TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 261, Number 1, September 1980

VANISHING THEOREMS AND KÄHLERITY FOR STRONGLY PSEUDOCONVEX MANIFOLDS

BY

VO VAN TAN[1]

ABSTRACT. A precise vanishing theorem of Kodaira-Nakano type for strongly pseudoconvex manifolds and Nakano semipositive vector bundles is established. This result answers affirmatively a question posed by Grauert and Riemenschneider. However an analogous version of vanishing theorem of Akizuki-Nakano type for strongly pseudoconvex manifolds and Nakano semipositive line bundles does not hold in general. A counterexample for this fact is explicitly constructed. Furthermore we prove that any strongly pseudoconvex manifold with 1-dimensional exceptional subvariety is Kählerian; in particular any strongly pseudoconvex surface is Kählerian.

Unless otherwise specified, all **C**-analytic manifolds are assumed to be noncompact, paracompact and of pure **C**-dim $= n \geqslant 1$. Let $\mathrm{Coh}(X)$ be the category of analytic coherent sheaves on the manifold X.

0. Preliminaries.

DEFINITION 1 **[1]**, **[6a]**. Let X be a **C**-analytic manifold. X is said to be *strongly pseudoconvex* (resp. *pseudoconvex*) if

(i) there exists a function $\phi \in C^\infty_{\mathbf{R}}(X)$ such that the Levi form

$$L(\phi)_x := \sum_{i,j} \frac{\partial^2\phi(x)}{\partial z_i \partial \bar{z}_j}\, dz_i\, d\bar{z}_j$$

is positive definite for all $x \in X \setminus K$, where K is some compact set $\subset X$ (resp. positive semidefinite for all $x \in X$),

(ii) $X_c := \{x \in X | \phi(x) < c\} \Subset X \ \forall c \in \mathbf{R}$.

REMARKS. (a) It can be shown (see **[1]**) that a **C**-analytic manifold X is strongly pseudoconvex if X is holomorphically convex and admits a maximal compact analytic subvariety S in the sense of **[1]**. From now on, we shall use the pair (X, S) to denote a strongly pseudoconvex manifold. In **[6a]**, **[6b]** pseudoconvex manifolds are called *weakly* 1-*complete* and in **[7]** strongly pseudoconvex manifolds are called 1-*convex*.

Received by the editors October 9, 1979 and, in revised form, February 6, 1980.

1980 *Mathematics Subject Classification.* Primary 32L20, 53C55; Secondary 32F10, 32F30.

Key words and phrases. Strongly pseudoconvex manifolds, Nakano semipositive vector bundles, weakly negative line bundles, precise vanishing theorem, non-Kählerian strongly pseudoconvex manifolds, Kähler metric.

[1]The author would like to thank Professor R. Wells and the referee for their helpful comments and suggestions.

0002-9947/80/0000-0416/$02.50

(b) Clearly any strongly pseudoconvex manifold is pseudoconvex. On the other hand $X := \mathbf{C}^1 \times \mathbf{P}_1$ is pseudoconvex but certainly X is not strongly pseudoconvex.

DEFINITION 2 **[6a]**, **[2b]**. A holomorphic vector bundle $\mathbf{V}$ with rank $\mathbf{V} = k \geqslant 1$, on a $\mathbf{C}$-analytic manifold X is called *Nakano positive* (resp. *Nakano semipositive*) if there exists a hermitian metric h on $\mathbf{V}$ (locally h is represented via a trivialization of $\mathbf{V}$ by a matrix $(h_{\alpha\beta})$) such that for any $x_0 \in X$:

(i) $(h_{\alpha\beta}(x_0)) = (\delta_{\alpha\beta})$;

(ii) $(dh_{\alpha\beta}(x_0)) = 0$;

(iii) the hermitian form

$$\sum \frac{\partial^2 h_{\alpha\beta}(x_0)}{\partial z_i \partial \bar{z}_j} \xi_{\alpha i} \bar{\xi}_{\beta j}$$

is negative definite (resp. seminegative definite) with $1 \leqslant i, j \leqslant n$ and $1 \leqslant \alpha, \beta \leqslant k$.

