TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 261, Number 1, September 1980

ON THE GROUP OF VOLUME-PRESERVING DIFFEOMORPHISMS OF $\mathbf{R}^n$

BY

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ABSTRACT. The group of all diffeomorphisms of $\mathbf{R}^n$ which preserve a given volume form is shown to be perfect when $n > 3$. Some useful factorizations of such diffeomorphisms are also obtained.

In this note we prove

THEOREM. *Let Ω be any volume form (that is, nonvanishing C^∞ n-form) on $\mathbf{R}^n$. Then the group* $\operatorname{Diff}_\Omega \mathbf{R}^n$ *of all C^∞ diffeomorphisms of $\mathbf{R}^n$ which preserve the form Ω is perfect, provided that $n \geqslant 3$.*

REMARK. It follows easily from Moser [7] (see [3]) that there are only two distinct cases of this theorem, namely $\operatorname{vol}_\Omega \mathbf{R}^n < \infty$ and $\operatorname{vol}_\Omega \mathbf{R}^n = \infty$.

When $n = 1$, the group $\operatorname{Diff}_\Omega \mathbf{R}$ is either trivial or isomorphic to $\mathbf{R}$, and the theorem is trivially true in the first case and trivially false in the second. On the other hand, the case $n = 2$ is potentially interesting. It has been shown by Thurston [8] and Banyaga [1] that this is the only dimension in which the identity component of the group of compactly supported volume-preserving diffeomorphisms of $\mathbf{R}^n$ is not perfect. Also, the volume-preserving and symplectic cases coincide in dimension 2. The present arguments do not work for the group of symplectic diffeomorphisms in any dimension. However the contact case is more tractable (see Banyaga and Pulido [2]).

A proof of the above theorem in the case when $\operatorname{vol}_\Omega \mathbf{R}^n = \infty$ is given in [6]. The present proof is easier and more direct. It also yields a factorization lemma (Lemma 1) which turns out to be crucial in working out the normal subgroups of $\operatorname{Diff}_\Omega \mathbf{R}^n$. This, together with the generalisation to manifolds other than $\mathbf{R}^n$, will be discussed elsewhere.

The present methods owe much to Ling [4] who used them to calculate the normal subgroups of the group of all diffeomorphisms of $\mathbf{R}^n$. Extra techniques are needed here in order to deal with the difficulties which are caused by the fact that the radial maps $x \mapsto \lambda(\|x\|)x$ do not preserve volume. In particular, in order to prove the factorization lemma when $n = 3$ we use a surprising but very elementary observation about knots in $\mathbf{R}^3$ (Lemma 8).

This paper is organised as follows. In the first section, we state the main lemmas and then use them to prove the Theorem. The factorization lemmas are proved in §2 and the lemmas about cells are proved in §3. For the convenience of the reader,

Received by the editors March 14, 1979 and, in revised form, October 2, 1979.
AMS (MOS) subject classifications (1970). Primary 57E99; Secondary 57D50.

0002-9947/80/0000-0405/$03.75

the needed results about extending volume-preserving diffeomorphisms are stated in an appendix.

1. We will say that an element $f \in \mathrm{Diff}_\Omega \mathbf{R}^n$ admits a *Ling factorization* with p factors if it may be written as a product $h_1 \cdots h_p$ of diffeomorphisms $h_j \in \mathrm{Diff}_\Omega \mathbf{R}^n$, each of which has support in a locally finite union $\amalg_{i>0} C_i$ of disjoint cells. (By definition, a *cell* is a smoothly embedded closed n-disc.) For short, we will often call such a union $\amalg_{i>0} C_i$ a disjoint union.

The main factorization lemma is the following.

LEMMA 1. *If $n \geqslant 3$, every element of* $\mathrm{Diff}_\Omega \mathbf{R}^n$ *has a Ling factorization.*

REMARK. This lemma is also true in the symplectic case when $n > 4$. However it is not clear that the disjoint unions $\amalg_{i>0} C_i$ which support the various factors are contained in disjoint unions of symplectic cells (i.e. cells which are symplectically embedded discs). Therefore Lemma 4 may fail in the symplectic case.

When $\mathrm{vol}_\Omega \mathbf{R}^n < \infty$, a second factorization lemma is useful.

LEMMA 2. *If $n \geqslant 3$, any diffeomorphism $f \in \mathrm{Diff}_\Omega \mathbf{R}^n$ with support in the interior of a cell C is the product $h_1 h_2 h_3$ of three elements $h_j \in \mathrm{Diff}_\Omega \mathbf{R}^n$ which are supported by the interiors of cells E_j, where $E_j \subset \mathrm{Int}\, C$ and $\mathrm{vol}_\Omega E_j < \frac{2}{3}\mathrm{vol}_\Omega C$.*

These lemmas are proved in §2. We also will need two results about cells which will be proved in §3.

