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ON A SIMPLICIAL COMPLEX ASSOCIATED TO THE MONODROMY

BY

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ABSTRACT. Suppose we have a complex analytic family, V_t, $|t| < 1$, such that the generic fibre is a nonsingular complex manifold of complex dimension n. Let T denote the monodromy induced from going once around the singular fibre and let I denote the identity map. We shall associate to the singular fibre a simplicial complex Γ, which is at most n-dimensional. Then under certain conditions on the family V_t (which are satisfied for the Milnor fibration of an isolated singularity or if the V_t are compact Kähler), there is an integer $N > 0$ such that $(T^N - I)^k H_k(V_t) = 0$ if and only if $H_k(\Gamma) = 0$.

1. Introduction.

1.1. In this article, we let V be a complex hypersurface having normal crossings, where the complex dimension of V is n. Then we shall construct a simplicial complex Γ corresponding to V, where the real dimension of Γ is at most n. In fact, to each integer $N > 0$, we shall associate to V a complex Γ_N, where $\Gamma_1 = \Gamma$. Then we shall see how these complexes have applications in studying the monodromy about V.

To be more specific, suppose we have V embedded in a complex analytic family V_t, $|t| < 1$, where $V_0 = V$, such that the nearby fibre is nonsingular and let T denote the monodromy induced from going once around the origin. We let a_k denote the number of terms of the type

$$\underbrace{\left.\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & 0 \\ 0 & & \ddots & & \\ & & & 1 & 1 \\ & & & 0 & 1 \end{bmatrix}\right\} k+1}_{k+1}$$

in the Jordan canonical form of the matrix of T acting on $H_k(V_t)$. Then if the analytic family is either (i) embeddable in a compact Kähler manifold with V_t being compact or (ii) is the Milnor fibration about an isolated singular point, then we shall show that there is an N such that $\dim H^k(\Gamma_N) = a_k$. Similar results are true for $k = n$ in case the components of V_0 satisfy certain conditions, regardless of the type of singularity.

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Throughout this article, we shall always denote $H_*(X)$ to mean homology with rational coefficients, for any space X. Moreover $H_*^F(X)$ or $H_*^c(X)$ will denote closed or compact support, respectively.

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2. Construction of the simplicial complex.

2.1. Let W be a complex manifold of complex dimension $n+1$ and let V be a hypersurface of W having normal crossings, i.e., $V = \bigcup_{i\in I} X_i$ where each X_i is a submanifold of W codimension$_{\mathbf{C}} = 1$ and V is locally given by $\prod_{i=1}^k z_i^{a_i} = 0$, where $a_i \geqslant 1$ is the multiplicity of $X_i = \{z_i = 0\}$. For $i_1, \ldots, i_k \in I$ put $X_{i_1\ldots i_k} = \bigcap_{j=1}^k X_{i_j}$. One can suppose that each $X_{i_1\ldots i_k}$ is connected. If not, take the monoidal transform with center on all but one of the components, which creates the new X_i's but $X_{i_1\ldots i_k}$ is now connected.

We also suppose that V is connected by looking at each component.

2.2. Then the simplicial complex Γ is constructed as follows. To each i with $X_i \neq \varnothing$, we associate a vertex σ_i. For $i<j$, we join σ_i to σ_j if and only if $X_{ij} \neq \varnothing$. Note that $X_{ijk} \neq \varnothing$ implies that each of the X_{ij}, X_{jk} and X_{ik} is nonempty. Then, we put a 2-simplex σ_{ijk} with boundary the σ_{ij}, σ_{ik} and σ_{jk} if and only if $X_{ijk} \neq \varnothing$. We continue in this fashion so that to each $X_{i_1\ldots i_{n+1}} \neq \varnothing$ we associate an n-dimensional face $\sigma_{i_1\ldots i_{n+1}}$ whose boundary will be the sum of the $\sigma_{i_1\ldots \hat{i}_j \ldots i_{n+1}}$.

