$L^p$ BEHAVIOR OF CERTAIN SECOND ORDER PARTIAL DIFFERENTIAL OPERATORS

BY

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Abstract. We give examples of bounded inverses of polynomials in $\mathbb{R}^n$, $n > 1$, which are not Fourier multipliers of $L^p(\mathbb{R}^n)$ for any $p \neq 2$. Our main tool is the Kakeya set construction of C. Fefferman. Using these results, we relate the invertibility on $L^p$ of a linear second order constant coefficient differential operator $D$ on $\mathbb{R}^n$ to the geometric structure of quadratic surfaces associated to its symbol $d$. This work was motivated by multiplier conjectures of N. Rivière and R. Strichartz.

0. Introduction. Let $D$ be a linear constant coefficient differential operator on $\mathbb{R}^n$, with symbol $d$. We assume throughout this paper that $D$ has order at most two, and we shall examine the $L^p$ spectrum and invertibility of $D$. (See §1 for the precise definitions we use.) Our main tool in this analysis will be the Kakeya set construction of Charles Fefferman, through which we shall relate invertibility of $D$ to the geometric structure of quadratic surfaces associated to $d$.

If $D$ is elliptic and $d$ has no real zeroes, $D$ is $L^p$ invertible for all $1 < p < \infty$. As we examine less trivial cases, great complexity seems to arise. The operator

$$\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 + i$$

is made tractable through the multiplier theorems of Marcinkiewicz [10] or Hörmander [6]; it is $L^p(\mathbb{R}^2)$ invertible if and only if $1 < p < \infty$. But a simple generalization

$$\partial^2/\partial x_1^2 - \sum_{j=2}^{n} \partial^2/\partial x_j^2 + i$$

cannot be $L^p(\mathbb{R}^n)$ invertible if $1 < p < 2(n - 1)/n$. Again, Littman, McCarthy and Rivière [9] showed that

$$i\partial/\partial x_1 + \sum_{j=2}^{n} \partial^2/\partial x_j^2 + i$$

cannot be $L^p(\mathbb{R}^n)$ invertible if $1 < p < 2n/(n + 1)$.

The complicated indices in the above results are illusory; it is the purpose of this paper to show that operators such as (2) and (3) above are $L^p(\mathbb{R}^n)$ invertible if and only if $p = 2$. The distinction between operators like (1) and those like (2) or (3)
will appear as a consequence of the distinct geometric properties of the surfaces 
\[ \xi_1^2 - \xi_2^2 = 0 \text{ versus } \xi_1^2 - \xi_3^2 = 0 \text{ or } \xi_1^2 - \xi_2^2 - \xi_3^2 = 0. \]

In §1 we collect preliminary results. We shall state and prove our main results in 
§§II and III; in §IV we discuss applications and motivations.

The work presented here was motivated by multiplier conjectures of N. Rivière 
[12] and R. Strichartz. Weaker versions of some of these results appeared previously in [16] and [17]. The main results of this paper were announced in [8].

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problems over the past five years.

I. Preliminaries. We begin with some facts on Fourier multipliers. A more 
extensive discussion may be found in Stein and Weiss [15].

Definition. Let \( m \) be a bounded measurable function. The operator \( T \) defined 
by \( Tf(\xi) = m(\xi)f(\xi) \) is said to be an \( L^p \)-multiplier if \( T \) has a bounded extension to 
\( L^p \).

Lemma 1. \( m \) is a multiplier of \( L^2 \) if and only if \( m \) is in \( L^\infty \).

Lemma 2. Let \( m \) be a multiplier of \( L^p \). Then \( m \) is a multiplier of \( L^p \) and of all \( L^r \) 
for \( r \) between \( p \) and \( p' \). In particular, \( m \) is in \( L^\infty \).

Lemma 3. \( m \) is a multiplier of \( L^1 \) if and only if there exists a finite Borel measure \( \mu \) 
with \( \hat{\mu} = m \).

Lemma 4 ([14] and [18]). Let \( m \) be continuous and a multiplier of \( L^p(\mathbb{R}^n) \) with 
operator norm \( C \). Then

(a) the restriction of \( m \) to any \( k \)-dimensional hyperplane is a multiplier of \( L^p(\mathbb{R}^n) \), 
with multiplier norm not exceeding \( C \).

(b) if \( A \) is any nonsingular linear transformation of \( \mathbb{R}^n \), \( m(A\xi) \) is a multiplier of 
\( L^p(\mathbb{R}^n) \) with norm \( C \).

We now recall some definitions and facts on \( L^p \)-behavior of constant coefficient 
partial differential operators \( D \). A more extensive discussion may be found in 
Schechter [14].