Notice that the matrix $-\{(\partial^2 h_{\alpha\beta}/\partial z_i \partial \bar{z}_j)(x_0)\}$ is the curvature matrix $\{\theta^i_{j\alpha\beta}\}$ in x_0.

EXAMPLE. Let X be the blowing up of $\mathbf{C}^n$ at the origin, with $n \geqslant 2$. Certainly X is a strongly pseudoconvex manifold with its maximal compact analytic subvariety $S \simeq \mathbf{P}_{n-1}$. Let $E := [S]$ be the line bundle on X determined by S and let $L := E^*$. One can check that L, equipped with some suitable metric, is actually Nakano positive.

1. Vanishing theorems. Following the results in **[6a]** and **[4]** one has the following "precise vanishing theorem".

THEOREM N. K. *Let X be a pseudoconvex manifold and let $\mathbf{V}$ be a holomorphic vector bundle on X which is Nakano positive. Then one has* $H^i(X, \Omega^n(\mathbf{V})) = 0$ $\forall i \geqslant 1$.

In the special case where rank $\mathbf{V} = 1$, an even sharper version than Theorem N. K. has been proved, namely

THEOREM N **[6b]**. *Let X be a pseudoconvex manifold and let L be a Nakano positive line bundle on X. Then one has $H^p(X, \Omega^q(L)) = 0$ for any $p + q \geqslant n + 1$.*

These two beautiful results extended the classical vanishing theorems for compact $\mathbf{C}$-analytic manifolds to the noncompact ones, namely the pseudoconvex manifolds. However there are two drawbacks in this context.

(i) Unless the base manifold X is Stein (see **[6b]**) it is quite rare to find Nakano positive vector bundles, with rank $\mathbf{V} > 1$, except the case when $\mathbf{V} = \bigoplus L_i$ where each L_i is a Nakano positive line bundle.

(ii) The Nakano positivity hypothesis of $\mathbf{V}$ (or L) in the above two results forces the base manifold X to be Kähler. Actually that Kählerian property played a crucial role in the proof of those two theorems. However, there are pseudoconvex manifolds which are not Kählerian. For instance $X := \mathbf{C}^1 \times H$, where H is the Hopf manifold, is a non-Kählerian pseudoconvex manifold.

Our main purpose here is to present a "parallel version" to Theorem N. K. within the framework of strongly pseudoconvex manifolds and Nakano semipositive vector bundles. One of the main advantages of our version is that the above two constraints do not occur in our context. First of all the following result will be needed.

THEOREM G. R. **[2b]**. *Let M be a Kählerian* $\mathbf{C}$*-analytic manifold, let U be a relatively compact strongly pseudoconvex domain with smooth boundary in M and let* $\mathbf{W}$ *be a Nakano semipositive vector bundle on M. Then one has* $H^i(U, \Omega^n(\mathbf{W})) = 0$ $\forall i \geqslant 1$.

We are now in a position to state the main result of this section.

THEOREM I. *Let (X, S) be a strongly pseudoconvex manifold and let* $\mathbf{V}$ *be a Nakano semipositive vector bundle on X. Then one has* $H^i(X, \Omega^n(\mathbf{V})) = 0$ $\forall i > 1$.

PROOF. W.l.o.g. one can assume that S is connected. Since X is strongly pseudoconvex, there exist a normal Stein space Y with only one isolated singular point $\{*\}$ and a proper morphism $\pi\colon X \to Y$ including a biholomorphism $X \setminus S \simeq Y \setminus \{*\}$. The main result in [3] asserts that there exists a monoidal transformation $f\colon \hat{X} \to Y$ with center $\{*\}$ such that

(a) $\hat{X}$ is nonsingular,

(b) there exists a proper morphism $h\colon \hat{X} \to X$ such that h induces a biholomorphism $\hat{X} \setminus T \simeq X \setminus S$ where $T := f^{-1}(*)$,

(c) $f = \pi \circ h$.