Then Γ is a simplicial complex. We formally define the boundary of Γ by $\partial\Gamma = \{\tau \mid \tau$ is a face of $\partial\sigma$ for some $\sigma \in \Gamma\}$. If I is finite, then Γ is compact. But Γ need not be locally finite.

2.3. We note that Γ is not necessarily n-dimensional. For example, let us start with $\bigcup_{i\in I} S_i$, which is a family of curves in general position. We let $V = \bigcup_{i\in I}(S_i \times \mathbf{C}P_1)$, i.e., $X_i = S_i \times \mathbf{C}P_1$. Then there are no triple points, hence Γ is a one-dimensional complex which is isomorphic to the usual dual of $\bigcup_{i\in I} S_i$.

2.4. As stated above, we have a subcomplex of Γ which we call its boundary. For example, if V_t is defined by $xyz = t$ in $\mathbf{C}^3$, then Γ will be a two-dimensional triangle.

Thus, in general, we have the exact sequence

$$0 \to H_n^c(\Gamma; \mathbf{Z}) \to H_n^c(\Gamma, \partial\Gamma; \mathbf{Z}) \xrightarrow{\partial_*} H_{n-1}^c(\partial\Gamma; \mathbf{Z}).$$

Recall, $H_*^c(X)$ denotes homology with compact support.

2.4.1 Definition. Let $B(\Gamma) \subset H_n^c(\Gamma, \partial\Gamma; \mathbf{Z})$ be the subgroup defined by $\alpha \in B(\Gamma)$ if and only if $\partial_*\alpha$ has a representative of the form $\Sigma \pm \sigma_{i_1\ldots i_n}$ such that $X_{i_1\ldots i_n}$ is a noncompact curve.

Then, of course, $H_n^c(\Gamma; \mathbf{Z}) \subset B(\Gamma)$. The two-simplex of the example $xyz = t$ is a member of $B(\Gamma)$.

2.5. Let $\pi\colon W \to D$ be a holomorphic function, where D is the unit disk in the complex t-plane. Suppose that if $V_t = \pi^{-1}(t)$, then $\pi\colon W - V_0 \to D - \{0\}$ is a locally trivial fibre bundle with fibre a complex manifold and that V_0 has normal crossings.

Let $f\colon D' \to D$ be given by $f(s) = s^N$ where D' is the unit disk in the complex s-plane. Then let $f^* W$ be the induced fibre space over D. Then Mumford **[12]** has shown that if N divides the $\mathrm{lcm}(a_1, \ldots, a_k)$ and one blows-up along centers over V_0, then one has a new V_0' with normal crossings and multiplicities equal to one, i.e., V_0' is locally given by $\prod_{i=1}^k z_i = 0$; the a_i's are equal to one.

2.5.1. If Γ is the complex which corresponds to V_0 and we take $D' \to D$ given by $s \to s^N$, we designate by Γ_N the simplicial complex which corresponds to V_0', where V_0' has multiplicities equal to one.

3. Applications of Γ to questions about monodromy.

3.1. Suppose we have $\pi\colon W \to D$ where $\pi^{-1}(0) = V_0$ has normal crossings with $\pi | W - V_0$ being a locally trivial fibre bundle whose fibre V_t is a nonsingular manifold. Assume further that V_0 is locally given by $\prod_{i=1}^k z_i = 1$, i.e., we study Γ_N and $V_0 = \bigcup_{i \in I} X_i$. Because Γ_N is induced from the N to one cover of $D' \to D$, the study of Γ_N corresponds to the study of T^N on the original $W - V_0$, which is T' on $f^* W - V_0'$.

Then we know from Clemens [3] that $(T^N - I)^{k+1} H_k(V_t) = 0$. Let a_k be the number of terms of the type

$$\underbrace{\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & 0 & \\ 0 & & \ddots & & \\ & & & 1 & 1 \\ & & & 0 & 1 \end{bmatrix}}_{k+1} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\end{matrix}}\right\} k+1$$

in the Jordan canonical form of the matrix T^N.