Definition. Let \( X \) be a Banach space, and let \( D_0 \) be a (possibly unbounded) 
linear operator from \( X \) to itself. A scalar \( \lambda \) is said to be in the resolvent set \( \rho(D_0) \) of 
\( D_0 \) if \( R(D_0 - \lambda) \) is dense in \( X \) and there is a constant \( C \) such that 
\[ \|X\| < C\|(D_0 - \lambda)X\|, \quad x \in D(D_0). \]

The spectrum \( \sigma(D_0) \) of \( D_0 \) consists of those scalars not in \( \rho(D_0) \). If \( D(D_0) \) is dense 
in \( X \), and \( 0 \in \rho(D_0) \), we say that \( D_0 \) is invertible.

As is well known, (see [14]) constant coefficient differential operators \( D \) are 
closable on \( L^p(\mathbb{R}^n) \), \( 1 < p < \infty \), and thus, have a minimal closed extension on
$L^p(\mathbb{R}^n)$, which we denote by $D_{0,p}$. When no confusion is likely to arise, we denote $D_{0,p}$ simply by $D_0$.

**Lemma 5** (See Theorem 4.1 of [14]). $\lambda \in \rho(D_{0,p})$ if and only if $(d(\xi) - \lambda)^{-1}$ is a multiplier of $L^p$ for $1 < p < \infty$.

**Corollary.** (a) $D_0$ is invertible on $L^2$ if and only if $|d(\xi)| > C_0$ for all $\xi \in \mathbb{R}^n$.
(b) If $D_0$ is invertible on $L^p$, then $|d(\xi)| > C_0$ for all $\xi \in \mathbb{R}^n$.
(c) If $n = 1$, then either $D_0$ is invertible on $L^p(\mathbb{R}^n)$ for all $p$, $1 < p < \infty$, or for no $p$.
(d) If $D$ is an elliptic operator of order $m$, $m > 1$, and $d(\xi) \neq 0$ for all $\xi \in \mathbb{R}^n$, then $D_0$ is invertible in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.

The remaining results are the essential constituents of our theorem. Lemma 6 is due to Marcinkiewicz and Zygmund [11]; Lemmas 7 and 8 are variants of the construction in Fefferman [2].

**Lemma 6.** Let $T$ be a linear operator with $\|Tf\|_p < C\|f\|_p$. Then

$$\left\| \left( \sum |Tf_j|^2 \right)^{1/2} \right\| < C \left\| \left( \sum |f_j|^2 \right)^{1/2} \right\|_p.$$  

**Lemma 7.** Let $N_k = 2^k \log k$. For each large $k$, there exists a set of $K \subset \mathbb{R}^2$ and a collection of $2^k$ disjoint rectangles $\{R_j\}$, such that for every $\gamma > 0$:
(a) The shorter and longer sides of each $R_j$ are bounded above by $N_k$, $N_k^2$, and below by $N_k/4$, $N_k^2/4$.
(b) $|K| \leq 20(\log \log N_k)^{-1} |R|_j$.
(c) Let $\vec{v}_j = (\cos \theta_j, \sin \theta_j)$ be the direction of the longer side of $R_j$. Then $|\theta_j - \pi/4| < \pi/8$.
(d) Let $\widetilde{R}_j$ denote $R_j + (1 + \gamma)\vec{v}_j$. Then $|\widetilde{R}_j|/4 < |\widetilde{R}_j \cap K|$.

**Lemma 8.** Let $N_k = 2^k \log k$, and let $\alpha, \beta, \gamma$ be given reals. For each $k > k_0(\alpha, \beta)$ there exists a set of $K \subset \mathbb{R}^2$ and a collection of $2^k$ disjoint rectangles $R_j$ with
(a) the longer and shorter sides of each $R_j$ are bounded above by $\beta N_k^2$, $\alpha N_k$, and below by $C\beta N_k^2$, $\alpha N_k$.
(b) $|K| \leq 20(\log \log N_k)^{-1} |R|_j$.
(c) Let $\vec{v}_j$ denote the direction of the longer side of $R_j$. Let $\widetilde{R}_j = R_j + (1 + \gamma)\vec{v}_j$. Then $|\widetilde{R}_j \cap K| > \frac{1}{4} |\widetilde{R}_j|$. Here $C = C(\alpha, \beta) = 2^{3/2} \beta (\alpha^2 + \beta^2)^{-1/2}$.

**II. Parabolic invariance.**

**Theorem A.** Let $\varphi$ be in $L^s \cap L^{\infty}(\mathbb{R}^1)$ for some $s$, $0 < s < \infty$, and assume $\varphi$ is not identically zero. Then the multiplier $m(\xi_1, \xi_2) = \varphi(\xi_2 - \xi_1^2)$ is $L^p(\mathbb{R}^2)$ bounded if and only if $p = 2$.

