TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 262, Number 2, December 1980

THE ASYMPTOTIC BEHAVIOR OF GAS IN AN n-DIMENSIONAL POROUS MEDIUM[1]

BY

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ABSTRACT. Consider the flow of gas in an n-dimensional porous medium with initial density $u_0(x) \geqslant 0$. The density $u(x, t)$ then satisfies the nonlinear degenerate parabolic equation $u_t = \Delta u^m$ where $m > 1$ is a physical constant. Assuming that $I \equiv \int u_0(x)dx < \infty$ it is proved that $u(x, t)$ behaves asymptotically, as $t \to \infty$, like the special (explicitly given) solution $V(|x|, t)$ which is invariant by similarity transformations and which takes the initial values $\delta(x)I$ ($\delta(x)$ = the Dirac measure) in the distribution sense.

1. Statement of the main result. Consider the Cauchy problem for $u(x, t)$:

$$u_t = \Delta u^m \qquad (x \in R^n, t > 0), \tag{1.1}$$

$$u(x, 0) = u_0(x) \qquad (x \in R^n). \tag{1.2}$$

The function u represents the density of a gas in a porous medium and m is a physical constant, $m > 1$. We assume that

$$\begin{gathered} u_0(x) \text{ is continuous}, \quad u_0(x) \geqslant 0, \quad u_0(x) \not\equiv 0, \quad u_0(x) \leqslant M, \\ u_0 \in L^1(R^n) \cap L^2(R^n) \ \ (M \text{ constant}), \end{gathered} \tag{1.3}$$

and set

$$I = \int_{R^n} u_0(x)dx. \tag{1.4}$$

By a weak solution of (1.1), (1.2) we mean a function u satisfying, for any $T > 0$,

$$\int_0^T \int_{R^n} \left[(u(x, t))^2 + |\nabla_x u^m(x, t)|^2 \right] dx\, dt < \infty$$

and

$$\int_0^T \int_{R^n} \left(u \frac{\partial f}{\partial t} - \nabla_x u^m \cdot \nabla_x f \right) dx\, dt + \int_{R^n} u_0(x) f(x) dx = 0$$

for any continuously differentiable function f with compact support in $R^n \times [0, T)$.

It is well known **[14]**, **[11]** that given u_0 satisfying (1.3), (1.4) there exists a unique generalized solution u of (1.1), (1.2) and

$$\int_{R^n} u(x, t)dx = I \quad \text{for all } t > 0; \tag{1.5}$$

Received by the editors August 27, 1979.

AMS (MOS) subject classifications (1970). Primary 35K55, 76S05; Secondary 35K15.

[1]This work is partially supported by National Science Foundation Grant MCS-7817204 and AFOSR Grant 78-3602.

a very general uniqueness theorem was recently proved by Brezis and Crandall [4]. By a recent result of Caffarelli and Friedman [5], [6] $u(x, t)$ is Hölder continuous in (x, t) uniformly in any strip $t \geq \delta$ $(\delta > 0)$.

In this paper we are interested in the asymptotic behavior of $u(x, t)$ as $t \to \infty$. To state the main result we first introduce the similarity solution V_L of Barenblatt and Prattle [13]. Let

$$G(s) = \left[(\beta^2 - c^2 s^2)^+ \right]^{1/(m-1)}$$

where

$$c^2 = \frac{l(m-1)}{2mn}, \qquad l = \frac{1}{m - 1 + 2/n}$$

and β is a positive constant such that $\int_{R^n} G(|x|)dx = 1$. Then, for any $L > 0$,

$$V_L(r, t) = L^{1/(m-1)} \frac{1}{L^l t^l} G\left(\frac{r}{L^{l/n} t^{l/n}} \right), \qquad r = |x|, \tag{1.6}$$

is a solution of

$$\frac{\partial}{\partial t} V_L = \Delta (V_L)^m$$

satisfying $V_L(r, 0) = \delta(x) L^{1/(m-1)}$ ($\delta(x)$ = the Dirac measure) in the sense of distributions. As easily verified,

$$\int_{R^n} V_L(r, t)dx = L^{1/(m-1)}. \tag{1.7}$$

We can now state the main result of the paper.