3.2 Theorem. *Suppose that the fibres V_t are compact and that there exists an embedding of $W \subset W' \times \mathbf{C}P^1$, where W' is a compact Kähler manifold. Suppose further that the following diagram commutes.*

$$\begin{tikzcd} W \arrow[r, phantom, "\subset"] \arrow[d, "\pi"] & W' \times \mathbf{C}P^1 \arrow[d, "\pi'"] \\ D \arrow[r, phantom, "\subset"] & \mathbf{C}P^1 \end{tikzcd}$$

where π' is the projection onto the second factor. Then $\dim H^k(\Gamma_N) = a_k$.

Proof of Theorem 3.2. We first prove it for W' being complex projective. We use the notation of Steenbrink **[14]**. We look at the limiting mixed Hodge structure on $H^k(X_\infty)$, whose weight filtration we denote by $\{W_i^k\}_{i=0}^n$. The filtration may be described in two different ways, which are the same by Steenbrink **[13**, §5**]**. In one way, $W_0^k = \mathrm{Im}(T - I)^k(H_k(V_t))$. I the other way there is a spectral sequence converging to $H^*(X_\infty)$ with $E_\infty^{k,0} = W_0^k$ in $H^k(X_\infty)$, cf. Steenbrink **[14**, §2.9**]**. Furthermore, $E_2^{k,0} = H^0(\tilde{X}_{i_1 \ldots i_{k+1}})$ with the differential being the alternating sum of the restriction maps. Here $\tilde{X}_{i_1 \ldots i_{k+1}}$ denotes the disjoint union of the $X_{i_1 \ldots i_{k+1}}$. But this cochain complex computes the cohomology of Γ_N: $E_2^{k,0} \xrightarrow{\sim} H^k(\Gamma_N, \mathbf{C})$. The spectral sequence degenerates at E_2, so that $E^{k,0} \xrightarrow{\sim} H^k(\Gamma_N, \mathbf{C})$.

However, if W' is a compact Kähler manifold, then the techniques still go through. Namely, if V_0 is a compact Kähler variety, then Hironaka [**8**] has shown that any complex analytic space can be resolved to normal crossings using monoidal transforms. Furthermore, Blanchard [**2**] has shown that the monoidal transform of a Kähler variety is still Kähler. Hence we stay in the category of compact Kähler varieties.

Then in this case, W_0^k is still the image of $(T - I)^k H_k(V_t)$, cf. Clemens [**3**]. Moreover, the spectral sequence still has $E_\infty^{k,0} = W_0^k$ as we are in the case of normal crossings, cf. Griffiths-Schmid [**7**, Chapter 4]. Furthermore, the spectral sequence still degenerates at E_2, which follows from the principle of the two types, cf. Gordon [**6**]. Q.E.D. for Theorem 3.2

3.3. Suppose V_0 dominates a hypersurface $\tilde{V} \subset \mathbf{C}^{n+1}$ for $n \geqslant 1$ and suppose that $\tilde{V}$ has an isolated singularity at the origin. Let X_0 denote the "inverse" image of the proper transform of $\tilde{V}$ in X_0. I.e., because the multiplicity of the proper transform of $\tilde{V}$ in any resolution is always one, there is only one component of V_0 which maps onto $\tilde{V}$. All the other components are sent to the isolated singularity. Let X_0 denote this component.

We let $\Gamma'_N \subset \Gamma_N$ be the simplicial complex associated to $\bigcup_{i>0} X_i$. This corresponds to removing the open star of the vertex $\sigma_0 \in \Gamma_N$.