Since $\hat{X}$ is a blowing up of Y which is Stein, $\hat{X}$ can be realized as a closed analytic submanifold of some $\mathbf{C}^N \times \mathbf{P}_M$; consequently $(\hat{X}, T)$ is a Kählerian strongly pseudoconvex manifold. Precisely one has the following diagram:

$$\begin{tikzcd}
\mathbf{V} \arrow[d, dotted] & & \mathbf{W} \arrow[d, dotted] \\
X \arrow[d, "\pi"] & & \hat{X} \arrow[ll, "h"'] \arrow[dll, dotted, "f"] \\
Y & &
\end{tikzcd}$$

Let $\{U\}$ be a fundamental system of arbitrary small Stein neighborhoods of $\{*\}$ in Y and let $\mathbf{W} := h^*(\mathbf{V})$. One can check that $\mathbf{W}$ is a Nakano semipositive vector bundle on $\hat{X}$ (see also [**2a**, Satz 1.4]). In view of Theorem G. R. one has, for any $i \geqslant 1$,

$$R^i f_*(\Omega^n_{\hat{X}}(\mathbf{W}))_{\{*\}} := \varinjlim_{U} H^i\big(f^{-1}(U), \Omega^n_{\hat{X}}(\mathbf{W})\big) = 0 \qquad (\dagger)$$

and

$$R^i h_*(\Omega^n_{\hat{X}}(\mathbf{W}))_{\pi^{-1}(*)} = 0.$$

In view of (c), one has the following Grothendieck spectral sequence (see e.g. P. Hilton and U. Stammbach, *A course in homological algebra*, Graduate Texts in Math., No. 4, Springer-Verlag, Berlin and New York, 1971)

$$E_1^{\nu,i} := \bigoplus R^\nu \pi_*\big(R^{i-\nu} h_*(\Omega^n_{\hat{X}}(\mathbf{W}))\big) \Rightarrow R^i f_*(\Omega^n_{\hat{X}}(\mathbf{W})). \qquad (\dagger\dagger)$$

Since $h_*(\Omega_{\tilde X}^n) = \Omega_X^n$ and $h_*(\mathbf{W}) = h_*h^*(\mathbf{V}) \simeq \mathbf{V}$, consequently, in view of (†) and (††), one has $\forall i \geqslant 1$,

$$R^i\pi_*\big(R^0h_*(\Omega_{\tilde X}^n(\mathbf{W}))\big) \simeq R^i\pi_*(\Omega_X^n(\mathbf{V})) = 0. \tag{§}$$

Since Y is Stein π induces the canonical morphism, $\forall i \geqslant 1$,

$$H^i(X, \Omega_X^n(\mathbf{V})) \simeq H^0\big(Y, R^i\pi_*(\Omega_X^n(\mathbf{V}))\big).$$

Hence in view of (§), one has for any $i \geqslant 1$, $H^i(X, \Omega^n(\mathbf{V})) = 0$. Q.E.D.

COROLLARY O (see also [**2a**]). *Let X be a strongly pseudoconvex manifold. Then one has $H^i(X, \Omega^n) = 0 \ \forall i \geqslant 1$.*

REMARK. Although in the proof we have used the crucial fact that any strongly pseudoconvex manifold is dominated by a Kählerian one, Theorem I always holds for *any arbitrary* strongly pseudoconvex manifold. Furthermore, the existence of non-Kählerian strongly pseudoconvex manifolds (see §2) was one of the main reasons to establish Theorem I. Another reason is that Corollary O holds for strongly pseudoconvex manifolds but not for pseudoconvex manifolds in general, since trivial bundles are semipositive (in any sense) but certainly not Nakano positive.

Also notice that Theorem I answers affirmatively a question posed by Grauert and Riemenschneider [**2a**, p. 278]. At this point one naturally is tempted to establish a "parallel version" to Theorem N within the framework of strongly pseudoconvex manifolds and Nakano semipositive line bundles. Such a result would be stated as follows.