**Remarks.** (1) The theorem is easily extended to higher dimensions using Lemma 4(a).
(2) The requirement $\varphi \in L^s$ is natural if we expect $m$ to correspond to the inverse of a differential operator, and some condition is needed to prevent $m$ from being constant. But arguments similar to those of [2] easily show that $\varphi(r) = |r|^s$
does not yield a multiplier of $L^p$ if $p \neq 2$, and Jodeit has remarked that a similar situation occurs if $\varphi(+\infty) \neq \varphi(-\infty)$. It seems likely that the requirement $\varphi \in L^1$ is not essential.

The proof is conceptually straightforward. Let $m_\xi(t_1, t_2^2, m_\xi = \text{sign}(t_1 t_2^2)$, and let $K$, $K_\infty$ be the corresponding convolution kernels. It is an immediate consequence of C. Fefferman's work on the disk that $m_\infty$ is not a multiplier of $L^p$. This is shown using a variant of Lemma 7, due to Yves Meyer and by showing that $e^{i\theta_2}K_\infty(\vec{x})$ is large on Kakeya set counterexample functions, for appropriate choices of $\vec{\omega}_j$. To prove Theorem A, we show that $t$ may be chosen so that $e^{i\theta_2}(K_t - K_\infty)$ is small on the above counterexamples.

**Proof of Theorem A.** We shall assume $m$ is a multiplier of $L^p(\mathbb{R}^2)$ for $p \neq 2$, and derive a contradiction. By Lemma 2, we may assume $p > 2$. The conjugate $\bar{m}$ is a multiplier, as are $m \pm \bar{m}$; we may assume $m$ is real: Powers of $m$ are also multipliers, as these correspond to iterates of the operator: we may assume $m$ is positive. Given that $\varphi$ is in $L^1 \cap L^\infty$, some power of $\varphi$ is in $L^1$, so we may assume $\varphi$ is in $L^1$.

In all, we may assume $\varphi > 0$ and $\int \varphi = 1$; this requires $\varphi$ not identically zero.

From Lemma 4 we see that dilates $m(t_1, t_2^2)$ are uniformly bounded multipliers; if $K_t = m_t$, we see from Lemma 6 that

$$\left\| \left( \sum |K_t \ast f|^2 \right)^{1/2} \right\|_p < A \left\| \left( \sum |f|^2 \right)^{1/2} \right\|_p.$$  

Let $K_t(\vec{x}) = e^{it_2}K_t(\vec{x})$; then

$$\left\| \left( \sum |K_t \ast f|^2 \right)^{1/2} \right\|_p < A \left\| \left( \sum |f|^2 \right)^{1/2} \right\|_p \quad (4)$$

where $A$ is independent of $\vec{\omega}_j$ and $t$.

Let $K, R_j, \vec{R}_j$ be as in the construction of Lemma 7. Let $f_j(\vec{x}) = \chi_{\vec{R}_j(\vec{x})}$; we shall show that for each $N_k$, and $t = \gamma_k N_k^2$,

$$|K_t \ast f_j(\vec{x})| > C_{\gamma} \text{ when } \vec{x} \text{ is in } \vec{R}_j.$$  

As in Fefferman [2], it immediately follows that

$$A > \frac{C_{\gamma}}{2} \left( \frac{\log \log N_k}{20} \right)^{1/2-1/p}. \quad (5)$$

As $N_k$ tends to infinity, we find that (4) cannot hold uniformly in $t$ for $p > 2$, which is the desired contradiction.

To prove (5) note that

$$\int_K \sum |K_t \ast f_j(x)|^2 > \sum \int_{\vec{R}_j \cap K} |K_t \ast f_j(x)|^2 > C_{\gamma} \sum |\vec{R}_j \cap K| > C_{\gamma} \sum |R_j|.$$  

But

$$\int_K \sum |K_t \ast f_j(x)|^2 < |K|^2 \left( \sum |K_t \ast f_j|^2 \right)^{1/2} \|_p$$

$$< |K|^{1-2/p} \left( \sum |f|^2 \right)^{1/2} \|_p$$

$$= |K|^{1-2/p} A^2 \left( \sum |R_j| \right)^{2/p} \text{ as the } R_j \text{ are disjoint.}$$
Then

\[ A^2 > \frac{C_n^2}{4} \left( \frac{\sum |R_i|}{|K|} \right)^{1-2/p} \]

and estimate (b) of Lemma 7 completes the proof.