Let $\tilde{V}$ be defined by $\{f = 0\}$. Recall that the Milnor fibre associated to $\tilde{V}$ is given by $V_t = \{z \in \mathbf{C}^{n+1} | f(z) = t \text{ and } |z| < \varepsilon\}$ for ε sufficiently small, $0 < |t| \ll \varepsilon$.

3.4 THEOREM. *If we are in the situation of* 3.3, *then* $\dim H^k(\Gamma'_N) = a_k$.

PROOF OF THEOREM 3.4. This follows from Steenbrink [**14**, Corollaries 3.9 and 3.10]. Namely to define the mixed Hodge structure on the Milnor fibre, one is forced to mod out by the contribution given by the proper transform. This corresponds to removing the star of the vertex σ_0 in Γ_N. Q.E.D. for Theorem 3.4

3.5. We note that one must pass to Γ'_N as seen in the example in 4.2 that follows.

3.6 DEFINITION. A complex manifold X is *quasi-Kähler* if there is a compact Kähler manifold $\bar{X}$ such that X embeds in $\bar{X}$ so that $\bar{X} - X$ is a subvariety (perhaps empty).

3.7 THEOREM. 3.7.1. *Without any restrictions on* V_t, *we have*

(i) *if we take homology with closed support on* V_t, *then* $\dim H_c^n(\Gamma_N) \geqslant a_n$,

(ii) *if we take homology with compact support on* V_t, *then* $\dim B(\Gamma_N) \geqslant a_n$.

3.7.2. *If each of the* $X_{i_1 \dots i_r}$ *is quasi-Kähler and we take homology with closed support on* V_t, *then* $\dim H_c^n(\Gamma_N) = a_n$.

3.7.3. *If each of the* $X_{i_1 \dots i_r}$ *is quasi-Kähler and in addition* V_0 *dominates an irreducible variety, then in homology with compact support we have* $\dim B(\Gamma_N) = a_n$.

PROOF OF THEOREM 3.7. 3.7.1 follows from the Leray spectral sequence associated to the inclusion of $W - V_0 \subset W$. The reason we must distinguish between compact and closed support is discussed in Gordon [**5**, §4].

3.7.2 follows because in Gordon [**6**] it is shown that under these hypotheses, the Leray spectral sequence degenerates at E_2.

To prove 3.7.3, if V_0 dominates an irreducible variety we let X_0 denote the proper transform. For $n > 2$, each curve $X_{i_1 \ldots i_n}$ will be compact if $i_j > 0$ for all j. This is because each curve will be of codimension greater than one. Hence for $n > 2$, $B(\Gamma_N) = H_n^c(\Gamma_N)$, as each closed n-simplex of Γ_N must have on its boundary at least one $\sigma_{i_1 \ldots i_n}$ with $i_j > 0$ for all j. Q.E.D. for Theorem 3.7

3.8. We need the hypothesis that V_0 dominates an irreducible variety, for if one considers $xyz = t$ in $\mathbf{C}^3$, then $N = 1$, and we find $B(\Gamma_N) \xrightarrow{\sim} \mathbf{Z}$ but $T = I$ on $H_2^c(V_t)$.

3.9. What Theorem 3.7 says is that to compute the space Image$(T^N - I)^n H_n(V_t)$, each n-tuple point of V_0 corresponds to a generator and the relations are given by the curves $X_{i_1 \ldots \hat{i_j} \ldots i_n}$ in closed support (but only the compact curves in compact support). Hence in dualizing to form Γ_N, each n-face of Γ_N is a generator, and the $(n-1)$-faces give relations; but to compute compact support, we allow relative cycles whose boundaries correspond to compact curves.