($) Let X be a strongly pseudoconvex manifold and let L be a Nakano semipositive line bundle on X. Then one has $H^p(X, \Omega^q(L)) = 0$ for any $p + q > n + 1$.

Obviously, for the case when $n = 2$, ($) is just a special case of our Theorem I. Unfortunately, the assertion ($) is false in general as one can easily see.

COUNTEREXAMPLE TO ($). Let B be the open unit ball centered at the origin in $\mathbf{C}^3$, let X be the blowing up of B at the origin and let $S \simeq \mathbf{P}_2$ be the proper transform of the origin. Clearly (X, S) is a strongly pseudoconvex manifold. Now $L := O_X$ is clearly a Nakano semipositive line bundle on X. Moreover, one can check easily (see also [**2a**]) that

$$H^4(S, \mathbf{C}) \simeq \bigoplus_{\substack{p+q=4\\ p,q>1}} H^{q,p} \tag{†}$$

where $H^{q,p} := H^p(X, \Omega^q)$.

If the assertion ($) were to be true, the right-hand side of (†) has to be equal to zero. But

$$H^4(S, \mathbf{C}) = H^4(\mathbf{P}_2, \mathbf{C}) \simeq \mathbf{C} \neq 0.$$

Contradiction!

However in the positive direction we have

PROPOSITION 1. *Let* (X, S) *be a strongly pseudoconvex manifold and let* L *be a line bundle on* X *such that* $L|S$ *is weakly positive* (*see Definition* 3). *Then* $H^p(X, \Omega^q(L)) = 0$ *for any* $p + q \geqslant n + 1$.

PROOF. In view of the Extension Lemma in [7] one can find some strongly pseudoconvex neighborhood Ξ of S in X and one can equip L with some suitable metric with which $L|\Xi$ is Nakano positive. Furthermore, the restriction map r: $H^i(X, \mathcal{F}) \to H^i(\Xi, \mathcal{F})$ is injective $\forall i \geqslant 1$ and $\forall \mathcal{F} \in \mathrm{Coh}(X)$ (see [7]). This fact combined with Theorem N above will give us the desired result. Q.E.D.

DEFINITION 3 [1], [7]. Let X be a compact $\mathbf{C}$-analytic space and let L be a holomorphic line bundle on X. Then L is said to be *weakly negative* if the zero section Σ of L admits some strongly pseudoconvex neighborhood U in L. L is said to be *weakly positive* if L^* is weakly negative.

2. The Kähler property. Naturally the previous discussion leads us to the question about the existence of a non-Kählerian strongly pseudoconvex manifold. The first known example of that type was exhibited explicitly in [7], namely we constructed a 3-dimensional strongly pseudoconvex manifold X which carries a compact cycle which is homologous to zero. Hence X fails to be Kähler. Our purpose here is to show that such an example is sharp with respect to dimension. Indeed, we have the following result.

THEOREM II. *Let* (X, S) *be a strongly pseudoconvex manifold. If* $\dim S = 1$, *then* X *is Kählerian. In particular, any strongly pseudoconvex surface is Kählerian.*

First of all we shall need the following technical result.

THEOREM M [5]. *Let* X *be a* $\mathbf{C}$-*analytic manifold and let* $\{*\}$ *be a point on* X. *If* $Z := X \setminus \{*\}$ *is Kählerian, then so is* X.

PROOF OF THEOREM II. Without loss of generality one can assume that S is connected and of pure $\mathbf{C}$-dim $= 1$. Since X is strongly pseudoconvex, there exist a normal Stein space with only one isolated singular point, say $\{*\}$, and a proper holomorphic map π: $X \to Y$ inducing a biholomorphism

$$X \setminus S \simeq Y \setminus \{*\}. \tag{§}$$

Since Y is Stein there exists a positive (in any sense) line bundle $\hat{E}$ on Y. Clearly, $E := \pi^*(\hat{E})$ is a Nakano semipositive line bundle on X. In view of (§), $E|X \setminus S$ is Nakano positive.