LEMMA 3. *Let $n \geqslant 2$. Suppose that $\amalg_{i>0} C_i$ is a disjoint union of cells and that $w_i \geqslant \mathrm{vol}_\Omega C_i$ for all $i \geqslant 0$. Suppose also that $\Sigma_i(w_i - \mathrm{vol}_\Omega C_i) \leqslant \mathrm{vol}_\Omega(\mathbf{R}^n - \amalg_{i>0} C_i)$, with strict inequality if both sides are finite. Then there is a disjoint union $\amalg_{i>0} D_i$ of cells D_i which have Ω-volume w_i, contain C_i and satisfy the condition $\mathrm{vol}_\Omega(\mathbf{R}^n - \amalg_{i>0} D_i) = \infty$ if $\mathrm{vol}_\Omega(\mathbf{R}^n - \amalg_{i>0} C_i) = \infty$.*

LEMMA 4. *If $\amalg_{i\geqslant 0} C_i$ and $\amalg_{i>0} D_i$ are disjoint unions such that $\mathrm{vol}_\Omega C_i = \mathrm{vol}_\Omega D_i$ for all i and $\mathrm{vol}_\Omega(\mathbf{R}^n - \amalg_{i>0} C_i) = \mathrm{vol}_\Omega(\mathbf{R}^n - \amalg_{i>0} D_i)$, then there is $g \in \mathrm{Diff}_\Omega \mathbf{R}^n$ such that $g(C_i) = D_i$ for all i.*

PROOF OF THEOREM. *Case* (i). $\mathrm{vol}_\Omega \mathbf{R}^n = \infty$. (This argument is adapted from Ling [4].)

By Lemma 1 it suffices to show that any element h with support in some disjoint union $\amalg_{i>0} C_i$ is in the commutator subgroup of $\mathrm{Diff}_\Omega \mathbf{R}^n$. By considering the restrictions of h to $\amalg_{i\in J} C_i$ and $\amalg_{i\notin J} C_i$ separately, for some suitable subset J of $\mathbf{N}$, we may suppose that $\mathrm{vol}_\Omega(\mathbf{R}^n - \amalg_{i>0} C_i) = \infty$. Then, by Lemma 3, we may replace the C_i by larger cells which satisfy the conditions $\mathrm{vol}_\Omega C_i < \mathrm{vol}_\Omega C_{i+1}$ for all i and $\mathrm{vol}_\Omega(\mathbf{R}^n - \amalg_{i>0} C_i) = \infty$. It now follows from Lemma 4 that there is $g \in \mathrm{Diff}_\Omega \mathbf{R}^n$ such that $g(C_i) \subseteq C_{i+1}$ for all i. (Take the D_i in that Lemma to be suitable subcells of C_{i+1}.)

Define $s \in \mathrm{Diff}_\Omega \mathbf{R}^n$ by

$$s(x) = h(ghg^{-1}) \cdots (g^i h g^{-i})(x) \quad \text{if } x \in C_i \text{ for some } i \geqslant 0,$$
$$= x \quad \text{otherwise.}$$

Then $\operatorname{supp} s \subseteq \amalg_{i\geq 0} C_i$ and $\operatorname{supp} gsg^{-1} \subseteq \amalg_{i\geq 1} C_i$, so that $[s, g] = sgs^{-1}g^{-1}$ has support in $\amalg_{i\geq 0} C_i$. Also, inspection shows that within each C_i we have $hgsg^{-1} = s$. Therefore, $[s, g] = h$, and h is a commutator as required.

Case (ii). $V = \operatorname{vol}_\Omega \mathbf{R}^n < \infty$.

We first show that $\operatorname{Diff}_\Omega \mathbf{R}^n$ is generated by elements h which are supported in the interior $\amalg_{i\geq 0}(\operatorname{Int} C_i)$ of some disjoint union of cells whose volumes $v_i = \operatorname{vol}_\Omega C_i$ satisfy

(i) $\frac{1}{2}v_i \leq v_{i+1} \leq v_i$ for all i and

(ii) $\Sigma_i v_i < \frac{1}{2}V$.

Notice to begin with that, by Lemma 1, $\operatorname{Diff}_\Omega \mathbf{R}^n$ is generated by elements h with support contained in some disjoint union $\amalg_{i\geq 0}(\operatorname{Int} C_i'')$. By Lemma 2 we can represent each h as a 3^4-fold product of elements h' which are supported in $\amalg_{i\geq 0} C_i'$, with $C_i' \subset C_i''$ and also $v_i' < (\frac{2}{3})^4 v_i'' < \frac{1}{4} v_i''$. (Here $v_i' = \operatorname{vol}_\Omega C_i'$ and $v_i'' = \operatorname{vol}_\Omega C_i''$.) Thus $\Sigma_i v_i' < \frac{1}{4}V$. After renumbering, we may assume that $v_0' \geq v_1' \geq \cdots$. Now define the numbers $v_0, v_1, \ldots$ with $v_i' \leq v_i \leq v_i' + \frac{1}{2}v_{i-1}$ inductively as follows. Let $v_0 = v_0'$. Put $v_{i+1} = v_{i+1}' \leq v_i' \leq v_i$ if $v_{i+1}' \geq \frac{1}{2}v_i$, and put $v_{i+1} = v_{i+1}' + \frac{1}{2}v_i < v_i$ otherwise. Clearly, the v_i satisfy condition (i). They also satisfy condition (ii). For, because $v_i \leq v_i' + \frac{1}{2}v_{i-1}$, induction shows that $v_i \leq \Sigma_{k=0}^i v_{i-k}'/2^k$. Summing over i, we obtain

$$\sum_i v_i \leq \sum_i \left(\sum_{k=0}^{\infty} 1/2^k \right) v_i' = \sum_i 2v_i' < 2\left(\tfrac{1}{4}V\right) = \tfrac{1}{2}V.$$

Thus, by replacing the cells C_i' by larger cells C_i with volumes v_i as in Lemma 3, we may suppose that the conditions (i) and (ii) are satisfied.