3.10. It would be nice to say that if each of the $X_{i_1 \ldots i_r}$ is quasi-Kähler, then if $H^k(\Gamma_N) \neq 0$, we have that $(T^N - I)^k H_k(V_t) \neq 0$, as in Theorem 3.2. However, this is not so. For example, let us consider the example of Kodaira **[10]** of a family of Hopf surfaces V_t such that the singular fibre V_0 is a surface acquiring a double curve. Here V_t is diffeomorphic to $S^1 \times S^3$. If we blow-up along this double curve, we obtain X_0 and X_1, nonsingular compact Kähler surfaces with $X_0 \cap X_1$ being two disjoint lines. Then blow-up along one of these lines obtaining X_2, so that $X_i \cap X_j$ is a line for $i \neq j$ and there are no triple points. Also, the multiplicity of X_1 is 2 and X_2 is 3. Thus $N = 6$. Then Γ is the boundary of a two-simplex, so that $H^2(\Gamma_6) = 0$ but $H^1(\Gamma_6; \mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}$. But $T^6 = I$ on $H_2(V_t)$. (In fact, $T^2 = I$.)

What happens is that if one forms the 1-cycle in V_0 by joining a point in $X_i \cap X_j$ by a real line c_i in X_i to $X_i \cap X_k$ for $\{i, j, k\} = \{0, 1, 2\}$, then as $z = t$ defines V_t near X_i, we can find a section over c_i in V_t, hence we get a one-cycle α_1 in V_t. Then $[\alpha_1] \neq 0$, since $0 \neq [c_0 + c_1 + c_2] \in H_1(V_0)$. Hence α_1 is a representative of the generator of $H_1(V_t)$. Let a_{ij} be the class which represents the one-cycle in V_t which is the fibre over a point in X_{ij}, cf. Gordon [5, 2.4] where $a_{ij} = g_*^{-1}(\gamma_{0,ij})$ in the notation of **[5]**. Then $(T^6 - I)[\alpha_1] = a_{10} + a_{02} + a_{12}$. But in this example, $a_{12} = a_{10}$, $a_{12} = a_{20}$, and also $a_{10} = 2a_{20}$. The last equality follows because X_{01} is not homologous to X_{02} in X_0. Hence, $a_{ij} = 0$, i.e., $(T^6 - I)[\alpha_1] = 0$.

Thus, $H_k(\Gamma_N; \mathbf{Z}) \neq 0$ implies one can find an $\alpha \in H_k(V_t)$ with $(T^N - I)^k \alpha = \alpha'$, but we do not a priori know if α' is a nonzero class of $H_k(V_t)$.

3.11. Given an element of $H^k(\Gamma_N)$, one can easily construct the element $\alpha \in H_k(V_t)$ with $(T^N - I)^k \alpha \neq 0$. The construction is the same as is done in Gordon **[5**, §§4 and 5], especially **[5**, Propositions 4.4.1 and 5.3]. An example is given in 4.1.2.

4. Some examples.

4.1. In this section, we shall give some examples of the simplicial complex. We shall first give an example of an isolated singularity for which $H_n(\Gamma_N) \neq 0$. Let us take the example of Malgrange **[11]**.

4.1.1 PROPOSITION. *Let $\tilde{V} \subset \mathbf{C}^3$ be defined by $x^2y^2z^2 + x^8 + y^8 + z^8 = 0$. Then there is a resolution of $\tilde{V}$ to V_0 such that the second betti number of Γ is 4.*

PROOF OF PROPOSITION 4.1.1. We shall always denote the proper transform by X_0.

First take the monoidal transform with center the origin in $\mathbf{C}^3$. Then $X_1 \cap X_0$ gives 3 lines each counted twice, and we blow-up along each of these lines yielding X_2, X_3 and X_4. This gives a resolution of $\tilde{V}$ with $X_1 \cap X_0 = \varnothing$. For $2 \leq i < j \leq 4$ we have that $X_i \cap X_j \cap X_0$ has two distinct points.

But to construct Γ, it is necessary that each component of X_{ijk} be connected, so we blow-up one of the two triple points for each of the X_{ij0}. This gives us X_5, X_6 and X_7 with the multiplicity of X_i for $i = 0, 1, 2, 3, 4, 5, 6, 7$ being 1, 6, 8, 8, 8, 2, 2, 2, respectively.

