. *Let $f = (f_0, \dots)$ be a form in $\Lambda^p(\mathcal{H}_0, M_{X_1})$. Then f is in the kernel of the canonical action iff*

$$f = \left(f_0, X_1^* f_0, (X_1^*)^2 f_0/2!, \dots, (X_1^*)^n f_0/n!, \dots\right).$$

PROOF. f is in the kernel of the canonical action iff $X_1 f = X_1^* f$ so

$$(f_1, 2f_2, \dots) = (X_1^* f_0, X_1^* f_1, \dots).$$

The sublemma follows easily from this.

It is now easily seen that the mapping $C_0^p \to \Lambda^p(\mathcal{H}_1, C^\infty(U_1))$ given by mapping $f = (f_0, \dots)$ into the restriction of f_0 to $\mathcal{H}_1$ is a continuous chain isomorphism with continuous inverse. It follows that $H^p(C_0^*)$ and $H^p(\mathcal{H}_1, C^\infty(U_1))$ are isomorphic, even as topological spaces. Our lemma follows.

We may finally finish Subcase II.A. It is clear that $H^p(\mathcal{H}_1, C^\infty(U_1))$ and $H^p((\mathcal{H}_1 + \operatorname{span}_{\mathbf{C}} Y)/Y, C^\infty(U_1/Y))$ are identical. It is easily verified that $\mathcal{H}_1 + \operatorname{span}_{\mathbf{C}} Y/Y$ is a maximal subordinate subalgebra in $(\mathfrak{N}_1)_c$ so by induction $H^p(\mathcal{H}_1, C^\infty(U_1))$ is zero unless p equals the deficit of $\phi_{\lambda_1/Y}$.

LEMMA 12. *ϕ_{λ_1} and $\phi_{\lambda_1/Y}$ have the same deficit.*

PROOF. Consider the subspace $\tilde{\mathcal{H}}$ of $\mathcal{H}_0$ spanned by $W = Y + iV$ and X_1. Since Y centralizes $\mathfrak{N}_1$, $\phi_\lambda(W, W) = 0$. We claim $\phi_\lambda(W, X_1) \neq 0$. Of course $\lambda([Y, X_1]) \neq 0$ since $X_1 \notin \mathfrak{N}_1$, while $\lambda([Y + iV, X_1]) = 0$. Hence $\lambda([iV, X_1]) = -\lambda([Y, X_1]) \neq 0$. Hence

$$\phi_\lambda(X_1, Y + iV) = -i\lambda([X_1, Y - iV]) = -i2\lambda([X_1, Y]) \neq 0.$$

It follows that ϕ_λ is nondegenerate on $\tilde{\mathcal{H}}$ and there is a canonical basis for ϕ_λ passing through $\tilde{\mathcal{H}}$. The matrix of ϕ_λ on $\tilde{\mathcal{H}}$ is of the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & 0 \end{pmatrix}$$

where $a = \phi_\lambda(X_1, X_1)$ is real and $b = \phi_\lambda(X_1, W)$. The characteristic polynomial is $x^2 - ax - |b|^2$. Hence A has one positive and one negative eigenvalue. On the other hand, W is in the radical of the restriction of ϕ_λ to $\mathcal{H}_1$. Hence on $\mathcal{H}_1$, ϕ_λ loses one positive eigenvalue and one negative eigenvalue and gains a zero eigenvalue. As the dimension of the subordinate subalgebra also decreases by one, this makes a net change of zero in the deficit. This proves the lemma.

It follows from the lemma that $H^p(\mathcal{H}, C^\infty(U))$ is zero if $p \neq q_0$ where q_0 is the deficit and $H^{q_0}(\mathcal{H}, C^\infty(U))$ is one dimensional. To finish Subcase II we need to show that $H^{q_0}(\mathcal{H}, C^\infty(U))$ is Hausdorff–i.e. the image of ∂ is closed in $\Lambda^{q_0}(\mathcal{H}, C^\infty(U))$. This will be the case if the image of ∂ is closed in $\Lambda^{q_0}(\mathcal{H}, M_{X_1})$ for the isomorphism of $H^{q_0}(\mathcal{H}, C^\infty(U))$ and $H^{q_0}(\mathcal{H}, M_{X_1})$ is equivalent with the statement that f is a boundary in $\Lambda^q(\mathcal{H}, C^\infty(U))$ iff $f \in D_{X_1}^{-1}(\partial\Lambda^{q-1}(\mathcal{H}, M_{X_1}))$. The closeness would then follow from the continuity of D_{X_1}. But from the proof of Lemma 11, $H^{q_0}(\mathcal{H}, M_{X_1})$ is *topologically* isomorphic with $H^{q_0}(\mathcal{H}_0, C^\infty(U_1))$. The closure property follows from this. This finishes Subcase II.A.