We now show that any h which satisfies these conditions is a product of commutators in $\operatorname{Diff}_\Omega \mathbf{R}^n$. First, use Lemma 3 to find a disjoint union $\amalg_{i\geq 1} D_i$ such that $C_i \subset D_i$ and $\operatorname{vol}_\Omega D_i = v_{i-1}$ for all $i \geq 1$. (There is enough room for this by (ii).) Next, use Lemma 4 to find a diffeomorphism $g \in \operatorname{Diff}_\Omega \mathbf{R}^n$ which takes C_i onto D_{i+1} for each $i \geq 0$. Because $(\frac{2}{3})^2 v_i \leq v_{i+1}$ by (i), it follows from Lemma 2 that any diffeomorphism with support in $\operatorname{Int} C_i$ may be written as the product of at most 9 diffeomorphisms which are supported by cells in $\operatorname{Int} C_i$ which have volume less than v_{i+1}. Therefore, we may construct elements s, l_j and k_j for $1 \leq j \leq 9$, such that

(a) $\operatorname{supp} s \subset \amalg_{i\geq 0}(\operatorname{Int} C_i)$,

(b) $s = k_1 \cdots k_9$ where $\operatorname{supp} k_j \subset \amalg_{i\geq 0}(\operatorname{Int} E_{ij})$ and where the cells $E_{ij} \subset \operatorname{Int} C_i$ satisfy $\operatorname{vol}_\Omega E_{ij} < v_{i+1}$,

(c) each l_j maps E_{ij} into $\operatorname{Int} C_{i+1}$ and satisfies $\operatorname{supp}(g^{-1}l_j) \subseteq \amalg_{i\geq 0} C_i$ so that l_j coincides with g outside the C_i and

(d) $s = h(l_1 k_1 l_1^{-1}) \cdots (l_9 k_9 l_9^{-1})$.

In fact, one can define these mappings inductively over the C_i, starting with $s = h$ on C_0. Given s on C_i, one chooses the k_j and l_j on C_i satisfying (b), (c), then defines s on C_{i+1} by (d). Now, comparing formulas (b) and (d) we have $k_1 \cdots k_9 = h(l_1 k_1 l_1^{-1}) \cdots (l_9 k_9 l_9^{-1})$ or, in other words,

$$h = k_1 \cdots k_9 \left(l_9 k_9 l_9^{-1}\right)^{-1} \cdots \left(l_1 k_1 l_1^{-1}\right)^{-1}.$$

Clearly the right side of this equation is congruent to the identity modulo the commutator subgroup. In fact it is not difficult to express it as a product of 9 commutators. □

REMARK. We will see that if $n \geq 3$ every $f \in \mathrm{Diff}_\Omega \mathbf{R}^n$ has a Ling factorization with at most 14 factors. It follows that there is a number M such that every element of $\mathrm{Diff}_\Omega \mathbf{R}^n$ is the product of at most M commutators. The present proof would give $M = 28$ if $\mathrm{vol}_\Omega \mathbf{R}^n = \infty$, and $M = 10206$ otherwise, although presumably one could do much better.

2. Proof of the factorization lemmas. We will call a continuous map F, from $[0, 1]$ to the group $\mathrm{Diff}_\Omega \mathbf{R}^n$ provided with the compact-open C^∞-topology, an *Ω-isotopy* from $F(0)$ to $F(1)$. Also, if X is an n-dimensional submanifold of $\mathbf{R}^n$ which is closed as a subset of $\mathbf{R}^n$, we will write $\mathrm{Diff}_{\Omega 0}(X, \mathrm{rel}\ \partial)$ for the subgroup of $\mathrm{Diff}_\Omega \mathbf{R}^n$ consisting of diffeomorphisms which are Ω-isotopic to the identity by an isotopy $t \mapsto f_t$ which is supported by the interior $\mathrm{Int}\, X = X - \partial X$ of X. For convenience, we will assume throughout this section that Ω is the standard volume form $dx_1 \wedge \cdots \wedge dx_n$ on $\mathbf{R}^n$ if $\mathrm{vol}_\Omega \mathbf{R}^n = \infty$, and otherwise, that it is spherically symmetric, that is that $\Omega(x) = \zeta(\|x\|^2) dx_1 \wedge \cdots \wedge dx_n$ for some nonvanishing smooth function ζ. This is permissible by the generalisation of Moser's theorem to noncompact manifolds [3].

The following result is surely well known. However I know of no proof in the literature.

LEMMA 5. *Every element of* $\mathrm{Diff}_\Omega \mathbf{R}^n$ *is Ω-isotopic to the identity.*

PROOF. If $\mathrm{vol}_\Omega \mathbf{R}^n = \infty$, an Ω-isotopy from f to the identity may be constructed as follows. First compose f with translations to take it to an element g which fixes 0. Then make g linear by the standard isotopy given by

$$g_t(x) = g(tx)/t, \qquad 0 < t \leq 1, \quad \text{and} \quad g_0(x) = \lim_{t \to 0} g(tx)/t.$$

Finally, join g_0 to the identity by a path in the connected group $SL(n, \mathbf{R})$.

