TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 262, Number 2, December 1980

REGULARITY OF CERTAIN SMALL SUBHARMONIC FUNCTIONS

BY

P. C. FENTON

ABSTRACT. Suppose that u is subharmonic in the plane and that $\underline{\lim}_{r\to\infty} B(r)/(\log r)^2 = \sigma < \infty$. It is known that, given $\varepsilon > 0$, there are arbitrarily large values of r such that $A(r) > B(r) - (\sigma + \varepsilon)\pi^2$. The following result is proved. Let u be subharmonic and let σ be any positive number. Then either $A(r) > B(r) - \pi^2\sigma$ for certain arbitrarily large values of r or, if this is false, then

$$\lim_{r\to\infty} (B(r) - \sigma(\log r)^2)/\log r$$

exists and is either $+\infty$ or finite.

1. Introduction. Let $u(z)$ be subharmonic in the plane and define

$$B(r) = B(r, u) = \max_{|z|=r} u(z),$$

$$A(r) = A(r, u) = \inf_{|z|=r} u(z).$$

In [3] the following result is proved.

THEOREM A. *Let $p > 1$ be given and suppose that $u(z)$ is subharmonic in the plane and satisfies*

$$\underline{\lim}_{r\to\infty} \frac{B(r)}{(\log r)^p} = \sigma < \infty.$$

Then, given $\varepsilon > 0$,

$$A(r) > B(r) - (\sigma + \varepsilon)\mathrm{Re}\{(\log r)^p - (\log r + i\pi)^p\} \tag{1.1}$$

for r outside an exceptional set E for which

$$\underline{\lim}_{r\to\infty} \frac{(p-1)}{(\log r)^{p-1}} \int_{E\cap[1,r]} \frac{(\log r)^{p-2}}{t}\, dt < \frac{\sigma}{\sigma+\varepsilon}.$$

Theorem A is related to certain results of P. D. Barry. (See Theorem 4 and the remarks in §7.4 of [1].) With Kjellberg's version of the cos $\pi\lambda$ Theorem [5], [6] in view we might expect that functions extremal for Theorem A would have some kind of regular asymptotic behaviour. In this direction we shall prove

THEOREM 1. *Suppose that $u(z)$ is subharmonic in the plane and that σ is any positive number. Then either*

$$A(r) > B(r) - \pi^2\sigma \tag{1.2}$$

Received by the editors April 24, 1979 and, in revised form, December 10, 1979.
AMS (MOS) subject classifications (1970). Primary 31A05.

for certain arbitrarily large values of r or, if this is not the case, then

$$\alpha = \lim_{r\to\infty} \frac{B(r) - \sigma(\log r)^2}{\log r} \tag{1.3}$$

exists and is either $+\infty$ *or finite.*

This result corresponds to $p = 2$ in Theorem A. It seems likely that, for the case of general p, the limit of (1.3) could be replaced by

$$\lim_{r\to\infty} \frac{B(r) - \sigma(\log r)^p}{(\log r)^{p-1}} > -\infty.$$

EXAMPLE. Any value of α admitted by Theorem 1 can in fact occur. Write $z = re^{i\theta}$, where $-\pi < \theta \leqslant \pi$. In the case $\alpha = +\infty$ we may let $u(z) = r\cos\theta$. If $\alpha = 0$ we set

$$u_0(z) = 0, \qquad r \leqslant 1,$$

$$u_0(z) = \max\big(\sigma(\log r)^2 - \sigma\theta^2, 0\big), \qquad r > 1.$$

Evidently $u_0(z)$ is continuous and subharmonic except possibly on the segment $r > 1$ of the negative real axis, since $\sigma(\log r)^2 - \sigma\theta^2 = \operatorname{Re}\sigma(\log r + i\theta)^2$ is harmonic except for negative real z. If we write θ_1, θ_2 for the values of $\arg z$ which satisfy respectively

$$-2\pi < \theta_1 < 0, \qquad 0 < \theta_2 < 2\pi,$$

and set

$$u_1(z) = \sigma(\log r)^2 - \sigma\theta_1^2, \qquad u_2(z) = \sigma(\log r)^2 - \sigma\theta_2^2,$$

then u_1 and u_2 are harmonic near the negative real axis and so

$$u_0(z) = \max(u_1(z), u_2(z))$$

is subharmonic there and so everywhere in the plane. Also, for any real α, we define c by $2\sigma\log c = \alpha$, and set $u(z) = u_0(cz) - \sigma(\log c)^2$. Then

$$B(r, u) = \sigma(\log r)^2 + \alpha\log r, \qquad r > 1/c,$$

$$A(r, u) = B(r, u) - \pi^2\sigma, \qquad r > e^{\pi}/c,$$

so that (1.2) fails and

$$\frac{B(r, u) - \sigma(\log r)^2}{\log r} = \alpha, \qquad r > 1/c.$$

From the Riesz representation theorem for subharmonic functions it follows that there is a unique nonnegative measure μ defined on all bounded, Borel measurable subsets of the plane such that, if R is a given positive number,

$$u(z) = h_R(z) + \int_{|\zeta|<R} \log\left|1 - \frac{z}{\zeta}\right| d\mu(\zeta) \tag{1.4}$$

for $|z| < R$, where $h_R(z)$ is harmonic in $|z| < R$. In the proof it is assumed that u is harmonic at 0 but this may be achieved without loss of generality to our results by replacing u in a disc about 0 by the Poisson integral of its boundary values on the disc. Throughout the paper we shall assume that $u(0) = 0$, as we may do without affecting the generality of our results. We define $\mu^*(r) = \mu(\{z: |z| < r\})$ for $r > 0$.