THEOREM 1.1. *As* $t \to \infty$,

$$t^l |u(x, t) - V_{L_0}(r, t)| \to 0 \tag{1.8}$$

uniformly with respect to x *in any set* $|x| \leq Ct^{l/n}$ $(C > 0)$, *where*

$$L_0 = I^{m-1}. \tag{1.9}$$

For $n = 1$ this theorem was proved by Kamin (Kamenomostkaya) [9], [10]. The proof for $n \geq 1$ given in this paper employs a different approach than in [9], [10] and exploits the continuity of the solution u; both methods use similarity transformations.

For $n = 1$ Peletier [12] and Van Duyan and Peletier [7] studied the asymptotic behavior in a half-plane $x > 0$ and in the whole space when $u_0(x) \sim A$ as $x \to \infty$, $u_0(x) \sim B$ as $x \to -\infty$; $A > 0$, $B > 0$. For a bounded domain Ω in $\mathbf{R}^n$, Aronson and Peletier [1] have recently obtained the asymptotic behavior of the solution in $\{(x, t) \in \Omega \times (0, \infty)\}$; they assume that $u(x, t) = 0$ if $x \in \partial\Omega$. Both the result and method are different from those obtained in the present work.

To facilitate the reading of the proof, which takes the rest of the paper, we have broken it into several parts. In §2 we derive some properties of the functions V_L. In §3 we study the function $L(t)$, where $L(t)$ is defined as the largest value of L such that, for some $\tau \geq 0$,

$$V_L(r, t + \tau) \leq u(x, t) \quad (\text{for all } x \in R^n).$$

In §4 we prove Theorem 1.1 in case $u_0(x)$ has compact support and, finally, in §5 we give the proof in the general case.

2. Auxiliary lemmas on V_L. Notice that

$$\text{support of the function } r \to V_L(r, t) \text{ is given by } r \leqslant \frac{L^{l/n}t^{l/n}}{c/\beta}. \tag{2.1}$$

LEMMA 2.1. *For fixed $L > 0$, $t > 0$, if $\varepsilon > 0$ is sufficiently small then*

$$V_L(r, t-\varepsilon) > V_L(r, t) \quad \text{for } 0 < r < \theta\frac{L^{l/n}t^{l/n}}{c/\beta},$$

$$V_L(r, t-\varepsilon) < V_L(r, t) \quad \text{for } \theta\frac{L^{l/n}t^{l/n}}{c/\beta} < r < \frac{L^{l/n}t^{l/n}}{c/\beta}$$

where

$$\theta = \theta_\varepsilon \to \theta_0 \quad \text{as } \varepsilon \to 0; \qquad \theta_0 = \sqrt{(m-1)l}\,. \tag{2.2}$$

Observe that

$$0 < \theta_0 < 1. \tag{2.3}$$

PROOF. We solve, for small ε, the equation in r:

$$V_L(r, t-\varepsilon) = V_L(r, t). \tag{2.4}$$

Thus

$$\left(\frac{t-\varepsilon}{t}\right)^{l(m-1)} G^{m-1}\left(\frac{r}{L^{l/n}t^{l/n}}\right) = G^{m-1}\left(\frac{r}{L^{l/n}(t-\varepsilon)^{l/n}}\right). \tag{2.5}$$

By the mean value theorem the right-hand side is equal to

$$G^{m-1}\left(\frac{r}{L^{l/n}t^{l/n}}\right) + \frac{r}{L^{l/n}}\left(\frac{1}{(t-\varepsilon)^{l/n}} - \frac{1}{t^{l/n}}\right)(G^{m-1})'\left(\frac{r}{L^{l/n}\tilde{t}^{l/n}}\right)$$

where $t - \varepsilon < \tilde{t} < t$; also

$$(G^{m-1})'(s) = -2c^2 s,$$

$$\left(\frac{t-\varepsilon}{t}\right)^{l(m-1)} \sim 1 - \frac{l(m-1)}{t}\varepsilon,$$

$$\frac{1}{(t-\varepsilon)^{l/n}} - \frac{1}{t^{l/n}} = -\frac{1}{t^{l/n}}\left(1 - \left(\frac{t}{t-\varepsilon}\right)^{l/n}\right) \sim \frac{1}{t^{l/n}}\,\frac{l/n}{t}\varepsilon.$$