Now let T be a set of all isolated singular points of S. Let $Z := X \setminus T$, let $L := E|Z$ and let $\{h_i\}$ be the induced metric on L. Since $\check{S} := S \setminus T$ is a closed Stein submanifold in Z, the metric $\{h_i\}$ can be suitably modified such that $L|\check{S}$ is Nakano positive (see [6b]). Precisely one has, for any $x \in \check{S}$,

$$\begin{aligned} &-\partial\bar{\partial} \log h_i(x) > 0 \quad \text{on } T_{\check{S},x}, \\ &-\partial\bar{\partial} \log h_i(x) \geqslant 0 \quad \text{on } N_{\check{S},x}, \\ &-\partial\bar{\partial} \log h_i(z) > 0 \quad \text{on } T_{Z,z} \text{ if } z \in Z \setminus \check{S} = X \setminus S \end{aligned} \tag{†}$$

where $T_{\check{S},x}$ is the Zariski tangent space to $\check{S}$ at x and $N_{\check{S},x}$ is the complement space of $T_{\check{S},x}$ in $T_{Z,x}$. Furthermore, in view of the strong pseudoconvexity of X, one can find (see [7]) a function $\phi \in C_{\mathbf{R}}^{\infty}(X)$ such that

$$\partial\bar{\partial}\phi(x) \geqslant 0 \quad \text{on } T_{X,x} \ \forall x \in X,$$

$$\partial\bar{\partial}\phi(x) > 0 \quad \text{on } T_{X,x} \ \forall x \in X \setminus S,$$

$$\partial\bar{\partial}\phi(x) > 0 \quad \text{on } N_{S,x} \ \forall x \in S \tag{††}$$

where $N_{S,x}$ is the complement space of $T_{S,x}$ in $T_{X,x}$.

In view of (†) and (††) one can check easily that L, equipped with the new metric $g_i := h_i e^{-\phi}$, is actually Nakano positive on Z. Obviously the positive (1, 1)-form $\Omega := -i\partial\bar{\partial} \log g_i$ will determine a Kähler metric on Z. Since T is finite, by repeatedly applying Theorem M it will follow that X itself is Kählerian. Q.E.D.

Notice that our Theorem II partially answers Problem 3a in [7]. Also a weaker form of Theorem II has been announced in [7, Theorem V].

References

1. H. Grauert, *Über Modifikationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

2. (a) H. Grauert and O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292.

(b) ______, *Kählersche Mannigfaltigkeiten mit hyper-q-konvexem Rand*, Problems in Analysis (Lectures at the Sympos. in honor of Solomon Bochner, Princeton Univ., Princeton, N. J., 1969), Princeton Univ. Press, Princeton, N. J., 1970, pp. 61–79.

3. H. Hironaka and H. Rossi, *On the equivalence of embeddings of exceptional complex spaces*, Math. Ann. **156** (1964), 313–333.

4. H. Kazama, *Approximation theorem and application to Nakano's vanishing theorem for weakly 1-complete manifolds*, Mem. Fac. Sci. Kyushu Univ. Ser. A **27** (1973), 221–247.

5. Y. Miyaoka, *Extension theorems for Kähler metrics*, Proc. Japan Acad. **50** (1974), 407–410.

6. (a) S. Nakano, *Vanishing theorems for weakly 1-complete manifolds*, Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Yasuo Akizuki, Kinokuniya, Tokyo, 1973, pp. 169–179.

(b) ______, *Vanishing theorems for weakly 1-complete manifolds*. II, Publ. Res. Inst. Math. Sci., Kyoto Univ. **10** (1974), 101–110.

7. Vo Van Tan, *On the embedding problem for 1-convex spaces*, Trans. Amer. Math. Soc. **256** (1979), 185–197.

Department of Mathematics, Tufts University, Medford, Massachusetts 02155

ABCDEFGHIJ–CM–89876543210