Then with all this data, we can give a presentation of Γ: there are 8 vertices, 18 edges, and 15 faces. Then Γ can be described as: We start with three one chains, σ_{23}, σ_{34} and σ_{24}, which form a one-cycle γ_1. Then we form two cones over γ_1 with vertices σ_0 and σ_1. This defines one 2-cycle. Over each of the boundaries of the 2-dim faces σ_{023}, σ_{034} and σ_{024}, we form the cones with vertices σ_5, σ_6 and σ_7 respectively. This gives 3 other 2-cycles and finishes the description of Γ. Q.E.D. for Proposition 4.1.1

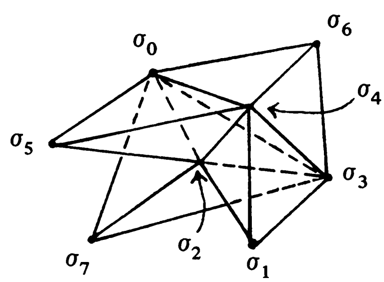

Figure 1

4.1.2. One can easily give a class $\gamma_2 \in H_2(V_t)$ with $(T^{24} - I)^2\gamma_2 \neq 0$ where $V_t = \{(x, y, z) | x^2y^2z^2 + x^8 + y^8 + z^8 = t\}$. Namely, take the cone with vertex σ_7. Locally at X_{ijk}, we have V_t is given by $x^{a_i}y^{a_j}z^{a_k} = t$ where $a_i = 1, 6, 8, 2$ for $i = 0, 2, 4, 7$, respectively. We note that if P is the triple point, then $[\tau_{1,i}\tau_{2,j}\tau_{3,k}(P)] \cap (x^{a_i}y^{a_j}z^{a_k} = t) = g_{\#}^{-1}(\gamma_{0,ijk})$ contains $\gcd(a_i, a_j, a_k)$ distinct cycles, cf. [5, §2.4] for the notation being used.

Hence, take the map of the unit disk into itself given by $s \to s^{24} = t$ where $24 = \operatorname{lcm}(1, 2, 6, 8)$. Then if we take the induced fibre product from this mapping, we find that the number of points mapping onto X_{ijk} is $\gcd(a_i, a_j, a_k)$, hence we get one point for X_{0jk} and two points for X_{247}. Similarly, X_{47} and X_{27} are covered two to one, so that we must add a new component $X_8 \approx X_7 \approx \mathbf{C}P_2$ with $X_{07} = X_{08} \approx \mathbf{C}P_1$ and $X_{28} \approx X_{48} \approx \mathbf{C}P_1$ and $X_{248} \neq \varnothing$. Hence when we blow-up along X_{08} to get normal crossings, we get an X_9 and the following configuration as a part of Γ_{24}.

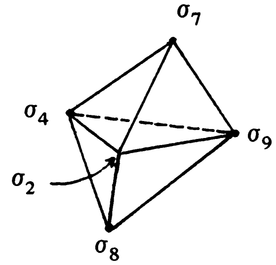

Figure 2

In each double curve, X_{ij}, where $(i, j) = \{(2, 4), (2, 7), (2, 9), (8, 9), (4, 8), (2, 8), (4, 9), (7, 9), (4, 7)\}$, take a real line c_{ij} between X_{ijk} and X_{ijl} where the triples run over 247, 248, 489, 289, 479 and 279. Then let $\gamma_i = \sum_j c_{ij}$, which is a one-dimensional cycle in X_i, for $i = 2, 4, 7, 8, 9$. Then $\gamma_i = \partial_{\#} c_i$ as noted in Gordon [5, §5]. Let $\tilde{c}_i$ be the cross-section in V_t over c_i, as discussed in [5, Proposition 5.3], and let $\alpha_1 = \sum_i \tilde{c}_i$. Then $(T^{24} - I)^2\alpha_1 \neq 0$.