THEOREM 2. *Suppose that $u(z)$ is subharmonic in the plane and that σ is a positive number such that*

$$A(r) \leqslant B(r) - \pi^2\sigma$$

for all large r. Suppose further that the limit α of (1.3) *is finite. Let*

$$u_1(z) = \int_{|\zeta|<\infty} \log\left|1 + \frac{z}{|\zeta|}\right| d\mu(\zeta) = \int_0^\infty \log\left|1 + \frac{z}{t}\right| d\mu^*(t)$$

and define $B_1(r) = \max_{|z|=r} u_1(z)$. Then

$$\lim_{r\to\infty} \frac{B_1(r) - \sigma(\log r)^2}{\log r} = \alpha.$$

THEOREM 3. *Under the conditions of Theorem* 2

$$\lim_{r\to\infty} (\mu^*(r) - 2\sigma \log r) = \alpha.$$

The first part of the paper is devoted to showing that, under the conditions expressed in the second alternative of Theorem 1, $\mu^*(r) = O(\log r)$ when

$$\underline{\lim}\, (B(r) - \sigma(\log r)^2)/\log r < \infty.$$

This is rather more drawn out than might be expected due to certain tiresome modifications to u that seem to be necessary in the subsequent parts of the proof. In §6 and §7 the growth properties of u and u_1 are considered and the theorems are proved more or less together.

2. Decomposition of u. In [2] Barry has put into subharmonic form results derived by Kjellberg [5, pp. 190–192] in the case $u(z) = \log|f(z)|$, where f is an entire function. Some of these are as follows.

With $\mu^*(t) = \mu(|z| < t)$ define

$$u_1(z, R) = \int_{|\zeta|<R} \log\left|1 - \frac{z}{\zeta}\right| d\mu(\zeta), \tag{2.1}$$

$$u_2(z, R) = \int_{|\zeta|<R} \log\left|1 + \frac{z}{|\zeta|}\right| d\mu(\zeta) = \int_0^R \log\left|1 + \frac{z}{t}\right| d\mu^*(t), \tag{2.2}$$

$$u_3(z, R) = u(z) - u_1(z, R). \tag{2.3}$$

Then, with $B_j(r, R) = \max_{|z|=r} u_j(z, R)$, $A_j(r, R) = \inf_{|z|=r} u_j(z, R)$, $j = 1, 2, 3$,

$$A_2(r, R) \leqslant A_1(r, R) \leqslant B_1(r, R) \leqslant B_2(r, R); \tag{2.4}$$

and

$$-\frac{4r}{R}B(2R) < A_3(r, R) < B_3(r, R) < \frac{4r}{R}B(2R). \tag{2.5}$$

for $0 \leqslant r \leqslant \frac{1}{2}R$.

We note finally the subharmonic analogue of Jensen's Theorem [4, p. 473]: for $r > 0$

$$u(0) + \int_0^r \log \frac{r}{t}\, d\mu^*(t) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta \leqslant B(r). \tag{2.6}$$

Concerning $u_2(z, R)$ we have the following lemma.

LEMMA 1. *Let R_1, R_2 and R be positive numbers satisfying $R_1 < R_2 < R$. Then*

$$\int_{R_1}^{R_2} \frac{A_2(t, R) - B_2(t, R)}{t}\, dt$$

$$= \mu^*(R)\int_0^{\pi} \operatorname{Arg}\left(1 - \frac{R_2}{R}e^{i\theta}\right) d\theta - \int_0^R \frac{\mu^*(t)}{t} \log\left|\frac{t + R_2}{t - R_2}\right| dt$$

$$- \mu^*(R)\int_0^{\pi} \operatorname{Arg}\left(1 - \frac{R_1}{R}e^{i\theta}\right) d\theta + \int_0^R \frac{\mu^*(t)}{t} \log\left|\frac{t + R_1}{t - R_1}\right| dt.$$

Also

$$I(R_1, R_2, R) = \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{A_2(t, R) - B_2(t, R)}{t}\, dt$$

$$= \mu^*(R)\int_{R_1}^{R_2} \frac{ds}{s} \int_0^{\pi} \operatorname{Arg}\left(1 - \frac{s}{R}e^{i\theta}\right) d\theta$$

$$- \int_0^R \frac{\mu^*(t)}{t}\, dt \int_{R_1}^{R_2} \frac{1}{s} \log\left|\frac{s + t}{s - t}\right| ds.$$

The first part is contained in the proof of the Lemma in [3] and the second part follows immediately from the first on integration.