If $\mathrm{vol}_\Omega \mathbf{R}^n < \infty$, we may identify $(\mathbf{R}^n, \Omega)$ with (D, Ω_0), where D is an open disc in $\mathbf{R}^n$ centered at 0 and $\Omega_0 = dx_1 \wedge \cdots \wedge dx_n$. Then, if f is an Ω_0-preserving diffeomorphism of D, by restricting the isotopy described above to a suitable neighbourhood U of 0 we get a path f_t of Ω_0-embeddings of U into D such that f_0 is the inclusion and $f_1 = f|U$. By Lemma A in the Appendix, there is an ambient Ω_0-isotopy h_t of D which equals f_t near 0. Thus f is Ω_0-isotopic to $h_1^{-1}f$, an element which is the identity near 0. (Observe that this procedure is valid even if f does not fix 0 since there is no need for the isotopy f_t to fix 0.)

Now notice that the subgroup of Ω_0-preserving diffeomorphisms of D which are the identity on some open disc $D_\varepsilon = \{x\colon \|x\| < \varepsilon\}$ may be identified with the group G_λ of diffeomorphisms of $\mathbf{R}^n - \{0\}$ which are the identity outside some open disc $D_\lambda = \{x\colon \|x\| < \lambda\}$ and which preserve Ω_0. (Indeed, to make this identification, it suffices to construct an Ω_0-preserving diffeomorphism $\psi\colon D - D_\varepsilon \to \overline{D}_\lambda - \{0\}$. Choosing λ so that $\mathrm{vol}_{\Omega_0} D_\lambda = \mathrm{vol}_{\Omega_0}(D - D_\varepsilon)$, we may take ψ to be a radial diffeomorphism of the form $x \to \theta(\|x\|)x$, where θ is a suitable diffeomorphism $[\varepsilon, \mu) \to (0, \lambda]$ and μ is the radius of D.) Any element f of the group G_λ is Ω_0-isotopic to the identity by the "Alexander" isotopy f_t, where $f_0 = \text{id}$ and f_t, $0 < t \leqslant 1$, is defined by $f_t(x) = tf(x/t)$. The result follows. □

We next show that, just as in the non-volume-preserving case, $\text{Diff}_\Omega\mathbf{R}^n$ is generated by diffeomorphisms which are supported by disjoint unions of annuli, $\amalg_{i \geqslant 0} A_i$. Here

$$A_i = \{x: \lambda_{2i} \leqslant \|x\| \leqslant \lambda_{2i+1}\},$$

where $0 = \lambda_0 < \lambda_1 < \cdots$ and $\lambda_i \to \infty$. (Observe that A_0 is in fact a disc.)

Lemma 6. *If $n \geqslant 2$, every $f \in \text{Diff}_\Omega\mathbf{R}^n$ may be written as a product $g_1 g_2$ where each g_j belongs to some group* $\text{Diff}_{\Omega 0}(\amalg_{i \geqslant 0} A_i, \text{rel } \partial)$.

Proof. By Lemma 5 there is an Ω-isotopy f_t, $0 \leqslant t \leqslant 1$, from $f_0 = \text{id}$ to $f_1 = f$. Let $\mu_1 = 1$ and choose a number μ_2 so that the image $f_t(\mu_1 S)$ of the unit sphere $\mu_1 S$ under the isotopy f_t lies inside the sphere $\mu_2 S$ of radius μ_2 for all t. Then, by Lemma A, there is an Ω-isotopy g_t defined inside $\mu_2 S$ which equals f_t near $\mu_1 S$ and the identity near $\mu_2 S$. Next, choose $\mu_3 < \mu_4$ so that $f_t(\mu_3 S)$ always lies outside $\mu_2 S$ and inside $\mu_4 S$. Then g_t may be extended to the interior of $\mu_4 S$ in such a way that it equals the identity near $\mu_2 S$ and $\mu_4 S$ and equals f_t near $\mu_3 S$. Continuing in this way, we construct g_t to equal f_t near each $\mu_{2i-1} S$ and to equal the identity near each $\mu_{2i} S$, where $\mu_1 < \mu_2 < \cdots$ and $\mu_i \to \infty$. Clearly, g_1 and $g_2 = g_1^{-1} f$ have the required form. □

Lemma 7. *Let A be the annulus $0 < \lambda_1 \leqslant \|x\| \leqslant \lambda_2$ and r be the ray segment $\{\lambda y: \lambda_1 \leqslant \lambda \leqslant \lambda_2\}$, where $\|y\| = 1$. Then, any element f of* $\text{Diff}_{\Omega 0}(A, \text{rel } \partial)$ *which equals the identity on r is the product of at most 5 elements of* $\text{Diff}_{\Omega 0}(A, \text{rel } \partial)$ *which are supported by cells.*