4.2. That we must pass to $\Gamma'_N \subset \Gamma_N$ for isolated singularities is given by the following example. Consider the singularity $\tilde{V} = \{(x^2 + y^3)(x^3 + y^2) + z^2 = 0\}$ in $\mathbf{C}^3$. Then $\tilde{V}$ is nonsingular except at 6 isolated points.

4.2.1 PROPOSITION. *There is a resolution of $\tilde{V}$ to V_0 such that the second betti number of Γ associated to V_0 is equal to one.*

PROOF OF PROPOSITION 4.2.1. We shall just briefly sketch the resolution. First blow-up the origin in $\mathbf{C}^3$. Then this creates X_1 with $X_1 \cap X_0$ a line in $\mathbf{C}P_2$ such that X_0 is singular along this line. Then blow-up this line creating X_2 with $X_2 \cap X_0$ being two lines meeting in two points P_1 and P_2. Next, blow-up each of the points P_1 and P_2 creating X_3 and X_4 such that $X_i \cap X_0 = X_i \cap X_2 =$ a line for $i = 3$ and 4. Then blow-up along $X_i \cap X_0$ for $i = 3$ and 4 which creates X_5 and X_6, respectively. Finally, we blow-up along one of the lines of $X_2 \cap X_0$, yielding an X_7, and X_0 is nonsingular in a neighborhood of $\bigcup_i X_i \cap X_0$.

Hence, we have a resolution of the origin with eight components X_i, $i = 0, 1, \ldots, 7$, of multiplicity $(1, 2, 4, 5, 5, 10, 10, 5)$ respectively. There are 8 vertices, 12 edges and 6 faces on Γ.

The other five points are given by a 2-fold covering branched along 2 curves intersecting transversely at smooth points of each of the curves, i.e., $0 = z^2 + (x + g_1(x, y))(y + g_2(x, y))$ is the local equation of $\tilde{V}$ at each of the other 5 isolated points, where the leading term of the g_i is of degree two. Hence, each of the five singular points will be resolved with one monoidal transform which contributes $X_8, X_9, X_{10}, X_{11}, X_{12}$, all of multiplicity two.

A representation of Γ is given in Figure 3. Q.E.D. for Proposition 4.2.1

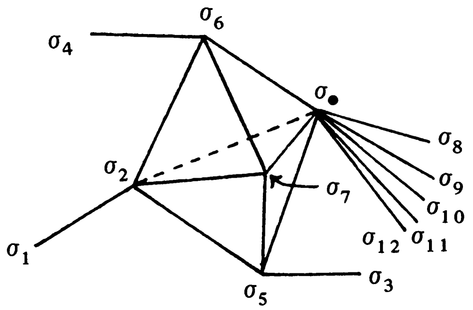

Figure 3

Thus, if $V_t = \{(x^2 + y^3)(x^3 + y^2) + z^2 = t\}$, then for t sufficiently small, and for all N, $(T^N - I)^2 H_2^c(V_t) \neq 0$. This follows because if for some N this were not true, i.e., $(T^N - I)^2 H_2^c(V_t) = 0$, it would also be zero if we replace N by $(20) \cdot N$. But by Proposition 4.2.1 and Theorem 3.2, $(T^{20 \cdot N} - I)^2 H_2^c(V_t) \neq 0$.

But, if we consider the local monodromy, i.e., $V_t \cap D_\varepsilon$, for D_ε a sufficiently small ball about the origin, we have that $(T^{20} - I)^2 H_2(V_t \cap D_\varepsilon) = 0$. This last fact follows from the results of Sebastiani-Thom **[12]** which state that the monodromy of $V_t \cap D_\varepsilon$ is minus the monodromy of $((x^2 + y^3)(x^3 + y^2) = t) \cap D_\varepsilon$.