Using these facts in (2.5), we get

$$\frac{l(m-1)}{t} c\varepsilon G^{m-1}\left(\frac{r}{L^{l/n}t^{l/n}}\right) \sim 2\frac{c^2 r^2}{L^{2l/n}t^{2l/n}}\,\frac{l/n}{t}\varepsilon,$$

so that

$$n(m-1)\left(\beta^2 - \frac{c^2r^2}{L^{2l/n}t^{2l/n}}\right) \sim \frac{2c^2r^2}{L^{2l/n}t^{2l/n}}.$$

It follows that the solution r of (2.4) is given by

$$\theta \frac{\beta}{c} L^{l/n} t^{l/n}$$

where

$$\theta = \theta_\varepsilon \to \left(\frac{n(m-1)}{n(m-1)+2} \right)^{1/2} = \theta_0 \quad \text{as } \varepsilon \to 0,$$

and the proof of the lemma follows.

LEMMA 2.2. *For any $x^0 \in R^n$ and τ real,*

$$t^l \left| V_L(|x - x^0|, t + \tau) - V_L(|x|, t) \right| \to 0 \quad \text{as } t \to \infty, \tag{2.6}$$

uniformly with respect to $x \in R^n$.

The proof is immediate from (1.6).

Choose a point $y^0 \in R^n$ such that $u_0(y^0) > 0$; then, for some $\delta_0 > 0$, $\eta_0 > 0$,

$$u_0(x) \geqslant \delta_0 \quad \text{if } |x - y^0| < \eta_0.$$

But then we can find $L_1 > 0$, $\tau_1 > 0$ such that

$$u_0(x) \geqslant V_{L_1}(|x - y^0|, \tau_1) \quad \text{for all } x \in R^n. \tag{2.7}$$

Indeed, we simply have to choose

$$L_1 \tau_1 = c_1, \qquad c_1 \text{ sufficiently small,}$$

and then choose L_1 sufficiently small (depending on c_1).

If we prove that

$$t^l \left| u(x, t) - V_{L_0}(|x - y^0|, t) \right| \to 0 \quad \text{as } t \to \infty, \tag{2.8}$$

uniformly in $x \in R^n$ in any set $|x| \leqslant Ct^{l/n}$, then, in view of Lemma 2.2, the assertion of Theorem 1.1 would follow.

For simplicity of notation we fix the origin at the point y^0. Thus (2.7) becomes

$$u_0(x) \geqslant V_{L_1}(r, \tau_1), \qquad r = |x|, \tag{2.9}$$

and the assertion (2.8) reduces to the assertion (1.8). From (2.9) it follows, by comparison, that

$$u(x, t) \geqslant V_{L_1}(r, t + \tau_1) \qquad (x \in R^n, t > 0). \tag{2.10}$$

From now on, until the end of §4, we impose the restriction:

$$u_0 \text{ has compact support.} \tag{2.11}$$

We can then find $L_2 > 0$, $\tau_2 > 0$ such that

$$u_0(x) \leqslant V_{L_2}(r, \tau_2) \quad \text{for all } x \in R^n. \tag{2.12}$$

Indeed, we simply take first $L_2 \tau_2 = c_2$ where c_2 is sufficiently large and then choose L_2 to be sufficiently large, depending on c_2.

From (2.12), we deduce that

$$u(x, t) \leqslant V_{L_2}(r, t + \tau_2) \qquad (x \in R^n, t > 0). \tag{2.13}$$

Suppose for some $t_0 > 0$, τ real, $t_0 + \tau > 0$,

$$u(x, t_0) \equiv V_L(r, t_0 + \tau). \tag{2.14}$$

Then, by uniqueness,

$$u(x, t) = V_L(r, t + \tau) \quad \text{if } t > t_0.$$

Recalling Lemma 2.2, the assertion of Theorem 1.1 then follows. Thus, in order to prove Theorem 1.1 (under the condition (2.11)), we may assume, without loss of generality, that

$$\text{for any } L > 0, \quad \tau \text{ real}, \quad t > 0, \quad t + \tau > 0,$$
$$u(x, t) \not\equiv V_L(r, t + \tau) \quad (x \in R^n). \tag{2.15}$$

For any fixed $t > 0$, denote by Σ_t the set of all points (L, τ) such that $L > 0$, $\tau \geqslant 0$, and

$$u(x, t) \geqslant V_L(r, t + \tau), \tag{2.16}$$

and set

$$L(t) = \sup_{(L,\tau) \in \Sigma_t} L. \tag{2.17}$$

3. Properties of $L(t)$.

LEMMA 3.1. *There exists a point $(L^*, \tau^*) \in \Sigma_t$ such that $L(t) = L^*$ and*

$$\tau^* \leqslant C(t + 1); \tag{3.1}$$

C is a constant independent of t.