3. Preliminaries. To prove the theorems we assume that for all large r

$$A(r) \leqslant B(r) - \pi^2\sigma \tag{3.1}$$

and that

$$\alpha = \varliminf_{r \to \infty} \frac{B(r) - \sigma(\log r)^2}{\log r} < +\infty \tag{3.2}$$

and aim to prove the existence of the finite limit (1.3). The first step is to show that (3.1) and (3.2) together imply

$$\mu^*(r) = O(\log r) \quad \text{as } r \to \infty \tag{3.3}$$

and in order to do this we assume that

$$\varlimsup_{r \to \infty} \frac{\mu^*(r)}{\log r} = +\infty \tag{3.4}$$

and deduce a contradiction. We have

LEMMA 2. *Suppose that* (3.1), (3.2) *and* (3.4) *hold. Then there exists a subharmonic function* $U(z)$ *which satisfies the following conditions*:

$$\text{(i)}\quad \varliminf_{r\to\infty} \frac{B(r, U) - \sigma(\log r)^2}{\log r} < -A < -\frac{4\sigma}{\pi^2}(\pi^2 + 90); \tag{3.5}$$

$$\text{(ii)}\quad A(r, U) = B(r, U) - \pi^2\sigma \tag{3.6}$$

holds for $e^{\pi} \leqslant r \leqslant r_0$ *and for* $2r_0 \leqslant r < \infty$, *for some* $r_0 > e^{\pi}$;

$$\text{(iii)}\quad \varlimsup_{r\to\infty} \frac{\mu^*(r, U)}{\log r} = \infty. \tag{3.7}$$

Set $v_1(z) = \max\{u(z), B(|z|) - \pi^2\sigma\}$. Since $B(|z|)$ is subharmonic [7, §3.20] so also is $v_1(z)$. Also, since v_1 and u have the same maximum on each circle about the origin, (3.2) holds for v_1; and it follows from (2.6) that (3.4) is equivalent to

$$\varlimsup_{r\to\infty} \frac{1}{(\log r)^2} \int_0^{2\pi} u(re^{i\theta})\, d\theta = \infty,$$

so that (3.4) persists likewise for v_1. Now set $v_2(z) = v_1(cz)$, where c is a positive constant. Then $B(r, v_2) = B(cr, v_1)$ so that

$$\varliminf_{r\to\infty} \frac{B(r, v_2) - \sigma(\log r)^2}{\log r} = \alpha + 2\sigma \log c,$$

and we choose c so that condition (i) obtains. Moreover

$$A(r, v_2) = B(r, v_2) - \pi^2\sigma$$

for all large r, say $r \geqslant R_0$, and (3.4) holds for v_2.

Since (3.4) holds we conclude that, given any positive number M, there is some number $r_0 > R_0 + 2$ such that

$$B(2r_0, v_2) - B(r_0, v_2) > M \log r_0.$$

For if there were no such r_0 we should have, for all large positive integers n,

$$\begin{aligned} B(2^n, v_2) &= B(2^n, v_2) - B(2^{n-1}, v_2) + B(2^{n-1}, v_2) \\ &\leqslant M(n-1)\log 2 + B(2^{n-1}, v_2) - B(2^{n-2}, v_2) + B(2^{n-2}, v_2) \\ &\leqslant \ldots \leqslant \tfrac{1}{2} n^2 M \log 2 + O(1) = O(\log 2^n)^2. \end{aligned}$$

Thus $B(r, v_2) = O(\log r)^2$ and so $B(r, u) = O(\log r)^2$. This together with (2.6) contradicts (3.4).

Now $w(z)$ defined by

$$\begin{aligned} w(z) &= 0, \qquad |z| \leqslant 1, \\ w(z) &= \max\{\sigma(\log|z|)^2 - \sigma(\operatorname{Arg} z)^2, 0\}, \qquad |z| > 1, \end{aligned}$$

is subharmonic in the plane (as was shown in the example following Theorem 1). We define a new function

$$U(z) = \begin{cases} w(z) + D, & |z| \leqslant r_0, \\ h(z), & r_0 \leqslant |z| \leqslant 2r_0, \\ v_2(z), & |z| \geqslant 2r_0, \end{cases} \tag{3.8}$$

where $D = B(r_0, v_2) - \sigma(\log r_0)^2 + \pi^2\sigma$ and $h(z)$ is the harmonic function in $r_0 < |z| < 2r_0$ taking boundary values $w(z) + D$ on $|z| = r_0$, $v_2(z)$ on $|z| = 2r_0$. Clearly $U(z)$ satisfies conditions (i) and (ii) of Lemma 2. Moreover, once we have shown that $U(z)$ is subharmonic, it is evident that (3.7) holds.

To show that U is subharmonic it is enough to show that h dominates both v_2 and $w + D$ for $r_0 \leqslant |z| \leqslant 2r_0$. For $|z| = r_0$,

$$h(z) = w(z) + D \geqslant \sigma(\log r_0)^2 - \pi^2\sigma + D = B(r_0, v_2) \geqslant v_2(z),$$

while for $|z| = 2r_0$,

$$\begin{aligned} h(z) &= v_2(z) \geqslant A(2r_0, v_2) = B(2r_0, v_2) - \pi^2\sigma > B(r_0, v_2) + M \log r_0 - \pi^2\sigma \\ &= \sigma(\log r_0)^2 + M \log r_0 + D - 2\pi^2\sigma > \sigma(\log 2r_0)^2 + D \geqslant W(z) + D \end{aligned}$$

provided M is large enough. This completes the proof.

