THE ASYMPTOTIC BEHAVIOR OF GAS IN AN
n-DIMENSIONAL POROUS MEDIUM

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ABSTRACT. Consider the flow of gas in an n-dimensional porous medium with
initial density $u_0(x) > 0$. The density $u(x, t)$ then satisfies the nonlinear degenerate
parabolic equation $u_t = Au^m$ where $m > 1$ is a physical constant. Assuming that
$I = \int u_0(x)dx < \infty$ it is proved that $u(x, t)$ behaves asymptotically, as $t \to \infty$,
like the special (explicitly given) solution $V(|x|, t)$ which is invariant by similarity
transformations and which takes the initial values $\delta(x)I$ ($\delta(x)$ = the Dirac mea-
sure) in the distribution sense.

1. Statement of the main result. Consider the Cauchy problem for $u(x, t)$:

\begin{align*}
  u_t &= Au^m \quad (x \in R^n, t > 0), \quad (1.1) \\
  u(x, 0) &= u_0(x) \quad (x \in R^n). \quad (1.2)
\end{align*}

The function $u$ represents the density of a gas in a porous medium and $m$ is a
physical constant, $m > 1$. We assume that

\begin{align*}
  u_0(x) &\text{ is continuous, } u_0(x) > 0, \quad u_0(x) \equiv 0, \quad u_0(x) < M, \\
  u_0 &\in L^1(R^n) \cap L^2(R^n) \quad (M \text{ constant}), \quad (1.3)
\end{align*}

and set

\[ I = \int_{R^n} u_0(x)dx. \quad (1.4) \]

By a weak solution of (1.1), (1.2) we mean a function $u$ satisfying, for any $T > 0$,

\[ \int_0^T \int_{R^n} \left[ (u(x, t))^2 + |\nabla u|^2 \right] dx \, dt < \infty \]

and

\[ \int_0^T \int_{R^n} \left( u \frac{\partial f}{\partial t} - \nabla u \cdot \nabla f \right) dx \, dt + \int_{R^n} u_0(x)f(x)dx = 0 \]

for any continuously differentiable function $f$ with compact support in $R^n \times [0, T)$.

It is well known [14], [11] that given $u_0$ satisfying (1.3), (1.4) there exists a unique
generalized solution $u$ of (1.1), (1.2) and

\[ \int_{R^n} u(x, t)dx = I \quad \text{for all } t > 0; \quad (1.5) \]
a very general uniqueness theorem was recently proved by Brezis and Crandall [4]. By a recent result of Caffarelli and Friedman [5], [6] $u(x, t)$ is Hölder continuous in $(x, t)$ uniformly in any strip $t > \delta$ ($\delta > 0$).

In this paper we are interested in the asymptotic behavior of $u(x, t)$ as $t \to \infty$. To state the main result we first introduce the similarity solution $V_L$ of Barenblatt and Prattle [13]. Let

$$G(s) = \left[(\beta^2 - c^2s^2)^+\right]^{1/(m-1)}$$

where

$$c^2 = \frac{l(m-1)}{2mn}, \quad l = \frac{1}{m-1 + 2/n}$$

and $\beta$ is a positive constant such that $\int_{R^n} G(|x|)dx = 1$. Then, for any $L > 0$,

$$V_L(r, t) = L^{1/(m-1)} \frac{1}{L^l} G\left(\frac{r}{L^{1/n}}\right), \quad r = |x|,$$  \hfill (1.6)

is a solution of

$$\frac{\partial}{\partial t} V_L = \Delta(V_L)^m$$

satisfying $V_L(r, 0) = \delta(x)L^{1/(m-1)}$ ($\delta(x)$ is the Dirac measure) in the sense of distributions. As easily verified,

$$\int_{R^n} V_L(r, t)dx = L^{1/(m-1)}. \hfill (1.7)$$

We can now state the main result of the paper.

**Theorem 1.1.** As $t \to \infty$,

$$t^\prime |u(x, t) - V_{L_0}(r, t)| \to 0 \hfill (1.8)$$

uniformly with respect to $x$ in any set $|x| < Ct^{l/n}$ ($C > 0$), where

$$L_0 = L^{m-1}. \hfill (1.9)$$

For $n = 1$ this theorem was proved by Kamin (Kamenomostkaya) [9], [10]. The proof for $n > 1$ given in this paper employs a different approach than in [9], [10] and exploits the continuity of the solution $u$; both methods use similarity transformations.