PROOF. The inequality (2.16) implies

$$\int_{R^n} V_L(r, t + \tau)dx \leqslant \int_{R^n} u(x, t)dx = I$$

where (1.5) was used. Recalling (1.7) we conclude that $L \leqslant I^{m-1}$. From (2.10) we also deduce that $L \geqslant L_1$. Thus, in seeking to find sup L in (2.17) we may restrict the L to lie in the interval

$$L_1 \leqslant L \leqslant I^{m-1}. \tag{3.2}$$

In view of (2.16) and (3.2),

$$\text{support of } x \to u(x, t) \text{ contains the set } r \leqslant C_1(t + \tau)^{l/n}$$

and in view of (2.13),

$$\text{support of } x \to u(x, t) \text{ is contained in the set } r \leqslant C_2(t + \tau_2)^{l/n};$$

both constants C_1, C_2 are positive and independent of t. It follows that

$$t + \tau \leqslant (C_2/C_1)^{n/l}(t + \tau_2), \quad \text{i.e.,} \quad \tau \leqslant C(t + 1) \qquad (C > 0).$$

Thus, in computing sup L in (2.17) we may restrict ourselves to $(L, \tau) \in \tilde{\Sigma}_t$, where

$$\tilde{\Sigma}_t = \Sigma_t \cap \left\{L_1 \leqslant L \leqslant I^{m-1}, 0 \leqslant \tau \leqslant C(t + 1)\right\}.$$

Since $\tilde{\Sigma}_t$ is a compact set, it follows that

$$\sup_{\tilde{\Sigma}_t} L = \max_{\tilde{\Sigma}_t} L = L^*,$$

with $(L^*, \tau^*) \in \tilde{\Sigma}_t$ rendering the maximum; this completes the proof.

LEMMA 3.2. *$L(t)$ is monotone decreasing in t, and there exists a sequence $t_j \uparrow \infty$ such that*

$$L(t_j) < L(t_{j+1}) \quad \textit{for all } j. \tag{3.3}$$

PROOF. Since (2.13) for a fixed t implies the same inequality for t replaced by t', $t' > t$, it follows that $L(t') \leqslant L(t)$. To prove (3.3), we fix point $t = t_0$ and construct a point $t_1 > t_0$ such that

$$L(t_0) < L(t_1); \tag{3.4}$$

this would establish (3.3).

We have

$$u(x, t_0) \geqslant V_L(r, t_0 + \tau) \quad \text{where } L = L(t_0). \tag{3.5}$$

We claim: there exists a $\xi > 0$ such that

$$u(x, t_0 + \xi) \not\equiv V_L(r, t_0 + \xi + \tau) \quad \text{for } r < (\beta/c)L^{l/n}(t_0 + \tau + \xi)^{l/n}. \tag{3.6}$$

Indeed, otherwise

$$u(x, t) \equiv V_L(r, t + \tau) \quad \text{whenever } t \geqslant t_0, V_L > 0. \tag{3.7}$$

Recalling (2.15) we conclude that, for some $x^0 \in R^n$, $\delta > 0$,

$$u(x, t_0) > 0, V_L(r, t_0 + \tau) = 0 \quad \text{if } |x - x^0| < \delta.$$

But then

$$u(x, t_0) > V_{L_*}(|x - x^0|, t_0 + \tau_*) \tag{3.8}$$

for some $L_* > 0$, $\tau_* > 0$.

Choose now a point $(\hat{x}, \hat{t})$ such that

$$|\hat{x}| = (\beta/c)L^{l/n}(\hat{t} + \tau)^{l/n}$$

(i.e., $\hat{x}$ lies on the boundary of the support $x \to V_L(r, \hat{t} + \tau)$) and

$$|\hat{x} - x^0| < (\beta/c)L^{l/n}(\hat{t} + \tau_*)^{l/n}$$

(i.e., $V_{L_*}(|\hat{x} - x^0|, \hat{t} + \tau_*) > 0$). By (3.8) and comparison we then deduce that $u(\hat{x}, \hat{t}) > 0$, whereas from (3.7) we get that $u(\hat{x}, \hat{t}) = 0$, a contradiction.