4. Modification of U. Let U be the function of Lemma 2. From (3.7) it follows that given any positive number K there are arbitrarily large values of r such that $B(r, U) > K(\log r)^2$. Given one such value, say r_1, take $2R$ to be the smallest number greater than r_1 such that

$$\frac{\sigma(\log 2R)^2 - B(2R, U)}{\log 2R} = A. \tag{4.1}$$

From (3.5) such an R exists; and clearly (if K is large enough) $r_1 < R$ so that

$$\frac{\sigma(\log R)^2 - B(R, U)}{\log R} < A.$$

That is

$$B(R, U) > \sigma(\log R)^2 - A \log R. \tag{4.2}$$

Also

$$B(R, U) < B(2R, U) = \sigma(\log 2R)^2 - A \log 2R. \tag{4.3}$$

We modify $U(z)$ in $|z| > R$ so as to obtain a subharmonic function which is not too large when $|z|$ is large. Let $h(z)$ be the harmonic function in $R < |z| < 2R$ which takes boundary values $U(z)$ on $|z| = R$ and $B(2R, U)$ on $|z| = 2R$. This function clearly dominates U on $R \leqslant |z| \leqslant 2R$ so that

$$U_1(z) = \begin{cases} U(z), & |z| \leqslant R, \\ h(z), & R \leqslant |z| \leqslant 2R, \end{cases}$$

is subharmonic in $|z| < 2R$. Further, $h(z) \geqslant h_1(z)$, where $h_1(z)$ is the harmonic function in $R < |z| < 2R$ taking boundary values $V_1 = \sigma(\log R)^2 - A \log R - \pi^2\sigma$ on $|z| = R$ and $V_2 = B(2R, U)$ on $|z| = R$. (This follows from (3.6) and (4.2).) $h_1(z)$ may be written explicitly as

$$h_1(z) = \frac{V_2 - V_1}{\log 2} \log \frac{|z|}{R} + V_1. \tag{4.4}$$

We define

$$U_2(z) = 2\sigma(\log r)^2 + B(2R, U) - 2\sigma(\log 2R)^2$$

and show that, if R is large, $U_2(z) \leqslant h(z)$ for $R < |z| \leqslant 2R$ so that

$$W_R(z) = \begin{cases} U_1(z), & |z| \leqslant 2R, \\ U_2(z), & z \geqslant 2R, \end{cases} \tag{4.5}$$

is subharmonic. For $R < |z| < 2R$ we obtain, after some simplification,

$$\begin{aligned} h(z) - U_2(z) &\geqslant h_1(z) - U_2(z) \\ &= \left(\frac{B(2R, U) + A \log R + \pi^2\sigma - \sigma(\log R)^2}{\log 2} - 2\sigma \log 2|z|R \right) \log \frac{|z|}{2R} \\ &\geqslant \left(\frac{\sigma(\log 2R)^2 - A \log 2 + \pi^2\sigma - \sigma(\log R)^2}{\log 2} - 2\sigma \log 2R^2 \right) \log \frac{|z|}{2R} \\ &> 0 \end{aligned}$$

if R is large enough.

We note that $W_R(z) = U_1(z) = U(z)$ for $|z| \leqslant R$ and that

$$B(R^2, W_R) = W_R(R^2) < 10\sigma(\log R)^2. \tag{4.6}$$

5. Behaviour of $\mu^*(r)$. Throughout this section the functions A, B, μ and μ^* will be understood to refer to the function W_R. Set $\rho = R^{3/2}$. Given $t \leqslant \frac{1}{2}R$ we have from (2.5)

$$A(t) = \int_{|\zeta|<\rho} \log\left|1 - \frac{z_1}{\zeta}\right| d\mu(\zeta) + O\left(\frac{t}{\rho} B(2\rho)\right),$$

$$B(t) = \int_{|\zeta|<\rho} \log\left|1 - \frac{z_2}{\zeta}\right| d\mu(\zeta) + O\left(\frac{t}{\rho} B(2\rho)\right),$$

where $|z_1| = |z_2| = t$, and thus

$$\begin{aligned} A(t) - B(t) &= \int_{|\zeta|<R} \log\left|1 - \frac{z_1}{\zeta}\right| d\mu(\zeta) - \int_{|\zeta|<R} \log\left|1 - \frac{z_2}{\zeta}\right| d\mu(\zeta) \\ &\quad + \int_{R \leqslant |\zeta| < \rho} \log\left|\frac{\zeta - z_1}{\zeta - z_2}\right| d\mu(\zeta) + O\left(\frac{t}{\rho} B(2\rho)\right) \\ &\geqslant A_1(t, R) - B_1(t, R) - \int_R^\rho \log\left|\frac{s+t}{s-t}\right| d\mu^*(s) + O\left(\frac{t}{\rho} B(2\rho)\right) \\ &\geqslant A_2(t, R) - B_2(t, R) - \mu^*(\rho) \log\left(\frac{R+t}{R-t}\right) + O\left(\frac{t}{\rho} B(2\rho)\right). \end{aligned}$$

Hence, for $1 < R_1 < R_2 = \frac{1}{2}R$,

$$\begin{aligned} \Delta(R_1, R_2) &= \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{A(t) - B(t)}{t}\, dt \\ &\geqslant \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{A_2(t, R) - B_2(t, R)}{t}\, dt \\ &\quad - \mu^*(\rho) \int_{R_1}^{R_2} \frac{ds}{s} \int_0^s \frac{1}{t} \log\left(\frac{R+t}{R-t}\right) dt + O\left(\frac{R_2}{\rho} B(2\rho)\right) \\ &= T_1 - T_2 + o(1), \end{aligned} \tag{5.1}$$

from (4.6), and we estimate T_1 and T_2 in turn. From Lemma 1

$$T_1 = \mu^*(R)\int_{R_1}^{R_2}\frac{1}{s}\,ds\int_0^{\pi}\operatorname{Arg}\left(1-\frac{s}{R}e^{i\theta}\right)d\theta$$
$$-\int_0^R\frac{\mu^*(t)}{t}\,dt\int_{R_1}^{R_2}\frac{1}{s}\log\left|\frac{t+s}{t-s}\right|ds.$$