To estimate \( K'_j \ast f_j(\vec{x}) \) on \( \vec{R}_j \), choose \( \vec{w}_j = (-\frac{1}{2} \cot \theta_j, \frac{1}{4} \cot^2 \theta_j) \), where \( \theta_j \) is as in Lemma 7. Note

\[ K'_j \ast f_j(\vec{x}) = t^{-1} \int_{\vec{R}_j} (x_2 - y_2)^{-1/2} \exp \{ i\phi((\vec{x} - \vec{y})/t) \} \tilde{\phi}((x_2 - y_2)/t) \, dy \]

where \( \phi(\vec{x} - \vec{y}) = (x_2 - y_2)/4((x_1 - y_1)/(x_2 - y_2) - \cot \theta_j)^2 \).

The quantity in brackets is \( \cot t - \cot \theta_j \), where \( t \) is the angle between \( \vec{x} - \vec{y} \) and the axis. As \( \vec{x} \) and \( \vec{y} \) vary, the difference \( \cot t - \cot \theta_j \) is easily maximized, and is bounded by \( 5(\gamma N_\delta)^{-1} \); then

\[ |\phi(\vec{x} - \vec{y})| < 10\gamma^{-1} \quad \text{and} \quad \exp i\phi = 1 + O(\gamma^{-1}). \]

Then \( K'_j \ast f_j(x) = M + E \) where the main term \( M \) is

\[ t^{-1} \int_{\vec{R}_j} (x_2 - y_2)^{-1/2} \tilde{\phi}((x_2 - y_2)/t) \, dy \]

and the error term \( E \) is

\[ O(t^{-1} \gamma^{-1}) \int_{\vec{R}_j} (x_2 - y_2)^{-1/2} |\tilde{\phi}((x_2 - y_2)/t)| \, dy. \]

For \( \vec{x} \in \vec{R}_j, \vec{y} \in \vec{R}_j \),

\[ |x_2 - y_2| - |x - y| < (2 + \gamma)N_k^2 < 4\gamma N_k^2, \quad |x_2 - y_2|/t < 2\gamma N_k^2/\gamma \delta N_k^2 = 2/\delta \]

when \( \gamma > 2 \); hence \( |x_2 - y_2|^{-1/2} > (4\gamma N_k^2)^{-1/2} \). As \( \tilde{\phi} \) is continuous, we may choose \( \delta \) sufficiently large such that \( \text{Re} \tilde{\phi}((x_2 - y_2)/t) > \tilde{\phi}(0)/2 = 1/2 \); hence

\[ \text{Re} M > t^{-1}(4\gamma N_k^2)^{-1/2} |R_j| \cdot \frac{1}{2} = (4\gamma^{3/2}\delta)^{-1}. \]

To estimate \( E \) from above, we have

\[ |x_2 - y_2| = |(\vec{x} - \vec{y}) \cdot \vec{e}_2| = |\vec{x} - \vec{y}| \cdot \sin \psi > \frac{1}{4} |\vec{x} - \vec{y}| > \frac{1}{4} \gamma N_k^2, \]

and

\[ |\tilde{\phi}((x_2 - y_2)/t)| < \|\tilde{\phi}\|_\infty < \|\tilde{\varphi}\|_1 = 1. \]

Thus

\[ |E| < \frac{100}{t \gamma} (\frac{1}{4} \gamma N_k^2)^{-1/2} |R_j| < 200(\gamma^{5/2}\delta)^{-1}. \]

If we choose \( \gamma > 1600, |M| > 2|E| \) and \( |K'_j \ast f_j(\vec{x})| > (8\gamma^{3/2}\delta)^{-1} = c_\gamma \). This completes the proof.

**III. Hyperbolic invariance.** Let \( r^2 = \Sigma_{i=1}^n x_i^2, s^2 = \Sigma_{j=n+1}^p x_j^2 \). We shall establish negative multiplier results for \( SO(n, m) \) invariant multipliers \( \varphi(r^2 - s^2) \), when \( \max(n, m) > 1 \). (If \( \max(n, m) = 1 \), the Marcinkiewicz multiplier theorem shows there are many \( \varphi \) which yield \( L^p \) multipliers.)
Lemma 4(a) allows us to restrict attention to the case \( n = 2, m = 1 \). Our results in this case are weaker than those for \( n = 3, m = 1 \), so that we state two separate cases. Lemma 4 allows us to use case (a) if \( \min(n, m) > 2 \), and case (b) if \( \min(n, m) = 2 \).

**Theorem B.** (a) Let \( \varphi \) be in \( L^q \cap L^\infty(\mathbb{R}^n) \) for some \( q, 0 < q < \infty \), and \( \varphi \equiv 0 \). Then \( m(x) = \varphi(r^2 - s^2) \) is \( L^p(\mathbb{R}^{n+m}) \) bounded iff \( p = 2 \), when \( \min(n, m) > 2 \).