Having proved (3.6) we can now write

$$u(x, t_0 + \xi) \not\equiv V_L(r, t_0 + \xi + \tau) \quad \text{if } r < (\tilde{\theta}\beta/c)L^{l/n}(t_0 + \tau + \xi)^{l/n} \equiv r_1 \tag{3.9}$$

if $0 < \tilde{\theta} < 1$, $1 - \tilde{\theta}$ sufficiently small.

We shall compare $u(x, t + \xi)$ with $V_L(r, t + \tau + \xi)$ for $r < r_1$, $t > t_0$. Notice that

$$V_L(r, t + \tau) > 0 \quad \text{if } r < r_1;$$

hence also u is positive. We therefore have, in the classical sense,

$$u_t = \Delta u^m, \qquad (V_L)_t = \Delta(V_L)^m.$$

It is easily seen that the function $w = u - V_L$ then satisfies in the cylinder, $r < r_1$, $t > t_0$,

$$w_t = a\Delta w + \sum b_i w_{x_i} + cw \tag{3.10}$$

for some smooth coefficients a, b_i, c; $a > 0$. Since $w \geqslant 0$ in the cylinder, the strong maximum principle implies that either

$$w(x, t) > 0 \quad \text{if } |x| < r_1, t > t_0,$$

or else $w \equiv 0$; the second possibility is ruled out by (3.9). Thus, for any $0 < \theta_1 < 1$ and for any $\eta > \xi$ ($\eta - \xi$ small enough)

$$u(x, t_0 + \eta) > V_L(r, t_0 + \tau + \eta) \quad \text{if } r < \theta_1 \cdot (\beta/c) L^{l/n}(t_0 + \tau + \eta)^{l/n}. \tag{3.11}$$

By Lemma 2.1,

$$V_L(r, t_0 + \tau + \eta) > V_L(r, t_0 + \tau + \eta - \varepsilon) \quad \text{if } r > \theta_\varepsilon \cdot (\beta/c) L^{l/n}(t_0 + \tau + \eta)^{l/n}$$

where $\theta_\varepsilon \to \theta_0$, $0 < \theta_0 < 1$. On the other hand,

$$|V_L(r, t_0 + \tau + \eta) - V_L(r, t_0 + \tau + \eta - \varepsilon)| = o(1) \to 0 \quad \text{if } \varepsilon \to 0,$$

uniformly in r. Taking $\theta_1 > \theta_0$ and choosing ε sufficiently small (depending on ξ, η) we conclude, using (3.11), that

$$u(x, t_0 + \eta) > V_L(x, t_0 + \tau + \eta - \varepsilon)$$

in a region $r < R$ containing the support of the function

$$r \to V_L(r, t_0 + \tau + \eta - \varepsilon).$$

But then, if $L' > L$ and $L' - L$ is sufficiently small,

$$u(x, t_0 + \eta) > V_{L'}(r, t_0 + \tau + \eta - \varepsilon).$$

It follows that

$$L(t_0 + \eta) \geqslant L' > L = L(t_0).$$

Setting $t_1 = t_0 + \eta$, (3.4) follows.

4. Proof of Theorem 1.1 for u_0 having compact support. Consider the one-parameter family of functions

$$u_k(x, t) = k^n u(kx, k^{n/l}t), \qquad k \geqslant 1. \tag{4.1}$$

From (2.13) we deduce that

$$u_k(x, t) \leqslant k^n V_{L_2}(kx, k^{n/l}t + \tau_2) = V_{L_2}(r, t + \tau_2/k^{n/l}). \tag{4.2}$$

Therefore, for any $\delta > 0$,

$$u_k(x, t) \leqslant C_\delta \qquad (x \in R^n, t \geqslant \delta, k \geqslant 1) \tag{4.3}$$

where C_δ is a constant depending on δ.