Also

$$\operatorname{Arg}\left(1-\frac{s}{R}e^{i\theta}\right) \geqslant -\operatorname{Arc\,sin}\frac{s}{R} \geqslant -\frac{\pi s}{2R}$$

so

$$T_1 \geqslant -\tfrac{1}{4}\pi^2\mu^*(R) - I(R_2) + I(R_1), \tag{5.2}$$

where

$$I(r) = \int_0^R\frac{\mu^*(t)}{t}\,dt\int_0^r\frac{1}{s}\log\left|\frac{t+s}{t-s}\right|ds. \tag{5.3}$$

We have

$$I(R_2) \leqslant \int_0^R\frac{\mu^*(t)}{t}\,dt\int_0^{\infty}\frac{1}{s}\log\left|\frac{t+s}{t-s}\right|ds$$
$$= \int_0^R\frac{\mu^*(t)}{t}\,dt\int_0^{\infty}\frac{1}{s}\log\left|\frac{s+1}{s-1}\right|ds$$
$$= \frac{1}{2}\pi^2\int_0^R\frac{\mu^*(t)}{t}\,dt \leqslant \frac{1}{2}\pi^2B(R) + O(1),$$

from (2.6), and combining this with (5.2) we obtain

$$T_1 \geqslant -\tfrac{1}{4}\pi^2\mu^*(R) - \tfrac{1}{2}\pi^2B(R) + I(R_1) + O(1). \tag{5.4}$$

A straightforward estimate yields

$$T_2 = \mu^*(\rho)\int_{R_1/R}^{R_2/R}\frac{ds}{s}\int_0^s\frac{1}{t}\log\left(\frac{1+t}{1-t}\right)dt \leqslant 2\mu^*(\rho) \tag{5.5}$$

and combining (5.1), (5.4) and (5.5) we obtain

$$\Delta(R_1, R_2) \geqslant -\left(2+\tfrac{1}{4}\pi^2\right)\mu^*(\rho) - \tfrac{1}{2}\pi^2B(R) + I(R_1) + O(1)$$
$$\geqslant -\frac{\left(4+\tfrac{1}{2}\pi^2\right)}{\log R}\int_{R^{3/2}}^{R^2}\frac{\mu^*(t)}{t}\,dt - \tfrac{1}{2}\pi^2B(R) + I(R_1) + O(1)$$
$$\geqslant -\frac{10}{\log R}B(R^2) - \tfrac{1}{2}\pi^2B(R) + I(R_1) + O(1)$$
$$\geqslant -110\sigma\log R - \tfrac{1}{2}\pi^2B(R) + I(R_1), \tag{5.6}$$

from (4.6). On the other hand $W(z) = U(z)$ for $|z| \leqslant R$ and thus from (3.6)

$$\Delta(R_1, R_2) \leqslant \int_{R_1}^{R_2}\left(\int_{e^{\pi}}^{s}\frac{-\pi^2\sigma}{t}\,dt + \int_{r_0}^{2r_0}\frac{\pi^2\sigma}{t}\,dt\right)\frac{ds}{s}$$
$$\leqslant -\tfrac{1}{2}\pi^2\sigma(\log R_2)^2 + \tfrac{1}{2}\pi^2\sigma(\log R_1)^2 + 2\pi^3\sigma\log R_2. \tag{5.7}$$

Since $2\pi^3 < 70$, (5.6) and (5.7) together yield

$$\tfrac{1}{2}\pi^2\big(B(R) - \sigma(\log R_2)^2\big) + 180\sigma \log R \geqslant I(R_1) - \tfrac{1}{2}\pi^2\sigma(\log R_1)^2. \tag{5.8}$$

Since $R_2 = \frac{1}{2}R$ the left-hand side of (5.8) is, from (4.3), no larger than

$$\tfrac{1}{2}\pi^2\Big(\sigma(\log 2R)^2 - A\log 2R - \sigma\big(\log \tfrac{1}{2}R\big)^2\Big) + 180\sigma \log R$$
$$= \log R\big(2\pi^2\sigma \log 2 - \tfrac{1}{2}A\pi^2 + 180\sigma\big) - \tfrac{1}{2}\pi^2 A \log 2 < 0$$

when R is large, from (3.5). Thus (5.8) reduces to

$$\tfrac{1}{2}\pi^2\sigma(\log R_1)^2 \geqslant I(R_1). \tag{5.9}$$

Now $W = U_1 = U$ for $|z| \leqslant R$ and U satisfies (3.7). Thus, by taking R sufficiently large, we can find R_1 such that $1 < R_1 < R_1^{3/2} < \frac{1}{2}R$ and $\mu^*(R_1^{1/2}) > 5\sigma \log R_1$. For such an R_1

$$I(R_1) \geqslant \int_{R_1^{1/2}}^{R_1^{3/2}} \frac{\mu^*(t)}{t}\,dt \int_0^{R_1} \frac{1}{s}\log\left|\frac{t+s}{t-s}\right| ds$$
$$\geqslant \mu^*\big(R_1^{1/2}\big)\int_{R_1^{1/2}}^{R_1^{3/2}} \frac{1}{t}\,dt \int_0^{R_1} \frac{1}{s}\log\left|\frac{t+s}{t-s}\right| ds$$
$$> 5\sigma \log R_1 \int_{R_1^{-1/2}}^{R_1^{1/2}} \frac{1}{t}\,dt \int_0^{1/t} \frac{1}{s}\log\left|\frac{s+1}{s-1}\right| ds$$
$$> 5\sigma \log R_1 \int_{R_1^{-1/2}}^{1} \frac{1}{t}\,dt \int_0^{1} \frac{1}{s}\log\left|\frac{s+1}{s-1}\right| ds$$
$$= \frac{5\sigma}{8}\pi^2(\log R_1)^2,$$

which contradicts (5.9). The assumption which leads to this contradiction, namely (3.4), is thus mistaken and we deduce (3.3).