(b) Let \( \varphi \) be as above, with \( \varphi(t) = O(t^{-\alpha}) \) for some \( \alpha > 0 \). Then \( m(x) = \varphi(r^2 - s^2) \) is \( L^p(\mathbb{R}^{n+m}) \) bounded iff \( p = 2 \) when \( \min(n, m) > 2 \).

**Proof.** We present the full proof only of part (a); the proof of part (b) is similar, but there are more error terms. Assume then that \( m \) is an \( L^p \) multiplier; we may assume \( p > 2 \), and by Lemma 4(a) we may assume \( n = 3, m = 1 \). Applying Lemma 4(a), (b), we see that the multiplier \( m_1(x, y, z) = \varphi((x^2 + y^2 + z^2) - 1) = \varphi(t(r^2 - 1)) \) are uniformly bounded on \( L^p(\mathbb{R}^3) \). As in the proof of Theorem A, we may assume \( \varphi \) is in \( L^1(\mathbb{R}^r) \), \( \varphi > 0 \) and \( \int \varphi = 1 \). Let

\[
T_t(R) = m_t(R) = \int_0^\infty \frac{J_{1/2}(2\pi rR)}{(rR)^{1/2}} \varphi(t(r^2 - 1)) r^2 \, dr
\]

We shall first show that

\[
T_t(R) = \frac{\Im}{2tR} e^{2\pi i R} \left( \frac{R}{2t} \right) + E_0(t, R)
\]

where \( E(t, r) = o(t^{-1} R^{-1}) \) as \( t \to R \) tend to infinity. For,

\[
T_t(R) = \frac{\Im}{R} \int_0^\infty e^{2\pi i R} \varphi(t(r^2 - 1)) r \, dr
\]

\[
= \frac{\Im}{2tR} \int_{-t}^t \exp \left( 2\pi i R \left( \frac{s}{t} + 1 \right)^{1/2} \right) \varphi(s) \, ds
\]

\[
= \frac{\Im}{2tR} \int_{-t^{1/4}}^{t^{1/4}} \exp \left( 2\pi i R \left( \frac{s}{t} + 1 \right)^{1/2} \right) \varphi(s) \, ds
\]

\[
+ o(t^{-1} R^{-1});
\]

the \( o \)-estimate follows from the integrability of \( \varphi \).

In the region \((- t^{1/4}; t^{1/4})\) Taylor series shows that

\[
(s/t + 1)^{1/2} = 1 + s/2t + O((s/t)^2) = 1 + s/2t + O(t^{-3/2}),
\]

hence

\[
\exp(2\pi i R(s/t + 1)^{1/2}) = e^{2\pi i R} e^{2\pi i R s/2t} + O(R t^{-3/2})
\]

and

\[
T_t(R) = \frac{\Im}{2tR} \int_{-t^{1/4}}^{t^{1/4}} e^{2\pi i R s/2t} \varphi(s) \, ds + O(t^{-9/2}) + o(t^{-1} R^{-1})
\]

\[
= \frac{\Im}{2tR} \int_{-\infty}^\infty e^{2\pi i R s/2t} \varphi(s) \, ds + o(t^{-1} R^{-1})
\]

as desired, if \( t \sim R \).
The proof of the remainder of the theorem is now similar to that of Theorem A, with obvious modifications to account for error terms and for the three-dimensional nature of the problem. We first construct three-dimensional analogues of the sets in Lemma 8. Let \( K, R_j, \overline{R_j} \) be as in Lemma 8, and let \( \delta > 0 \) be given. Let

\[
S_j^0 = \{ \tilde{y} \in R_j \mid |\tilde{v}_j \cdot \tilde{y} - n| < \delta \text{ for some } n \},
\]

\[
\overline{S_j^0} = \{ \tilde{x} \in \overline{R_j} \mid |\tilde{v}_j \cdot \tilde{x} - (n + \frac{1}{4})| < \delta \text{ for some } n \}.
\]

Let \( L = K \times [0, \alpha N_k] \), \( S_j = S_j^0 \times [0, \alpha N_k] \), etc.

Now let \( \tilde{v}_j \) denote the direction of the longest side of \( S_j \), and let \( \tilde{w}_j \) be any perpendicular to \( \tilde{v}_j \). For the remainder of the proof, we shall assume \( \tilde{x} \in \overline{S_j} \) and \( \tilde{y} \in S_j \). We then have estimates analogous to those of Lemma 8.

\[
\alpha^2 \beta \delta N_k^4 < |S_j| < 2 \alpha^2 \beta \delta N_k^4. \tag{i}
\]

\[
(1/\delta)|S_j| < |\overline{S_j} \cap L|. \tag{ii}
\]

\[
|L| < \frac{6}{\delta} (c \log \log N_k)^{-1} \sum |S_j|. \tag{iii}
\]

\[
\gamma N_k^2 < |\tilde{x} - \tilde{y}| < ((\beta + \gamma)^2 + 2\alpha^2)^{1/2} N_k^2. \tag{iv}
\]

\[
\gamma N_k^2 < |(\tilde{x} - \tilde{y}) \cdot \tilde{v}_j| < (2\beta + \gamma)N_k^2. \tag{v}
\]

\[
(\tilde{x} - \tilde{y}) \cdot \tilde{w}_j < \sqrt{2} \alpha N_k. \tag{vi}
\]

(i), (iii), (iv), (v), (vi) are obvious, while (ii) follows from the proof of Lemma 8, as given in [2].