Applying the continuity result of Caffarelli and Friedman **[5]**, **[6]** we deduce that

$$\text{the } u_k(k, t) \text{ are equicontinuous in } (x, t) \in R^n \times [\delta, \infty), \quad \text{for } k \geqslant 1. \tag{4.4}$$

Hence from any sequence $k_i^* \uparrow \infty$ we can extract a subsequence k_i such that, for any $\delta > 0$,

$$u_{k_i}(x, t) \to w(x, t) \text{ uniformly in } (x, t) \text{ in compact subsets of } R^n \times [\delta, \infty); \tag{4.5}$$

w, the limit function, may a priori depend on the sequence.

By the energy inequality [14], [9] for (1.1),

$$\int_\delta^T \int_{R^n} |\nabla (u_k)^m|^2 \, dx \, dt \leq C \quad (C \text{ constant}).$$

Hence,

$$\nabla u_{k_i} \to \nabla w \text{ weakly in } L^2_{\mathrm{loc}}[R^n \times (0, \infty)].$$

Since each u_{k_i} is a generalized solution, we deduce that also

$$w \text{ is a generalized solution.} \tag{4.6}$$

Define

$$L_0 = \lim_{t\to\infty} L(t). \tag{4.7}$$

We have

$$u_k(x, t) = k^n u(kx, k^{n/l} t)$$

and

$$u(x, t) \geq V_{L(t)}(r, t + \tau_t), \qquad \tau_t \leq Ct \; (t \geq 1).$$

Hence

$$u_k(x, t) \geq k^n V_{L(k^{n/l}t)}\big(kr, k^{n/l} t + \tau_{k,t}\big) \tag{4.8}$$

where

$$\tau_{k,t} \leq \frac{1}{k^{n/l}} \tau_{k^{n/l}t} \leq Ct.$$

For each t there is a subsequence $k_{i'}$ of k_i for which

$$\tau_{k_{i'},t} \to \tilde{\tau}_t, \qquad \tilde{\tau}_t \leq C.$$

Taking $k = k_{i'} \to \infty$ in (4.8) and using (4.5), (4.7), we get

$$w(x, t) \geq V_{L_0}(r, t + \tilde{\tau}_t). \tag{4.9}$$

LEMMA 4.1. *For any $t > 0$,*

$$w(x, t) = V_{L_0}(r, t + \tilde{\tau}_t). \tag{4.10}$$

PROOF. Suppose otherwise; then for some $t > 0$,

$$w(x, t) \not\equiv V_{L_0}(r, t + \tilde{\tau}_t) \quad \text{for } x \in R^n.$$

Proceeding as in Lemma 3.2 (recall that w is a solution of (1.1)) we find that for some $\eta > 0$ and for sufficiently small ε, $0 < \varepsilon < \eta$,

$$w(x, t + \eta) > V_{\hat{L}}(r, t + \tilde{\tau}_t + \eta - \varepsilon) \quad \text{on supp } V_{\hat{L}},$$

for some $\hat{L} > L_0$. Recalling (4.5) we deduce that

$$u_{k_i}(x, t + \eta) \geq V_{\hat{L}}(r, t + \tilde{\tau}_t + \eta - \varepsilon)$$

if k_i is sufficiently large. Hence

$$L\big(k_i^{n/l}(t + \eta)\big) \geq \hat{L}$$

and, consequently, also $L_0 \geq \hat{L}$, a contradiction.

LEMMA 4.2. $\tilde{\tau}_t = 0$.

PROOF. Since w is a generalized solution, the same is true of the function

$$(x, t) \to V_{L_0}(r, t + \tilde{\tau}_t).$$

Since also

$$(x, t) \to V_{L_0}(r, t + \tilde{\tau}_\delta) \qquad (\delta > 0)$$

is a solution, and both solutions agree on $t = \delta$, it follows that they agree for all $t \geqslant \delta$. Hence $\tilde{\tau}_t = \tilde{\tau}_\delta$ if $t \geqslant \delta$; thus $\tilde{\tau}_t = \text{const} = \tau^*$.