Proof. *Case* (i). $n \geqslant 4$. Taking the derivative df_{ty} of f at points ty of r, one gets a loop $t \mapsto df_{ty}$, $\lambda_1 \leqslant t \leqslant \lambda_2$, in $SL(n, \mathbf{R})$. This loop is contractible because f is Ω-isotopic to the identity relative to the boundary of A. Since $f = \text{id}$ on r, these derivatives df_{ty} in fact lie in the subgroup G_n of $SL(n, \mathbf{R})$ consisting of matrices with first row $(1, 0, \ldots, 0)$. Because $G_n \simeq SL(n-1, \mathbf{R})$ and

$$\pi_1 SL(n-1, \mathbf{R}) \xrightarrow{\cong} \pi_1 SL(n, \mathbf{R}) \cong \mathbf{Z}/2\mathbf{Z} \quad \text{if } n > 4,$$

this loop contracts in G_n as well. It follows that there is $h \in \text{Diff}_{\Omega 0}(A, \text{rel } \partial)$ which has support in a cell containing r such that $fh = \text{id}$ near r. To see this, first construct a (not necessarily volume-preserving) isotopy f_t with $f_0 = f$, $f_t = f$ outside a small neighbourhood of r in A for all t, and $f_1 = \text{id}$ in a tubular neighbourhood T of r. Then, by Lemma A, there is an Ω-isotopy h_t in $\text{Diff}_{\Omega 0}(A, \text{rel } \partial)$ extending $f^{-1} f_t | \partial T$, which is supported by a cell containing r and is such that $h_1 = f^{-1}$ on T. Clearly, we may take $h = h_1$. Thus fh has support in a cell of the form $A -$ (nbhd of r), and so $f = (fh)h^{-1}$ may be factored into 2 elements of the required type.

Case (ii). $n = 3$. In this case the loop $t \mapsto df_{ty}$, $\lambda_1 \leqslant t \leqslant \lambda_2$, need not contract in G_3. Instead it is homotopic in G_3 to a loop in $SO(2)$ which represents an element of the kernel of the homomorphism $\mathbf{Z} \cong \pi_1 SO(2) \to \mathbf{Z}/2\mathbf{Z} \cong \pi_1 SL(3, \mathbf{R})$. For each integer k, let γ_k be the loop representing the element $2k \in \mathbf{Z} \cong \pi_1 SO(2)$ which is defined as follows. $\gamma_k(t)$, for $t \in [\lambda_1, \lambda_2]$, is the rotation through the angle $4\pi\theta(k, t)$, where $\theta(k, \cdot)$ is a smooth function of t which has graph as in Figure 1 if $k > 0$. We put $\theta(0, t) = 0$ for all t, and $\theta(-k, t) = -\theta(k, t)$.

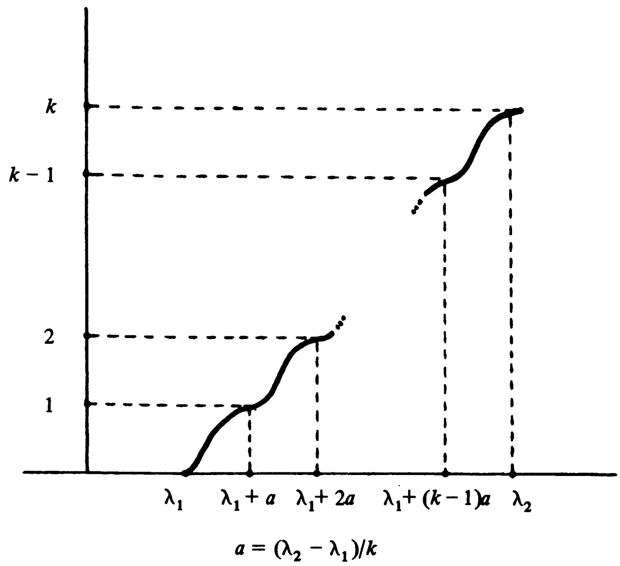

FIGURE 1. Graph of $\theta(k, \cdot\,)$ for $k > 0$

Consider the diffeomorphism g_k of A which rotates each sphere of radius t, $\lambda_1 \leqslant t \leqslant \lambda_2$, about the axis r and through the angle $4\pi\theta(k, t)$. (Thus $g_0 = \mathrm{id}$ and $g_{-k} = g_k^{-1}$.) Since Ω is spherically symmetric, g_k preserves Ω. Also, because γ_k contracts in $SL(3, \mathbf{R})$, it is not hard to see that $g_k \in \mathrm{Diff}_{\Omega 0}(A, \mathrm{rel}\ \partial)$. Given any $f \in \mathrm{Diff}_{\Omega 0}(A, \mathrm{rel}\ \partial)$, we may choose k so that the loop $t \mapsto d(g_k^{-1}f)_{ty}$ contracts in G_3. The argument of case (i) then applies to show that $g_k^{-1}f$ may be factored into 2 elements which are supported by cells. Therefore, in order to factor f into 5 factors, it suffices to factor each g_k into 3 factors.

When $k = 1$ this may be done as follows. Clearly, it suffices to find two elements $h_1, h_2 \in \mathrm{Diff}_{\Omega 0}(A, \mathrm{rel}\ \partial)$ which are supported by cells of the form A − (nbhd of ray) and are such that $h_1 h_2 = g_1$ near the ray γ, since then we have $g_1 = h_1 \cdot h_2 \cdot (h_1 h_2)^{-1} g_1$. Such h_1 and h_2 may be found because of the well-known fact that a ribbon in $\mathbf{R}^3$ with a total twist of 4π between its fixed ends E and F may be untwisted by passing it once around F. (See Figure 2.) Thus g_1, and hence also $g_{-1} = g_1^{-1}$, has the required factorization.