Subcase II.B. In this case $Y \in \mathcal{H} + \overline{\mathcal{H}}$ but $Y \notin \mathcal{H}_1 + \overline{\mathcal{H}}_1$. Then there is a $V \notin \mathcal{N}_1$ such that $X_1 = Y + iV \in \mathcal{H}$. We may pick the X_0 used above equal to $2cV$ where c is a real constant chosen so that $\lambda([2cV, Y]) = 1$. Unfortunately, Subcase II.B now splits into two cases, depending on the sign of c.

Assume first that $c > 0$.

LEMMA 13. *The canonical action of X_1 on $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ is surjective and open. Its kernel is isomorphic with $\Lambda^p(\mathcal{H}_0, C^\infty(U_1))$ under the mapping $f \to f(0)$.*

PROOF. The openness will follow from the open mapping theorem once surjectivity is shown. We first consider $p = 0$. As mappings on $C^\infty(\mathbf{R}, C^\infty(U_1))$, $\partial U(X_0)$ is d/ds and $\partial U(Y)$ is multiplication by is.

$$\partial U(X_1) = i(s + (2c)^{-1} d/ds).$$

g is in the image of $\partial U(X_1)$ iff $-2cig = (d/ds)f + 2csf$. A solution of this equation is

$$f(s) = -2ice^{-cs^2} \int_0^s e^{ct^2} g(t)\, dt$$

provided the f so defined belongs to $C^\infty(U)$. To see that f does define an element of $C^\infty(U)$, note that it is easily computed by differentiation under the integral that f is valued in $C^\infty(U_1)$. The fundamental theorem of calculus combined with differentiation under the integral shows that f is a C^∞ mapping of $\mathbf{R}$ into $C^\infty(U_1)$. Hence we need only show that f and its derivatives are polynomially bounded. However

$$(2c)^{-1}\|f(s)\| \leqslant e^{-cs^2} \int_0^s e^{ct^2} \|g(t)\|\, dt \leqslant e^{-cs^2} \int_0^s Me^{ct^2}.$$

By l'Hospital's rule this tends to zero as $s \to \pm\infty$. The polynomial boundedness of f follows similarly using the polynomial boundedness of g. The polynomial boundedness of the derivatives of f is proven using the Leibnitz rule for differentiation and l'Hospital's rule as above.

The kernel of $\partial U(X_1)$ is obviously all elements of the form ve^{-ct^2} where $v \in C^\infty(U_1)$. The isomorphism with $C^\infty(U_1)$ is clear.

For $p > 0$, the first two statements of the lemma follow as in Lemma 10. To prove the statement about the kernel, suppose f is in the kernel of the canonical action. Then

$$\partial U(X_1)f - X_1^* f = 0$$

i.e.

$$(d/ds + 2cs + 2icX_1^*)f = 0.$$

To solve this equation we need some notation. If S and T are arbitrary elements of $\mathfrak{N}_c$, we let

$$e^S(T) = \sum_{n=0}^{\infty} (\operatorname{ad} S)^n(T)/n!.$$

If ω is a p-form on $\mathfrak{H}_0$, valued in some arbitrary vector space, we set

$$e^S\omega(Y_1, \ldots, Y_n) = \omega\big(e^S(Y_1), \ldots, e^S(Y_n)\big)$$

whenever S normalizes $\mathfrak{H}_0$. The relevance of this to our current problem is that

$$(d/ds)\big(e^{sX_1}f(Y_1, \ldots, Y_n)\big) = X_1^* e^{sX_1}f(Y_1, \ldots, Y_n),$$

as the reader may easily verify. For $f \in \Lambda^p(\mathfrak{H}_0, C(U))$ we set

$$\tilde{f}(s) = e^{2sciX_1}f(s).$$

Then the equation we wish to solve is equivalent with

$$(d/ds + 2sc)\tilde{f}(s) = 0.$$

From the above comments we obtain the solution

$$f(s) = e^{-cs^2}e^{-cisX_1}v$$

where v is a fixed element of $\Lambda^p(\mathfrak{H}_0, C^\infty(U_1))$. This shows that $f \to f(0) = v$ is an isomorphism onto $\Lambda^p(\mathfrak{H}_0, C^\infty(U_1))$ as claimed. This proves the lemma.