Next, by (2.10),

$$u_k(x, t) = k^n u(kx, k^{n/l}t) \geqslant k^n V_{L_1}(kr, k^{n/l}t + \tau_1)$$
$$= V_{L_1}(r, t + \tau_1/k^{n/l}).$$

Taking $x = 0$, $t = \delta$ we get

$$u_k(0, \delta) \geqslant V_{L_1}(0, \delta + \tau_1/k^{n/l}) \geqslant \beta^{2/(m-1)}/2\delta^l$$

if k is sufficiently large. On the other hand

$$u_{k_i}(0, \delta) \to w(0, \delta) = V_{L_0}(0, \delta + \tau^*).$$

Therefore

$$V_{L_0}(0, \delta + \tau^*) \geqslant \beta^{2/(m-1)}/2\delta^l.$$

Taking $\delta \to 0$ we deduce that τ^* must be equal to zero.

We have proved so far that

$$u_{k_i}(x, t) \to V_{L_0}(r, t) \tag{4.11}$$

uniformly in (x, t) in compact subsets of $R^n \times (0, \infty)$. Since the supports of the functions

$$x \to u_{k_i}(x, t)$$

are uniformly bounded, by (2.13), we conclude that

$$\int_{R^n} u_{k_i}(x, t)dx \to \int_{R^n} V_{L_0}(r, t)dx.$$

Observing that

$$\int_{R^n} u_k(x, t)dx = k^n \int_{R^n} u(kx, k^{n/l}t)dx = I,$$

the assertion (1.9) follows. Thus the limit in (4.11) is independent of the sequence k_i^* that we have started with (just before (4.5)). It follows that

$$u_k(x, t) \to V_{L_0}(r, t) \quad \text{as } k \to \infty, \tag{4.12}$$

where the convergence is uniform in (x, t) in compact subsets of $R^n \times (0, \infty)$; in view of (2.13), the convergence is in fact uniform in $(x, t) \in R^n \times [\delta, \infty)$, for any $\delta > 0$.

Taking $t = 1$ in (4.12) we deduce that, as $k \to \infty$,

$$k^n u(kx, k^{n/l}) - V_{L_0}(r, 1) \to 0 \quad \text{uniformly in } x \in R^n,$$

or

$$k^n\left[u(kx, k^{n/l}) - V_{L_0}(kr, k^{n/l})\right] \to 0$$

uniformly in $x \in R^n$. Replacing kx by x and setting $t = k^{n/l}$, the assertion (1.8) follows. This completes the proof of Theorem 1.1 in case u_0 has compact support.

5. Proof of Theorem 1.1. We shall now remove the restriction (2.11). The following estimate due to Benilan [3] and Veron [15] will be needed:

LEMMA 5.1. *There exists a positive constant C such that, for any nonnegative initial data u_0 in $L^1(R^n)$,*

$$\sup_{x \in R^n} u(x, t) \leqslant \frac{C}{t^l}\left[\int u_0(x)dx\right]^{2l/n}. \tag{5.1}$$

For any $N = 1, 2, \ldots$, let $u_0^N(x)$ be an initial data satisfying (1.3), such that

$$u_0^N(x) = u_0(x) \quad \text{if } |x| < N,$$
$$u_0^N(x) = 0 \quad \text{if } |x| > N + 1,$$
$$u_0^N(x) \leqslant u_0(x) \quad \text{if } N \leqslant x \leqslant N + 1.$$

Denote by $u^N(x, t)$ the solution of (1.1), (1.2) corresponding to u_0^N and set

$$u_k^N(x, t) = k^n u^N(kx, k^{n/l}t),$$

$$I_N = \int_{R^n} u_0^N(x)dx.$$

Then $I_N \to I$.

By Lemma 5.1,

$$u_k(x, t) \leqslant (C/t^l)I^{2l/n}. \tag{5.2}$$

Hence, by [5], [6], the u_k are equicontinuous in compact subsets of $R^n \times (0, \infty)$. It follows that for any given sequence $k_i^* \uparrow \infty$ there exists a subsequence k_i such that

$$u_{k_i}(x, t) \to w(x, t) \text{ uniformly in compact subsets of } R^n \times (0, \infty); \tag{5.3}$$

the function w may a priori depend on the sequence.

By what we have proved in §4, for any N,

$$u_k^N(x, t) \to V_{L_N}(r, t)$$

uniformly in compact subsets of $R^n \times (0, \infty)$, where

$$L_N = (I_N)^{m-1}.$$

Since $u \geqslant u^N$, we also have $u_k \geqslant u_k^N$, so that

$$w(x, t) \geqslant V_{L_N}(r, t).$$

Taking $N \to \infty$ we find that

$$w(x, t) \geqslant V_{L_0}(r, t) \tag{5.4}$$

where L_0 is defined by (1.9).