When $|k| > 1$, observe that g_k is the product of k "disjoint copies" of g_1 or g_{-1}, one supported in each annulus $A_i = \{x \colon \lambda_1 + ia < \|x\| < \lambda_1 = (i + 1)a\}$. Therefore, each g_k also factors as the product of 3 elements of $\mathrm{Diff}_{\Omega 0}(A, \mathrm{rel}\ \partial)$ which are supported by cells. (Each of these cells will contain the union of k disjoint cells, one in each A_i.) □

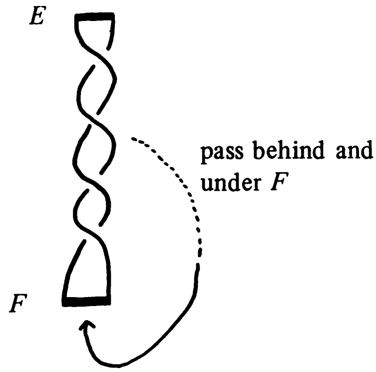

FIGURE 2

The last ingredient in the proof of Lemma 1 is the following result about knots.

LEMMA 8. *Let γ_1 and γ_2 be two smooth arcs from P to Q in $\mathbf{R}^3$ whose interiors are disjoint. Then there is a smooth arc γ_0 from P to Q which is unknotted with respect to both γ_1 and γ_2.*

PROOF. Arrange the knot $\gamma_1 \cup \gamma_2$ so that all crossings are 2-fold, and none involve γ_2. (See Figure 3.) Call a crossing an over-crossing 0 if, when γ_1 is traversed from P to Q, one goes over the crossing before going under it. Otherwise call the crossing an under-crossing U. Then $\gamma_1 \cup \gamma_2$ may be unknotted by changing all under-crossings to over-crossings.

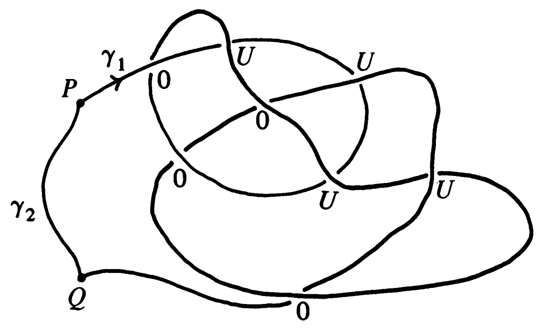

FIGURE 3

Let γ_0 be a smooth arc from P to Q which lies vertically above γ_1 and very close to it except near the under-crossings, where it goes over instead of under. It is easy to check that $\gamma_0 \cup \gamma_2$ is unknotted. To see that $\gamma_0 \cup \gamma_1$ is also unknotted, notice that it bounds an immersed disc which intersects itself only at the under-crossings. If we shrink the disc starting with the end Q, it always happens that the first time we reach an under-crossing we are in the process of contracting the two inner strands. (See Figure 4.) This may always be done. When we return to this under-crossing there is no longer any obstruction there and the disc may be shrunk further towards P. □

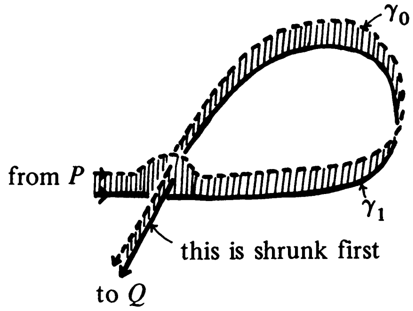

FIGURE 4

PROOF OF LEMMA 1. By Lemma 6, it clearly suffices to show that if A is an annulus in $\mathbf{R}^n$, then any element f of $\mathrm{Diff}_{\Omega 0}(A, \text{rel } \partial)$ is a product of at most 7 diffeomorphisms each of which has support in some cell contained in A. Let r be the intersection of the ray $\{\lambda y: \lambda > 0\}$ with A. Then, by altering f near r (that is, by replacing f by ff_1 where $f_1 \in \mathrm{Diff}_{\Omega 0}(A, \text{rel } \partial)$ is supported by a small neighbourhood of r), we may suppose that the intersection $r \cap f(r)$ consists of two connected arcs. It follows easily (using Lemma 8 when $n = 3$) that there is a smooth arc r_0, connecting the two boundary components of A and disjoint from $r \cup f(r)$, which is unknotted with respect to both r and $f(r)$. (See Figure 5.) This means that there is an Ω-isotopy $t \mapsto g_t \in \mathrm{Diff}_{\Omega 0}(A, \text{rel } \partial)$ which is the identity near r_0 and is such that $g_1(r) = f(r)$. By altering g_1 near r we may suppose that $g_1 = f$ on r. Then, by Lemma 7, $g_1^{-1}f$ is the product of 5 elements of $\mathrm{Diff}_{\Omega 0}(A, \text{rel } \partial)$ which are supported by cells. Since g_1 is supported by a cell of the form $A -$ (nbhd of r_0), it follows that f has the required factorization into 7 factors. (The extra factor comes from the preliminary modification of f.) □

REMARK. The proof shows that every element of $\mathrm{Diff}_\Omega \mathbf{R}^n$ has a Ling factorization with at most 14 factors. No attempt has been made to find the smallest number of factors which are necessary.