We are now in a position to apply Lemmas 2 and 3 with $\delta = \delta_{X_1}$ and $X = X_1$ in the surjective case. The module C_0^p is the intersection of the kernel of the canonical action and the kernel of δ_{X_1} so under the isomorphism of Lemma 13, C_0^p is chain isomorphic with $\Lambda^p(\mathfrak{H}_1, C^\infty(U_1))$ where X_1 acts trivially. It follows from Lemmas 2 and 3 that $H^p(\mathfrak{H}_0, C^\infty(U))$ is topologically isomorphic with $H^p(\mathfrak{H}_1, C^\infty(U_1)) = H^p(\mathfrak{H}_1 + Y/Y, C^\infty(U_1/Y))$. As $\mathfrak{H}_1 + Y$ is maximal subordinate in $\mathfrak{N}_1/Y$, we are in a position to apply induction. Our sub-subcase will be done if we show that the deficits of ϕ_λ and ϕ_{λ_1} are the same. Now

$$\phi_\lambda(X_1, X_1) = -i\lambda\big([Y + iV, Y - iV]\big) = c^{-1} > 0.$$

As X_1 is orthogonal to the rest of $\mathfrak{H}_1$, it follows that ϕ_{λ_1} has one fewer positive

eigenvector than ϕ_λ. Since the dimension of $\mathcal{H}_1$ is one less than that of $\mathcal{H}_0$, the deficit remains constant.

Next suppose $c < 0$. In this case we have

LEMMA 14. *The canonical action of X_1 on $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ is injective and X_1 has a continuous inverse defined on its image. A form f is in the image of X_1 iff*

$$\int_{-\infty}^{\infty} e^{s^2c} e^{2iscX_1} f(s)\, ds = 0.$$

PROOF. As in the proof of Lemma 13, f is in the image of X_1 iff

$$-2icf = (d/ds + 2cs + 2ciX_1^*)g. \qquad (**)$$

Solving as before we have a solution of the form

$$g(s) = -2ice^{-s^2c} e^{-2icsX_1} \int_{-\infty}^{s} e^{t^2c} \tilde{f}(t)\, dt$$

where $\tilde{f}(s) = e^{2icsX_1} f(s)$. Suppose

$$\int_{-\infty}^{\infty} e^{ct^2} \tilde{f}(t)\, dt = 0.$$

We claim that in this case g represents an element of $C^\infty(U_1)$. Since e^{2iscX_1} is polynomial in s, one can see that for g to be in $C^\infty(U)$ it suffices to show that

$$h(s) = e^{-s^2c} \int_{-\infty}^{s} e^{t^2c} \tilde{f}(t)\, dt$$

defines an element of $C^\infty(U)$.

We claim that $h(s)$ is bounded in $C^\infty(U_1)$. To see this let ϕ be an element of the continuous dual $C^{-\infty}(U_1)$. From the Banach-Steinhaus theorem, it suffices to show that $\langle h(s)(Y_1, \ldots, Y_n), \phi\rangle$ is bounded for all such ϕ and all $Y_1, \ldots, Y_n \in \mathcal{H}_0$. But

$$\langle h(s)(Y_1, \ldots, Y_n), \phi\rangle = e^{-cs^2} \int_{-\infty}^{s} e^{ct^2} \langle \tilde{f}(t)(Y_1, \ldots, Y_n), \phi\rangle\, ds.$$

From l'Hospital's rule, the limit of the left-hand side of the above as $s \to \infty$ is equal to

$$\lim_{s\to\infty} (2cs)^{-1} \langle \tilde{f}(s)(Y_1, \ldots, Y_n), \phi\rangle = 0.$$

Note that to apply l'Hospital's rule we need $\int_{-\infty}^{\infty} = 0$. It also follows from l'Hospital's rule that the $-\infty$ limit is zero.

Similarly we may prove the polynomial boundedness of h. Also, by using the Leibnitz rule for differentiating products and l'Hospital's rule we can show the polynomial boundedness of the derivatives of h. This proves the sufficiency part of the lemma.

Conversely, suppose f is in the image of X_1. Let g be as above. g is the unique solution of $(**)$ which satisfies $\lim_{s\to-\infty} g(s) = 0$. In order for g to define an element of $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ we must have $\lim_{s\to\infty} h(s) = 0$ where

$$h(s) = (2ic)^{-1} e^{2icsX_1} g(s).$$

This necessitates the condition on the integral of $\tilde{f}$.