Let \( f_j(\tilde{y}) = \chi_{S}(\tilde{y}) \cos[2\pi(\tilde{v}_j \cdot \tilde{y})] \). As in the proof of Theorem A, we shall show that when \( t = dN_k^2 \),

\[
|T_{ij}f_j(\tilde{x})| > C(\alpha, \beta, \gamma, \delta, d) > 0 \tag{6}
\]

independently of \( N_k \); it then follows that

\[
\|T_{ij}\|_{L^p \rightarrow L^p} > C(\alpha, \beta, \gamma, \delta, d)(C \log \log N_k)^{1/2 - 1/p},
\]

which will yield a contradiction, when \( N_k \to \infty \).

The \( T_{ij} \) commute with rotations, so that it suffices to prove (6) when \( \tilde{v}_j = (1, 0, 0) \). Let \( \tilde{x} = (x_1, x_2, x_3) \), and similarly for \( \tilde{y} \). Then

\[
T_{ij}f_j(\tilde{x}) = \frac{\text{Im}}{2t} \int_{S_j} e^{2\pi i \tilde{x} \cdot \tilde{y}} \left| \frac{\tilde{x} - \tilde{y}}{2t} \right|^{-1} \cos 2\pi \tilde{v}_j \cdot \tilde{y} \, d\tilde{y} + \int_{S_j} E_0(t, |\tilde{x} - \tilde{y}|) \cos 2\pi (\tilde{v}_j \cdot \tilde{y}) \, d\tilde{y} = M_1 + E_1.
\]

Recall that \( E_0(t, R) = o(t^{-1} R^{-1}) \) and

\[
|\tilde{x} - \tilde{y}| > \gamma N_k^2, \quad |S_j| < 2 \alpha^2 \beta \delta N_k^4, \quad t = dN_k^2,
\]
then $E_1 = o(1)$. To estimate $M_1$, let $\tilde{\phi} = F + iG$; then

$$M_1 = \frac{1}{2t} \int_{S_j} \sin 2\pi|\tilde{x} - \tilde{y}|F\left(\frac{|\tilde{x} - \tilde{y}|}{2t}\right)|\tilde{x} - \tilde{y}|^{-1} \cos 2\pi\tilde{v}_j \cdot \tilde{y} \, d\tilde{y}$$

$$+ \frac{1}{2t} \int_{S_j} \cos 2\pi|\tilde{x} - \tilde{y}|G\left(\frac{|\tilde{x} - \tilde{y}|}{2t}\right)|\tilde{x} - \tilde{y}|^{-1} \cos 2\pi\tilde{v}_j \cdot \tilde{y} \, d\tilde{y}$$

$$= M_2 + E_2.$$

To estimate $E_2$, note that as $\tilde{x}$ and $\tilde{y}$ vary,

$$\frac{|\tilde{x} - \tilde{y}|}{2t} \text{ varies from } 0 \text{ to } \frac{((\beta + \gamma)^2 + 2\alpha^2)^{1/2}}{2d} = \epsilon.$$

Let $\tilde{G} = \sup_{0 < s < \epsilon} |G(s)|$. Then

$$|E_2| \leq \left(\frac{1}{2dN_k^2}\right) |S_j| \left(\frac{\gamma N_k^2}{\alpha^2}\right)^{-1} \tilde{G} \leq \left(\frac{\alpha^2\beta\delta}{\gamma d}\right) \tilde{G}.$$