The reason this happens is as follows: At all the triple points $x^{a_1} y^{a_2} z^{a_3} = t$, we have $\gcd(a_1, a_2, a_3) = 1$, so that when we form the 20 to one cover, $20 = \operatorname{lcm}(1, 2, 4, 5, 10)$, we add no new triple points, hence no new faces are added when we form Γ_{20}.

However, if one forms the one-cycle γ_0 in X_0 by joining X_{026} to X_{067} by a real line in X_{06}, joining X_{067} to X_{057} by a real line in X_{07}, X_{057} to X_{025}, and finally, X_{025} back to X_{026}, then $\gamma_0 = \partial_\# c_0$. Furthermore, c_0 is a compact 2-chain in X_0 such that if π: $V \to \tilde{V}$ is the resolution, then $\pi_\# c_0$ is a compact 2-chain in $\tilde{V} - S$, where S is the 6 points which constitute the singular locus of $\tilde{V}$. But $\pi_\# c_0$ cannot be deformed so that it lies in $D_\varepsilon \cap X_0$.

However, as in 4.1.2, we can form $\alpha_2 \in H_2(V_t)$ and $(T^{20} - I)\alpha_2 = \alpha \in H_2(V_t)$ such that α_2 has a representative which lies in $V_t \cap D_\varepsilon$, i.e., $\alpha_2 \in H_2(V_t \cap D_\varepsilon)$ and $T^N \alpha_2 \neq \alpha_2$ for all N.

Analogous to this, one can show

4.2.2 Proposition. *Suppose P is an isolated singularity of a hypersurface in $\mathbf{C}^{n+1}$ and there is a resolution of the isolated singularity so that $H^n(\Gamma_N) \neq 0$ for some N. Then for all M, $(T^M - I)^{n-1} \neq 0$, where T denotes the local monodromy about P.*

4.3. One can also give examples with $n > 1$ where $(T^N - I)^2 \neq 0$, but $H_n(\Gamma) = 0$, i.e., one must go to Γ_N for $H_n(\Gamma_N) \neq 0$. For example, take the example of Karras given by $z^3 = (x + y^6)(x^2 + y^6)$. Then the monodromy of the Milnor fibre has $(T^{990} - I)^2 = 0$ and $H^2(\Gamma) = 0$. But $H^2(\Gamma_{990}) \neq 0$, which means that if one looks at the monodromy in all of $\mathbf{C}^3$, $(T^{990} - I)^2 \neq 0$.

4.4. One can give a large class of examples where $H^n(\Gamma_N) \neq 0$. Namely, let $V(n_0, \ldots, n_n; m_0, \ldots, m_n) = \{(z_0, \ldots, z_n) | \prod_{i=0}^n z_i^{n_i} + \sum_{i=0}^n z_i^{m_i} = 0\}$ with each $m_i > 1$ and each $n_i > 0$. Then by the same method used by Malgrange [**11**], one can show that if all the n_i and m_i are even, then there is an $N > 0$ such that for the monodromy of the Milnor fibre,

$$(T^N - I)^n \neq 0 \quad \text{if and only if } \sum_{i=0}^n \frac{n_i}{m_i} < 1.$$

After having made the calculation for several other examples, it appears that $H^n(\Gamma_N(n_0, \ldots, n_n; m_0, \ldots, m_n)) \neq 0$ when $\sum_{i=0}^n n_i/m_i < 1$, even if the n_i or m_i are odd.

There seems to be a connection between these examples and with the concept of modality of Arnol'd [**1**]. For example, $x^p + y^q + z^r + xyz$ is unimodal if $1/p + 1/q + 1/r < 1$. For all of these, we have that $H^2(\Gamma_N(1, 1, 1; p, q, r)) \neq 0$ if $1/p + 1/q + 1/r < 1$. However, $x^3 + y^3 + z^4 + xyz^2$ is also unimodal (it is U_{12} in [**1**, p. 227]), but $T^{24} = I$ for this singularity.

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