6. Bounds on the growth of $B(r)$. The remainder of the paper is concerned only with the functions u and u_1 occurring in the statement of the theorems. In this section we aim to show that (3.1) and (3.2) together imply

$$-\infty < \alpha = \varliminf_{r\to\infty} \frac{B(r) - \sigma(\log r)^2}{\log r} < \beta = \varlimsup_{r\to\infty} \frac{B(r) - \sigma(\log r)^2}{\log r} < +\infty. \tag{6.1}$$

As we have shown, (3.3) holds and we may thus introduce

$$u_1(z) = \lim_{R\to\infty} u_2(z, R) = \operatorname{Re}\left(z\int_0^\infty \frac{\mu^*(t)}{t(t+z)}\,dt\right).$$

We write $B_1(r) = \max_{|z|=r} u_1(z)$, $A_1(r) = \inf_{|z|=r} u_1(z)$ and take the limit as $R \to \infty$ in the first equation of Lemma 1 to obtain for all large $R_1 < R_2$

$$0 \geqslant \int_{R_1}^{R_2} \frac{A_1(t) - B_1(t) + \pi^2\sigma}{t}\,dt = \nu(R_1) - \nu(R_2), \tag{6.2}$$

where

$$\nu(r) = \int_0^\infty \frac{\mu^*(t)}{t}\log\left|\frac{t+r}{t-r}\right| dt - \pi^2\sigma \log r. \tag{6.3}$$

We conclude that $\nu(r)$ is nondecreasing for all large r. Integrating (6.3) over $(1, r)$ we obtain

$$\int_0^\infty \frac{\mu^*(t)}{t}\, dt \int_1^r \frac{1}{s} \log\left|\frac{t+s}{t-s}\right| ds - \frac{1}{2}\pi^2\sigma(\log r)^2 = \int_1^r \frac{\nu(s)}{s}\, ds$$

so that

$$\int_0^\infty \frac{\mu^*(t)}{t}\, dt \int_0^r \frac{1}{s} \log\left|\frac{t+s}{t-s}\right| ds - \frac{1}{2}\pi^2\sigma(\log r)^2 = \int_1^r \frac{\nu(s)}{s} + O(1). \tag{6.4}$$

Thus

$$\begin{aligned} B_1(r) &= r\int_0^\infty \frac{\mu^*(t)}{t(t+r)}\, dt \\ &= \sigma(\log r)^2 + \frac{2}{\pi^2}\int_1^r \frac{\nu(t)}{t}\, dt + J(r) + O(1), \end{aligned} \tag{6.5}$$

where

$$J(r) = \int_0^\infty \frac{\mu^*(t)}{t}\left(\frac{r}{r+t} - \frac{2}{\pi^2}\int_0^r \frac{1}{s} \log\left|\frac{s+t}{s-t}\right| ds\right) dt. \tag{6.6}$$

Two changes of variable in the integrals of (6.6) yield

$$J(r) = \int_0^\infty \frac{\mu^*(rt)}{t} h(t)\, dt, \tag{6.7}$$

where

$$h(t) = \frac{1}{1+t} - \frac{2}{\pi^2}\int_0^{1/t} \frac{1}{s} \log\left|\frac{1+s}{1-s}\right| ds.$$

We have

LEMMA 3. *$h(t) \geqslant 0$ for $t \geqslant 1$ and $h(1/t) = -h(t)$ for $t \geqslant 1$.*

For $t > 1$

$$\begin{aligned} (t+1)\frac{2}{\pi^2}\int_0^{1/t} \frac{1}{s} \log\left|\frac{1+s}{1-s}\right| ds &= (t+1)\frac{2}{\pi^2}\int_0^{1/t}\left(2 + \frac{2s^2}{3} + \frac{2s^4}{5} + \dots\right) ds \\ &= (t+1)\frac{4}{\pi^2}\left(\frac{1}{t} + \frac{1}{3^2t^3} + \frac{1}{5^2t^5} + \dots\right) \\ &= \frac{4}{\pi^2}\left(1 + \frac{1}{t} + \frac{1}{3^2t^2} + \frac{1}{3^2t^3} + \dots\right) \\ &< \frac{8}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) = 1. \end{aligned}$$

Thus $h(t) > 0$ for $t > 1$ and certainly $h(1) = 0$. Further, for $t > 1$,

$$\begin{aligned} h\left(\frac{1}{t}\right) &= \frac{t}{1+t} - \frac{2}{\pi^2}\int_0^t \frac{1}{s} \log\left|\frac{s+1}{s-1}\right| ds \\ &= \frac{t}{1+t} - \frac{2}{\pi^2}\int_{1/t}^\infty \frac{1}{s} \log\left|\frac{s+1}{s-1}\right| ds \\ &= \frac{t}{1+t} - \frac{2}{\pi^2}\left(\frac{\pi^2}{2} - \int_0^{1/t} \frac{1}{s} \log\left|\frac{s+1}{s-1}\right| ds\right) = -h(t) \end{aligned}$$

and this proves Lemma 3.