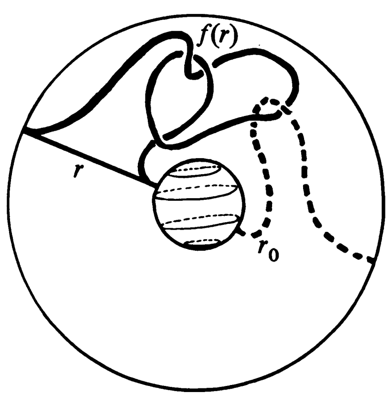

FIGURE 5

PROOF OF LEMMA 2. Suppose that $f \in \mathrm{Diff}_\Omega \mathbf{R}^n$ is supported by Int C, where, without loss of generality, we assume that C is a disc in $\mathbf{R}^n$. Choose a closed region $W \subset C$ with volume $\frac{1}{3}$ $\mathrm{vol}_\Omega C$ which is bounded by a hyperplane intersected with C and let V be an open ε-neighbourhood of W in C such that $\mathrm{vol}_\Omega(V \cup fV) < \frac{2}{3}$ $\mathrm{vol}_\Omega C$. (See Figure 6. Such V exists because $f = \mathrm{id}$ near ∂C so that W overlaps fW.) Then there is a (non-volume-preserving) isotopy g_t with support in $V \cup fV$ such that $g_0 = \mathrm{id}$ and also, $g_1 = f$ on W. (For instance, one might take $g_t = fp_t^{-1}f^{-1}p_t$, where p_t is an isotopy with support in V which shrinks W so close to ∂C that $f = \mathrm{id}$ on $p_1(W)$.) Since $(V \cup fV) - W$ is connected, it follows from Lemma A that we may actually choose g_t to preserve volume. Then $g_1^{-1}f$ preserves volume and, because it equals the identity on W and near ∂C, it has support in a cell of volume less than $\frac{2}{3}$ $\mathrm{vol}_\Omega C$. It will be one of the factors of f. We will complete the proof by expressing g_1 as a product of 2 factors of the required type.

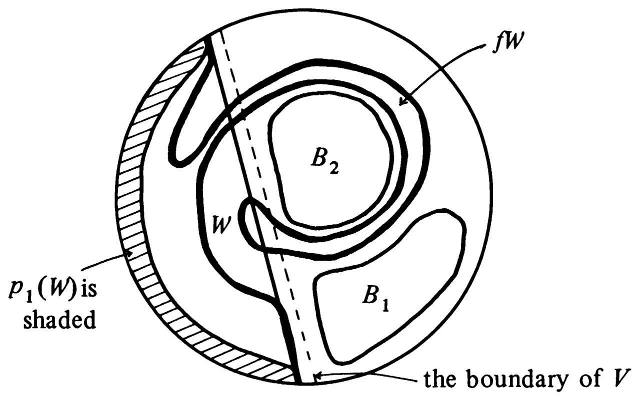

FIGURE 6

By construction, g_1 has support in $V \cup fV$. Since $\mathrm{vol}_\Omega(V \cup fV) < \frac{2}{3}$ $\mathrm{vol}_\Omega C$ there are disjoint cells $B_1, \ldots, B_m$ in $(\mathrm{Int}\ C) - (V \cup fV)$ such that $\frac{1}{3}$ $\mathrm{vol}_\Omega C = \Sigma_{i=1}^m \mathrm{vol}_\Omega B_j$. (See Figures 6, 7.) Join ∂C to B_1 by a smooth arc γ_1 and, for $1 < j \leqslant m$, join ∂B_{j-1} to ∂B_j by a smooth arc γ_j. We may suppose that these arcs are disjoint and do not meet V or the boundaries ∂C and ∂B_j except possibly at their endpoints. Thus the complement in C of a suitable neighbourhood of $\bigcup_j B_j \cup \gamma_j$ will be a cell. Clearly, $g_1 = \mathrm{id}$ near $\bigcup_j B_j$. The idea now is to modify g_1 so that it equals the identity near $\bigcup_j B_j \cup \gamma_j$ and hence has support in a cell.

As a first step, notice that we may assume that the arcs $g_1\gamma_j$ all lie outside W. For, if they do not, we may find an isotopy k_t (which we may assume to be volume preserving because $n \geqslant 3$) which pushes the arcs $g_1\gamma_j$ outside W, and has support in V. Then we may factor $k_1 g_1 k_1^{-1}$ instead of g_1.

Now, let h be a (non-volume-preserving) diffeomorphism which is the identity near $\bigcup_j(B_j \cup \gamma_j \cup g_1\gamma_j)$ and near ∂C, and which pushes the support of g_t outside W. Then hg_th^{-1} is an isotopy with support in $\mathrm{Int}(C - W)$ such that $hg_1h^{-1} = g_1$ near $\bigcup_j B_j \cup \gamma_j$. By Lemma A there is an element $q \in \mathrm{Diff}_\Omega \mathbf{R}^n$ with support in $\mathrm{Int}(C - W)$ which equals g_1 near $\bigcup_j B_j \cup \gamma_j$. Then

$$\operatorname{supp} q^{-1}g_1 \subset \operatorname{Int}\left[C - \left(\bigcup_j B_j \cup \gamma_j\right)\right]$$

so that both q and $q^{-1}g_1$ are supported by cells in Int C with volume less than $\frac{2}{3}\operatorname{vol}_\Omega C$. Thus $f = q(q^{-1}g_1)(g_1^{-1}f)$ has the required factorization. □

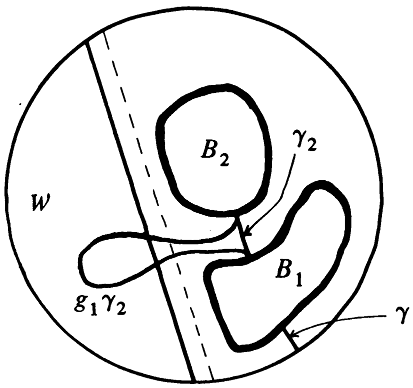

FIGURE 7

3.