From (5.3) we obtain, using Fatou's lemma,

$$\int_{R^n} w(x, t)dx \leqslant \lim \int_{R^n} u_{k_i}(x, t)dx = I;$$

consequently,

$$\int_{R^n}\left[w(x, t) - V_{L_0}(r, t)\right]dx \leqslant I - L_0^{m-1} = 0.$$

Comparing with (5.4) it follows that

$$w(x, t) = V_{L_0}(r, t).$$

We deduce that the entire family u_k is convergent to the same limit function, namely V_{L_0}.

We have thus proved that

$$u_k(x, t) = k^n u(kx, k^{n/l}t) \to V_{L_0}(r, t)$$

uniformly in compact subsets of $R^n \times (0, \infty)$.

Choosing $t = 1$ and replacing kx by x, the assertion (1.8) follows.

Remark 1. The method of proof of Theorem 1.1 can be used to prove similar results for other equations. We illustrate this in the case of the heat equation

$$u_t = \Delta u \text{ in } R^n \times (0, \infty)$$

with the Cauchy data

$$u(x, 0) = u_0(x) \qquad (x \in R^n),$$

assuming that

$$u_0(x) \text{ is continuous}, \quad u_0(x) \not\equiv 0,$$

$$0 \leqslant u_0(x) \leqslant C_0 e^{-\alpha|x|^2} \quad (\text{for some } C_0 > 0, \alpha > 0).$$

We shall show that, as $t \to \infty$,

$$t^{n/2}\left[u(x, t) - \frac{L_0}{(4\pi t)^{n/2}} e^{-|x|^2/4t}\right] \to 0 \tag{5.5}$$

uniformly in x, $|x|^2 \leqslant Ct$ (for any $C > 0$), where

$$L_0 = \int_{R^n} u_0(x)dx.$$

(This result, for any $u_0 \in L^1(R^n)$, can of course be proved directly from the formula

$$u(x, t) = \frac{1}{(4\pi t)^{n/2}}\int_{R^n} e^{-|x-y|^2/4t}u_0(y)dy.\Big)$$

We define

$$V_L(r, t) = \frac{L}{(4\pi t)^{n/2}} e^{-r^2/(4t)}$$

and set

$$L(t) = \inf_{(L,\tau)\in\Sigma_t} L$$

where Σ_t consists of all pairs (L, τ) such that $L \geqslant 0, \tau \geqslant 0$,

$$u(x, t) \leqslant V_L(r, t + \tau) \qquad (x \in R^n).$$

Then $L(t)$ is strictly increasing (assuming (2.15)); we use here an analog of Lemma 2.1, namely, if

$$V_L(r, \tau + \varepsilon) = V_L(r, \tau)$$

then

$$r = r_\varepsilon \to (2n\tau)^{1/2} \quad \text{as } \varepsilon \to 0.$$

Representing $u(x, t)$ in terms of the fundamental solution and the initial data, we find that

$$u(x, t) \geqslant (c/t)e^{-\beta|x|^2} \quad \text{if } t \geqslant 1 \qquad (c > 0, \beta > 0).$$

We now define

$$u_k(x, t) = k^n u(kx, k^2 t)$$

and continue as in the preceding proof.

REMARK 2. For the equation (1.1) with $(n - 2)/n < m < 1$ the assertion (1.5) is still valid. Furthermore, P. Benilan (oral communication) proved a uniform Lipschitz continuity in any interval (δ, ∞), $\delta > 0$, with coefficient depending on C, where $\|u_0\|_{L_1} \leqslant C$. Defining $V_L(r, t)$ as in (1.6), but with c^2 replaced by $-c^2$, we see that the support of $x \to V_L(r, t)$ is all of R^n. The method of Remark 1 extends with minor changes to the present case, showing that (1.8) is valid.

The same procedure can be applied to nonlinear parabolic equations, such as

$$u_t = \alpha u_{xx} + \beta|u_{xx}| \qquad (\alpha > |\beta| > 0)$$

studied in **[8]**, **[2]**; similarity solutions are constructed in **[2]**.

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