To estimate $M_2$, note that

$$|\tilde{x} - \tilde{y}| = |x_1 - y_1| \left[1 + \frac{(x_2 - y_2)^2 + (x_3 - y_3)^2}{(x_1 - y_1)^2}\right]^{1/2}$$

$$= |x_1 - y_1| + D(x, y)$$

where

$$|D(x, y)| \leq \frac{\alpha^2}{\gamma} + o\left(N_k^{-1}\right).$$

Then

$$M_2 = \frac{1}{2t} \int_{S_j} \sin 2\pi|x_1 - y_1|\cos 2\pi D) \cdot |\tilde{x} - \tilde{y}|^{-1} F\left(\frac{|\tilde{x} - \tilde{y}|}{2t}\right) \cos 2\pi\tilde{v}_j \cdot \tilde{y} \, d\tilde{y}$$

$$+ \frac{1}{2t} \int_{S_j} \cos 2\pi|x_1 - y_1|\sin 2\pi D) \cdot |\tilde{x} - \tilde{y}|^{-1} F\left(\frac{|\tilde{x} - \tilde{y}|}{2t}\right) \cos 2\pi\tilde{v}_j \cdot \tilde{y} \, d\tilde{y}$$

$$= M_3 + E_3.$$

Note that $|E_3| \leq (2dN_k^2)^{-1} |S_j| (2\pi\alpha^2/2\gamma) (\gamma N_k^2)^{-1} ||F||_\infty + o(1)$. But $||F||_\infty \leq ||F + iG||_\infty < ||\phi||_1 = 1$, so that

$$|E_3| \leq \left(\frac{\alpha^2\beta\delta}{\gamma d}\right) (\pi\alpha^2/\gamma) + o(1).$$

To estimate $M_3$, note that $S_j$ and $\tilde{S}_j$ may all be oriented so that $x_1 - y_1 > 0.$ As $\tilde{v}_j \cdot \tilde{y} = y_1$,

$$M_3 = \frac{\sin \frac{2\pi x_1}{2t}}{2t} \int_{S_j} \cos 2\pi y_1 \cos 2\pi D) \cdot |\tilde{x} - \tilde{y}|^{-1} F\left(\frac{|\tilde{x} - \tilde{y}|}{2t}\right) \, d\tilde{y}$$

$$- \frac{\cos \frac{2\pi x_1}{2t}}{2t} \int_{S_j} \sin 2\pi y_1 \cos 2\pi D) \cdot |\tilde{x} - \tilde{y}|^{-1} F\left(\frac{|\tilde{x} - \tilde{y}|}{2t}\right) \, d\tilde{y}$$

$$= M_4 + E_4.$$
To estimate $E_4$, recall that $|x_1 - (n + \frac{1}{4})| < \delta$ and $|y_1 - n| < \delta$. If $\delta$ is small, $|\cos 2\pi x_1| < 2\pi \delta$ and $|\sin 2\pi y_1| < 2\pi \delta$, and therefore

$$|E_4| < 2\pi \delta (2dN_k)^{-2}\|S_j\|_{L^\infty} \|F\|_\infty$$

or $|E_4| < (\alpha^2 \beta \delta / \gamma d)(4\pi^2 \delta^2)$.

We shall now compute a lower bound for $M_4$. If we require $\delta < 10^{-3}$, $\sin 2\pi x_1 > \frac{1}{2}$ and $\cos 2\pi y_1 > \frac{1}{2}$. We shall also require $\alpha^2 / 2\gamma < 10^{-2}$, so that $\cos 2\pi D > \frac{1}{2}$. Then if $\bar{F} = \inf_{0 < s < \epsilon} F(s)$,

$$M_4 > (4dN_k)^{-1}|S_j| \frac{1}{2} \cdot [(\beta + \gamma)^2 + 2\alpha^2]^{-1/2} N_k^{-2} \bar{F}$$

$$> \frac{\alpha^2 \beta \delta}{\gamma d} \left( \frac{\gamma \bar{F}}{8((\beta + \gamma)^2 + 2\alpha^2)^{1/2}} \right).$$

In all, $T_{sf}(x) = M_4 + E_2 + E_3 + E_4 + o(1)$ where $|E_2 + E_3 + E_4| < (\alpha^2 \beta \delta / \gamma d)(\bar{G} + \pi \alpha^2 / \gamma + 4\pi^2 \delta^2)$. We shall now choose $\beta = \gamma = \frac{1}{2}$; we still must require $\delta < 10^{-3}$ and now $\alpha^2 < 10^{-2}$. Then $\gamma((\beta + \gamma)^2 + 2\alpha^2)^{-1/2} > 1/4$ and $M_4 > (\alpha^2 \delta / d)(\frac{1}{2}) \bar{F}$. Since $e = ((\beta^2 + \gamma^2) + 2\alpha^2)^{1/2} / 2d < d^{-1}$, $e$ is small if $d$ is large. We shall use the continuity of $\phi = F + iG$ to estimate $\bar{F}$ and $\bar{G}$. $1 = \phi(0)$, so that if $d$ is large, $\inf_{0 < s < \epsilon} F(s) > \frac{1}{2}$ and $\sup_{0 < s < \epsilon} |G(s)| < 1/6.128$. Then $M_4 > \alpha^2 \delta / d \cdot 1 / 64$. To insure that the error terms are only a fraction of $M_4$, choose $\delta$ so small that $4\pi^2 \delta^2 < 1 / 6.128$ (still requiring $\delta < 10^{-3}$) and $\alpha$ so small that $\pi \alpha^2 / \gamma < 1 / 6.128$ (still requiring $\alpha^2 < 10^{-2}$, $\gamma = \frac{1}{2}$).