To see the injectivity of X_1, it suffices by the proof of Lemma 6 to consider X_1 on $C^\infty(U)$. The kernel of X_1 is the space of functions of the form $f(s) = e^{-cs^2}v$ for v fixed in $C^\infty(U_1)$. However, such functions are not square integrable unless $v = 0$. Clearly, the integral defining g is a continuous inverse to X_1 on its image. This proves the lemma.

Now we may apply Lemmas 2 and 3 with $C^p = \Lambda^p(\mathcal{H}_0, C^\infty(U))$ and $\delta = \delta_{X_1}$ so X is given by the canonical action of X_1. The module C_1^{p-1} is $\delta(C^p)/\delta(C^p) \cap XC^{p-1}$. The image of $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ under δ_{X_1} is isomorphic with $\Lambda^{p-1}(\mathcal{H}_1, C^\infty(U))$. The above lemma implies that the mapping D of $\delta(C^p)$ into $\Lambda^{p-1}(\mathcal{H}_1, C^\infty(U_1))$ given by

$$D: f \to \int_{-\infty}^{\infty} e^{s^2c} e^{2sciX_1} f(s)\, ds$$

induces a vector space isomorphism of C_1^{p-1} onto $\Lambda^{p-1}(\mathcal{H}_1, C^\infty(U_1))$.

LEMMA 15. *D is a chain mapping.*

PROOF. Let $f \in \Lambda^{p-1}(\mathcal{H}_1, C^\infty(U))$. We define

$$\tilde{f}(s) = e^{2iscX_1} f(s)$$

as before. Note that $2iscX_1 = -sX_0 + 2iscY$. As Y is central in $\mathfrak{N}_1$ and $\mathcal{H}_1 \subset \mathfrak{N}_1$, it follows that $\tilde{f}(s) = e^{-sX_0} f(s)$. Recall that we have defined $Y^s = e^{sX_0}(Y)$ for $Y \in \mathfrak{N}_1$. The following computation now follows for $Y_1, \ldots, Y_p \in \mathcal{H}_1$.

$$\begin{aligned}(\partial f)^{\sim}(Y_1, \ldots, Y_p)(s) &= \partial f(Y_1^{-s}, \ldots, Y_p^{-s})(s) \\ &= \sum_{i=1}^{p} (-1)^i \partial U(Y_1^{-s}) f(Y_1^{-s} \cdots Y_i^{-s} \cdots Y_p^{-s})(s) \\ &\quad - \sum_{i<j} (-1)^{i+j} f\big([X_i, Y_j]^{-s}, Y_1^{-s} \cdots Y_i^{-s} \cdots Y_j^{-s} \cdots Y_p^{-s}\big)(s).\end{aligned}$$

Noting that $\partial U(Y_1^{-s})h(s) = \partial U_1(Y_1)h(s)$ for any $h \in C^\infty(U)$, we see that $(\partial f)^{\sim}(s) = \partial_1(\tilde{f}(s))$ where ∂_1 refers to the $\Lambda^p(\mathcal{H}_1, C^\infty(U_1))$ coboundary operator. Multiplying by e^{cs^2} and integrating we get the desired conclusion, thus proving the lemma.

It now follows from Lemma 14 that C_1^{p-1} is chain isomorphic with $\Lambda^{p-1}(\mathcal{H}_1, C^\infty(U_1))$. Hence Lemmas 2 and 3 imply the following

COROLLARY 16. *$H^p(\mathcal{H}_0, C^\infty(U))$ is topologically isomorphic with $H^{p-1}(\mathcal{H}_1, C^\infty(U_1))$ for $p > 0$. If $p = 0$, $H^p(\mathcal{H}_0, C^\infty(U))$ is zero.*

Again, this will prove our theorem once we show that the deficit ϕ_{λ_1} is one unit less than that of ϕ_λ. This is a calculation similar to that done in the $c > 0$ case. We shall omit it here. This finishes Subcase II.B.

Now we proceed to the final case

Case III. $Y \notin \mathcal{H}_0 + \overline{\mathcal{H}}_0$. In this case a very fortunate circumstance occurs which aids immensely in computing the cohomology.