Now, from (6.7) and Lemma 3,

$$\begin{aligned} J(r) &= \int_0^1 \frac{\mu^*(rt)}{t} h(t)\,dt + \int_1^\infty \frac{\mu^*(rt)}{t} h(t)\,dt \\ &= \int_1^\infty \frac{\mu^*(r/t)}{t} h(1/t)\,dt + \int_1^\infty \frac{\mu^*(rt)}{t} h(t)\,dt \\ &= \int_1^\infty \frac{\mu^*(rt) - \mu^*(r/t)}{t} h(t)\,dt \geqslant 0. \end{aligned} \tag{6.8}$$

Combining this with (6.5) we obtain

$$B_1(r) \geqslant \sigma(\log r)^2 + \frac{2}{\pi^2}\int_1^r \frac{\nu(t)}{t}\,dt + O(1).$$

Hence

$$\alpha_1 = \underline{\lim}_{r\to\infty} \frac{B_1(r) - \sigma(\log r)^2}{\log r} \geqslant \underline{\lim}_{r\to\infty}\left(\frac{2}{\pi^2 \log r}\int_1^r \frac{\nu(t)}{t}\,dt\right) > -\infty, \tag{6.9}$$

since $\nu(t)$ is nondecreasing. Moreover, since for any $k > 1$

$$\begin{aligned} B_1(r) &= r\int_0^\infty \frac{\mu^*(t)}{t(t+r)}\,dt \\ &\leqslant \int_0^{kr} \frac{\mu^*(t)}{t}\,dt + O\left(r\int_{kr}^\infty \frac{\log t}{t(t+r)}\,dt\right) \\ &\leqslant B(kr) + O\left(\int_k^\infty \frac{\log r + \log t}{t(t+1)}\,dt\right) = B(kr) + O(\log r), \end{aligned} \tag{6.10}$$

we deduce at once from (6.9) that

$$\alpha = \underline{\lim}_{r\to\infty} \frac{B(r) - \sigma(\log r)^2}{\log r} > -\infty. \tag{6.11}$$

But $\alpha < \infty$ by hypothesis and from this together with (6.10) we deduce that $\alpha_1 < \infty$. Thus, from (6.9),

$$\lim_{r\to\infty} \nu(t) = A_0$$

exists and is finite. From (6.5), then,

$$\beta_1 = \overline{\lim}_{r\to\infty} \frac{B_1(r) - \sigma(\log r)^2}{\log r} = \frac{2}{\pi^2}A_0 + \overline{\lim}_{r\to\infty} \frac{J(r)}{\log r}. \tag{6.12}$$

But

$$\begin{aligned} J(r) &= \int_1^\infty \frac{1}{t}(\mu^*(rt) - \mu^*(r/t))h(t)\,dt \leqslant \int_1^\infty \frac{\mu^*(rt)}{t(1+t)}\,dt \\ &= O\left(\int_1^\infty \frac{\log t + \log r}{t(1+t)}\,dt\right) = O(\log r), \end{aligned}$$

so, from (6.12), $\beta_1 < \infty$. Since $\beta \leqslant \beta_1$ we deduce finally (6.1).

7. Conclusion of the proofs of Theorems 1 and 2. We prove

LEMMA 4. $\mu^*(r) = 2\sigma \log r + O(1)$.

Since $\lim_{r\to\infty} \nu(r)$ exists and is finite we deduce from (6.2) that

$$\int_1^\infty \frac{A_1(t) - B_1(t) + \pi^2\sigma}{t}\, dt > -\infty.$$

Hence

$$A_1(r) > B_1(r) - 2\pi^2\sigma \tag{7.1}$$

outside a set E of finite logarithmic measure, and thus

$$\int_0^r \frac{\mu^*(t)}{t}\, dt = \frac{1}{2\pi}\int_0^{2\pi} u_1(re^{i\theta})\, d\theta = B_1(r) + O(1)$$
$$= \int_0^\infty \frac{r\mu^*(t)}{t(t+r)}\, dt + O(1) \tag{7.2}$$

as $r \to \infty$ outside E. Rearranging (7.2) yields

$$\int_1^\infty \frac{\mu^*(rt) - \mu^*(r/t)}{t(t+1)}\, dt = O(1).$$

Using (6.8) and the fact that $h(t) \leqslant 1/(1+t)$, we deduce that $J(r) = O(1)$, and so (6.5) yields

$$B_1(r) = \sigma(\log r)^2 + \frac{2}{\pi^2}\int_1^r \frac{\nu(t)}{t}\, dt + O(1) \tag{7.3}$$

as $r \to \infty$ outside E. Combining (7.2) and (7.3) we obtain

$$\int_0^r \frac{\mu^*(t)}{t}\, dt = \sigma(\log r)^2 + \frac{2}{\pi^2}\int_1^r \frac{\nu(t)}{t}\, dt + O(1)$$

as $r \to \infty$ outside E. Now, given r outside E, we may choose k satisfying $3 > k > 2$ such that kr is also outside E (this follows since E has finite logarithmic measure). Thus, for r outside E,

$$\mu^*(r)\log k \leqslant \int_r^{kr} \frac{\mu^*(t)}{t}\, dt$$
$$= 2\sigma \log k \log r + \sigma(\log k)^2 + \frac{2}{\pi^2}\int_r^{kr} \frac{\nu(t)}{t}\, dt + O(1)$$
$$\leqslant 2\sigma \log k \log r + \sigma(\log k)^2 + \frac{2}{\pi^2}\nu(kr)\log k + O(1)$$

and so

$$\mu^*(r) \leqslant 2\sigma \log r + O(1) \tag{7.4}$$

for r outside E. Since E has finite logarithmic measure and $\mu^*(r)$ increases with r we easily deduce that (7.4) holds for all large r. Quite similarly we obtain $\mu^*(r) \geqslant 2\sigma \log r + O(1)$ and this proves the lemma.