Then $|E_2 + E_3 + E_4| < \alpha^2 \delta / d \cdot 1 / 128$, whence $|T_{sf}(\bar{x})| > \alpha^2 \delta / 64 d + o(1)$. If $N_k$ is large, $|T_{sf}(\bar{x})| > \alpha^2 \delta / 128 d = C(\alpha, \beta, \gamma, \delta, d) > 0$. This completes the proof.

IV. Applications and extensions. (1) A. P. Calderón observed that every bounded rational function of a real variable gives rise to an $L^p(\mathbb{R}^1)$ multiplier if $1 < p < \infty$. The work of Littman, McCarthy and Rivière [9] showed this cannot hold in higher dimensions, but Rivière [12] conjectured that bounded rational functions must be $L^p$ multipliers for some $p \neq 2$. Theorem A shows this conjecture is false.

(2) Let $G$ be a noncompact connected semisimple Lie group with finite center, Lie algebra $g$ and Killing form $B$. Greenleaf, Moskowitz and Rothchild [5] showed that there are no nontrivial finite measures on $g$ invariant under the action of $\text{Ad } G$. This means that there are no nontrivial multipliers of $L^1(g)$ which commute with the action of $\text{Ad } G$. R. Strichartz conjectured this holds for all $p \neq 2$. Theorem B provides evidence in favor of the conjecture. Defining a Fourier transform on $L^p(g)$ by $\hat{f}(y) = \int_g e^{iB(xy)} f(x) dx$, a multiplier is seen to commute with $\text{Ad } G$ if and only if $m$ is $\text{Ad } G$ invariant. If $\text{Ad } G$ acted transitively on the level sets $B(x, x) = \text{constant}$, it would follow that $m(x) = \phi(B(x, x))$. Diagonalizing $B$ and observing $\dim g > 3$, we see that Theorem B applies, and $m$ is indeed not a multiplier of $L^p(g)$. In general, $\text{Ad } G$ does not act transitively, but there is a three-dimensional subspace on which $B$ has signature $(+, -, -)$ and on which $\text{Ad } G$ acts transitively. We use Lemma 4(a) to restrict $m$ to this subspace, and then apply Theorem B. This provides a partial solution to Strichartz’ conjecture, with
restrictions on the growth of \( m \). But \( m(x) = |B(x, x)|^\mu \) is easily seen not to be a multiplier, but fails to satisfy our growth restrictions. The general case of Strichartz’ conjecture therefore remains open.

(3) The results in this paper were adapted to the analysis of second order differential operators, or, more generally, multipliers constant on quadratic surfaces. The techniques may nonetheless be used to obtain unboundedness of multipliers of the form \( \varphi(\xi - \varphi(\eta)) \) where \( \varphi \) has growth restrictions and \( \varphi \) curvature restrictions. As A. Cordoba has kindly pointed out, positive and negative results for such operators may be obtained using the methods of [1]. He has informed us that an analysis of such operators has been carried out in the doctoral dissertation of his student, A. Ruiz.

(4) The techniques of this paper have more general applicability for the study of the \( L^p \) spectrum of constant coefficient differential operators. We shall consider a class of “good” differential operators for which our techniques are applicable. Using Lemma 5, the central result of this paper may be phrased as a classification.

**Classification.** Let \( D \) be a “good” operator. Then either

(a) \( \sigma(D_{0,p}) = \text{range } d \) for all \( p, 1 < p < + \infty \), or
(b) \( \sigma(D_{0,2}) = \text{range } d \) when \( p = 2 \), while \( \sigma(D_{0,p}) = \mathbb{C} \) for \( 1 < p < \infty, p \neq 2 \).

The basic problem is to find the largest class of good operators for which the classification holds. It is clear that we cannot expect this for all operators; for example, the operator with symbol \((\xi_2 - \xi_1^2 + i)(\xi_2 + \xi_1^2 + i)\) is \( L^p(\mathbb{R}^2) \) invertible for some \( p \neq 2 \), but not for all \( p \) (see [7] and [13]).

Using Theorems A and B, and some simple variants of the Marcinkiewicz and Hörmander multiplier theorems, one can see that if \( d \) is a polynomial of degree 2, with real coefficients, then \( D \) is a “good” operator. It seems plausible that this holds for arbitrary 2nd order operators. We have made some progress on this question, and shall return to the problem in a later paper.

**References**


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