LEMMA 17. *There is a real element $X_0' \in \mathcal{H} \cap \mathfrak{N} \sim \mathfrak{N}_1$ such that $\lambda([X_0', Y]) = 1$.*

PROOF. Let X_0 be as above–i.e. $X_0 \in \mathfrak{N} \sim \mathfrak{N}_1$, $\lambda([X_0, Y]) = 1$. Since $(\mathfrak{N}_1)_c + \mathcal{H} = \mathfrak{N}_c$ we may write $X_0 = Y_1 + H_1$ where $Y_1 \in (\mathfrak{N}_1)_c$ and $H_1 \in \mathcal{H}$. We define a new linear functional λ' on $\operatorname{span}_{\mathbf{C}}(\mathcal{H}_0 + \overline{\mathcal{H}}_0 + Y + \mathfrak{Z}(\mathfrak{N}_c))$ by

(i) $\lambda'(X) = \lambda([Y_1, X])$, $X \in \operatorname{span}_{\mathbf{C}}(Y + \mathcal{H}_0 + \mathfrak{Z}(\mathfrak{N}_c))$,

(ii) $\lambda'(X) = \lambda'(\overline{X})^-$, $X \in \overline{\mathcal{H}}_0$.

This is well defined, for if $X \in \mathcal{H}_0 \cap \overline{\mathcal{H}}_0$, then

$$\lambda([Y_1, X]) = \lambda([Y_1 + H_1, X]) = \lambda([X_0, X]).$$

Hence, since X_0 is real,

$$\lambda([Y_1, \overline{X}]) = \lambda([X_0, \overline{X}]) = \lambda([X_0, X])^- = \lambda([Y_1, X])^-.$$

Clearly λ' satisfies $\lambda'(\overline{X}) = \lambda'(X)^-$. Thus λ' is a real linear functional. We extend λ' linearly to $\mathfrak{N}$ and then by complex linearity to $\mathfrak{N}_c$.

Now, the form B_λ is nondegenerate on $\mathfrak{N}/\mathfrak{Z}(\mathfrak{N})$ and λ' is trivial on $\mathfrak{Z}(\mathfrak{N})$. Hence there is a $Y_2 \in \mathfrak{N}$ such that $\lambda'(X) = B_\lambda(Y_2, X)$. Now for $X \in \mathcal{H}_0$

$$\lambda'(X) = B_\lambda(Y_1, X) = B_\lambda(Y_2, X).$$

Thus $Y_1 - Y_2$ annihilates $\mathcal{H}_0$ relative to B_λ. Hence from the maximality of $\mathcal{H}$, $Y_1 - Y_2 \in \mathcal{H}$. It follows that in the decomposition $X_0 = Y_1 + H_1$ we may take $Y_1 = Y_2$. Thus $X_0' = X_0 - Y_1$ is a real element of $\mathfrak{N}$ which also belongs to $\mathcal{H}$. Furthermore, it follows from $\lambda'(Y_2) = 0$ that $Y_1 \in \mathfrak{N}_1$ so $\lambda([X_0', Y]) = 1$ as desired. This proves the lemma.

We, of course, now choose $X_0 = X_0'$. As X_0 acts on $C^\infty(U)$ as d/ds, it is clear that $\partial U(X_0)$ is injective on $C^\infty(U)$ and hence on $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ by Lemma 6. As in the proof of Lemma 14, it is easily seen that the canonical action of X_0 has a continuous inverse and a form f in $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ is in the image of the canonical action of X_0 iff

$$\int_{-\infty}^{\infty} e^{-sX_0} f(s)\, ds = 0.$$

It now follows as in Lemma 15 and Corollary 16 that $H^p(\mathcal{H}_0, C^\infty(U))$ and $H^{p-1}(\mathcal{H}_1 + Y/Y, C^\infty(U/Y))$ are isomorphic for $p > 0$. Case III is then completed by computing the deficit of ϕ_{λ_1}. We leave the details to the reader.

This finishes the proof of Theorem 1. Q.E.D.

CONCLUDING REMARKS. Having been through the proofs, the reader may wonder at this point about the roles played by the square integrability and the local injectivity. The roles are, in fact, the same. The local injectivity is to preclude the Lie algebra cohomology of $\mathcal{H}_0$ from entering the picture. The necessity of this precaution is exemplified by the case where U is one dimensional so $\mathcal{H} = \mathfrak{N}_c$. Then $C^\infty(U) = \mathbf{C}$ and $\mathcal{H}_0$ acts trivially. Then $\Lambda^p(\mathcal{H}_0, C^\infty(U))$ is just the pth Lie algebra cohomology space of $\mathcal{H}_0$. Certainly our Theorem 1 need not be true in this circumstance.

The role of the square integrability is to insure that the representation U_1/Y used in the induction scheme is locally injective. It seems plausible that there is a generalization of Theorem 1 which assumes neither square integrability nor local

injectivity. Such a theorem probably would require a knowledge of the Lie algebra cohomology of the kernel of U. Of course, it is only the locally-injective, square-integrable case which is relevant to holomorphic induction.

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DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907