From Lemma 4 and (6.8) it follows that $J(r) = O(1)$ as $r \to \infty$ and hence, from (6.5), $\alpha_1 = \beta_1$. From (7.1)

$$B(r) > A(r) > A_1(r) > B_1(r) - 2\pi^2\sigma$$

for r outside E. But E has finite logarithmic measure and so, given any r, we may choose $k = k(r) > 1$ such that r/k is outside E and $k(r) \to 1$ as $r \to \infty$. Hence $B(r) > B(r/k) > B_1(r/k) + O(1)$ and finally

$$\alpha = \varliminf_{r\to\infty} \frac{B(r) - \sigma(\log r)^2}{\log r} \geqslant \varliminf_{r\to\infty} \frac{B_1(r/k) - \sigma(\log r)^2}{\log r}$$
$$= \alpha_1 - \lim_{r\to\infty} 2\sigma \log k = \alpha_1.$$

But $\alpha \leqslant \beta \leqslant \beta_1 = \alpha_1$ so $\alpha = \beta = \alpha_1 = \beta_1$ and this completes the proof of Theorems 1 and 2. Let us note that, from (7.3), $\alpha = (2/\pi^2)A_0$.

8. Proof of Theorem 3. We write $\mu^*(r) = 2\sigma \log r + \varepsilon(r)$, where $\varepsilon(r)$ is bounded for $r \geqslant 1$. From (6.3) we obtain

$$\int_0^\infty \frac{\mu^*(t) - 2\sigma \log t}{t} \log\left|\frac{t+r}{t-r}\right| dt = \nu(r). \tag{8.1}$$

We introduce $\varepsilon_1(t) = \varepsilon(t)$ for $t \geqslant 1$, $\varepsilon_1(t) = \varepsilon(1)$ for $0 \leqslant t \leqslant 1$, and write

$$\nu_1(r) = \nu(r) + \int_0^1 (\varepsilon(1) - \varepsilon(t))\log\left|\frac{t+r}{t-r}\right| dt/t.$$

Then $\nu_1(r) \to A_0$ as $r \to \infty$ and (8.1) may be rewritten as

$$\int_0^\infty \frac{\varepsilon_1(t)}{t} \log\left|\frac{t+r}{t-r}\right| dt = \nu_1(r).$$

With a change of variable we obtain

$$\int_{-\infty}^\infty \varepsilon_1(e^t)\log\left|\frac{e^{r-t}+1}{e^{r-t}-1}\right| dt = \nu_1(e^r).$$

Now $\varepsilon_1(e^t) = \mu^*(e^t) - 2\sigma t$ so

$$\varliminf_{\substack{x\to\infty\\ y>x\\ (y-x)\to 0}} (\varepsilon_1(e^y) - \varepsilon_1(e^x)) \geqslant 0;$$

that is, $\varepsilon_1(e^t)$ is slowly decreasing in the sense of [**8**, p. 209]. Moreover an application of contour integration yields

$$\int_{-\infty}^\infty e^{ixt} \log\left|\frac{e^t+1}{e^t-1}\right| dt = \frac{\pi}{|x|}\,\frac{1-e^{-\pi|x|}}{1+e^{-\pi|x|}}$$

for $x \neq 0$. We are thus able to apply Theorem 10a of [**8**, p. 211], to deduce that

$$\varepsilon_1(e^t) \to \frac{2A_0}{\pi^2} = \alpha$$

as $t \to \infty$ and this completes the proof of Theorem 3.

NOTE ADDED IN PROOF. The conjecture that follows the statement of Theorem 1 is false. Details of a correct theorem for $p \neq 2$ should appear in due course.

References

1. P. D. Barry, *The minimum modulus of small integral and subharmonic functions*, Proc. London Math. Soc. **12** (1962), 445–495.

2. ______, *On a theorem of Kjellberg*, Quart. J. Math. Oxford Ser. **15** (1964), 179–191.

3. P. C. Fenton, *The infimum of small subharmonic functions*, Proc. Amer. Math. Soc. **78** (1980), 43–47.

4. W. K. Hayman, *The minimum modulus of large integral functions*, Proc. London Math. Soc. **2** (1952), 469–512.

5. Bo Kjellberg, *On the minimum modulus of entire functions of lower order less than one*, Math. Scand. **8** (1960), 189–197.

6. ______, *A theorem on the minimum modulus of entire functions*, Math. Scand. **12** (1963), 5–11.

7. T. Rado, *Subharmonic functions*, Springer-Verlag, Berlin and New York, 1937.

8. D. V. Widder, *The Laplace transform*, Princeton Math. Ser., vol. 6, Princeton Univ. Press, Princeton, N. J., 1941.

Department of Mathematics, University of Otago, Dunedin, New Zealand