PROOF OF LEMMA 3. There is a (non-volume-preserving) diffeomorphism h of $\mathbf{R}^n$ which takes each cell C_i onto the disc C_i' of radius $\frac{1}{4}$ centred at $x_i = (i, 0, \ldots, 0)$. (To see this, first choose a diffeomorphism which takes the centre of each C_i to x_i and then shrink the images of the C_i down.) Therefore, the problem reduces to finding a locally finite collection of disjoint discs D_i' which contain the C_i' and whose $(h_*\Omega)$-volumes satisfy the given conditions. Care is needed when $\operatorname{vol}_\Omega(\mathbf{R}^n - \amalg_i C_i) < \infty$ since in this case the D_i' must be chosen so that

$$\operatorname*{vol}_{h_*\Omega}\left(\mathbf{R}^n - \coprod_i D_i'\right) = \operatorname*{vol}_{\Omega}\left(\mathbf{R}^n - \coprod_i C_i\right) - \sum_i\left(w_i - \operatorname*{vol}_{\Omega} C_i\right).$$

However, because we can choose the D_i' to have nice geometric shapes, we can make them fill up as much or as little of $\mathbf{R}^n$ as necessary. For instance, we may take them to be annuli $\mu \leq \|x\| \leq \lambda$, with a hole drilled along the negative x_1-axis so that they are cells, and distorted along the positive x_1-axis so that $C_i' \subseteq D_i'$ for all i. (See Figure 8.)

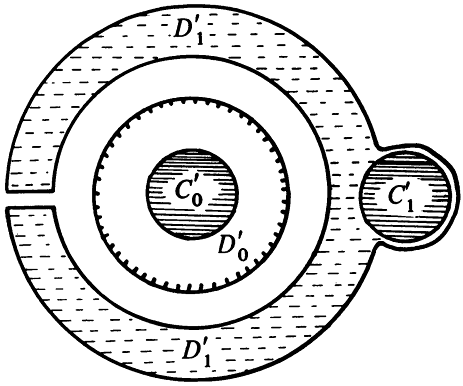

FIGURE 8

PROOF OF LEMMA 4. As in the proof of Lemma 3, we may identify the cells C_i with the set of discs centred at the points x_i and with radius $\frac{1}{4}$. It is now not difficult to construct an increasing sequence of cells C_i'' with $C_i'' \subset \operatorname{Int} C_{i+1}''$ for all i, whose union is $\mathbf{R}^n$ and which are such that $C_i \subset \operatorname{Int}(C_{i+1}'' - C_i'')$ for all i. Moreover, if the w_i are any positive numbers such that $w_i \to \operatorname{vol}_\Omega \mathbf{R}^n$ as $i \to \infty$ and $\operatorname{vol}_\Omega C_i < w_{i+1} - w_i$ for all i, then we may clearly choose the C_i'' so that $\operatorname{vol}_\Omega C_i'' = w_i$. Let D_i'' be a similar sequence for the cells D_i with $\operatorname{vol}_\Omega D_i'' = w_i$ also.

Now use Lemma A to construct an Ω-isotopy f_t such that $f_1(C_i'') = D_i''$ for all i. (Construct an isotopy $\tilde{f}_t$ in $\operatorname{Diff}_\Omega \mathbf{R}^n$ so that $\tilde{f}_1(C_1'') = D_1''$. Then extend $\tilde{f}_t | C_1''$ to an isotopy $\tilde{\tilde{f}}_t$ in such a way that $\tilde{\tilde{f}}_1(C_2'') = D_2''$, and so on.) By Lemma A again, there is an Ω-isotopy h_t with support in $\coprod_{i \geq 0} \operatorname{Int}(D_{i+1}'' - D_i'')$ which moves each cell $f_1(C_i)$ onto D_i. Clearly, $g = h_1 f_1$ satisfies the required conditions. □

Appendix. The following lemma gives all the information we need about extending volume-preserving isotopies. It is proved by Krygin in [5].

LEMMA A. *Let W be an $(n-1)$-dimensional compact submanifold of $\mathbf{R}^n$ and suppose that g_t, $0 \leq t \leq 1$, is a smooth family of embeddings of W into some open subset U of $\mathbf{R}^n$, with g_0 equal to the inclusion. If each component of $\mathbf{R}^n - g_1(W)$ has the same Ω-volume as the corresponding component of $\mathbf{R}^n - W$, there is an isotopy f_t in $\operatorname{Diff}_\Omega \mathbf{R}^n$ with support in U and such that $f_0 = \mathrm{id}$, and $f_1 = g_1$ on W. Moreover, if g_t is defined on some components V of $\mathbf{R}^n - W$, and if it preserves Ω on V and preserves the total volume of the other components of $\mathbf{R}^n - W$ either for all t or when $t = 1$, then we may suppose that $f_t = g_t$ on V for those values of t.*

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