For $n = 1$ Peletier [12] and Van Duyan and Peletier [7] studied the asymptotic behavior in a half-plane $x > 0$ and in the whole space when $u_0(x) \sim A$ as $x \to \infty$, $u_0(x) \sim B$ as $x \to -\infty$; $A > 0$, $B > 0$. For a bounded domain $\Omega$ in $R^n$, Aronson and Peletier [1] have recently obtained the asymptotic behavior of the solution in $\{(x, t) \in \Omega \times (0, \infty)\}$; they assume that $u(x, t) = 0$ if $x \in \partial\Omega$. Both the result and method are different from those obtained in the present work.

To facilitate the reading of the proof, which takes the rest of the paper, we have broken it into several parts. In §2 we derive some properties of the functions $V_L$. In §3 we study the function $L(t)$, where $L(t)$ is defined as the largest value of $L$ such that, for some $\tau > 0$,

$$V_L(r, t + \tau) < u(x, t) \quad \text{for all } x \in R^n.$$
In §4 we prove Theorem 1.1 in case $u_0(x)$ has compact support and, finally, in §5 we give the proof in the general case.

2. Auxiliary lemmas on $V_L$. Notice that

$$\text{support of the function } r \rightarrow V_L(r, t) \text{ is given by } r < \frac{L^{1/n} L^{1/n}}{c/\beta}. \quad (2.1)$$

**Lemma 2.1.** For fixed $L > 0$, $t > 0$, if $\varepsilon > 0$ is sufficiently small then

$$V_L(r, t - \varepsilon) > V_L(r, t) \quad \text{for } 0 < r < \theta \frac{L^{1/n} L^{1/n}}{c/\beta},$$

$$V_L(r, t - \varepsilon) < V_L(r, t) \quad \text{for } \theta \frac{L^{1/n} L^{1/n}}{c/\beta} < r < \frac{L^{1/n} L^{1/n}}{c/\beta}$$

where

$$\theta = \theta_\varepsilon \rightarrow \theta_0 \text{ as } \varepsilon \rightarrow 0; \quad \theta_0 = \sqrt{(m - 1)l}. \quad (2.2)$$

Observe that

$$0 < \theta_0 < 1. \quad (2.3)$$

**Proof.** We solve, for small $\varepsilon$, the equation in $r$:

$$V_L(r, t - \varepsilon) = V_L(r, t). \quad (2.4)$$

Thus

$$\left(\frac{t - \varepsilon}{t}\right)^{(m - 1)} G^{-1}\left(\frac{r}{L^{1/n} L^{1/n}}\right) = G^{-1}\left(\frac{r}{L^{1/n} (t - \varepsilon)^{1/n}}\right). \quad (2.5)$$

By the mean value theorem the right-hand side is equal to

$$G^{-1}\left(\frac{r}{L^{1/n} L^{1/n}}\right) + \frac{r}{L^{1/n}} \left(\frac{1}{(t - \varepsilon)^{1/n}} - \frac{1}{t^{1/n}}\right) \left(G^{-1}\left(\frac{r}{L^{1/n} L^{1/n}}\right)\right)'$$

where $t - \varepsilon < \tilde{t} < t$; also

$$(G^{-1})'(s) = -2c^2s,$$

$$\left(\frac{t - \varepsilon}{t}\right)^{(m - 1)} \sim 1 - \frac{l(m - 1)}{t} \varepsilon,$$

$$\frac{1}{(t - \varepsilon)^{1/n}} - \frac{1}{t^{1/n}} = -\frac{1}{t^{1/n}} \left(1 - \left(\frac{t}{t - \varepsilon}\right)^{1/n}\right) \sim \frac{1}{t^{1/n}} \frac{l/n}{t} \varepsilon.$$  

Using these facts in (2.5), we get

$$\frac{l(m - 1)}{t} c \varepsilon G^{-1}\left(\frac{r}{L^{1/n} L^{1/n}}\right) \sim 2 - \frac{c^2 r^2}{L^{2l/n} L^{2l/n}} \frac{l/n}{t} \varepsilon,$$

so that

$$n(m - 1) \left(\beta^2 - \frac{c^2 r^2}{L^{2l/n} L^{2l/n}}\right) \sim \frac{2c^2 r^2}{L^{2l/n} L^{2l/n}}.$$
It follows that the solution \( r \) of (2.4) is given by

\[
\theta \frac{\beta}{c} L^{\frac{1}{l/n}}
\]

where

\[
\theta = \theta \rightarrow \left( \frac{n(m-1)}{n(m-1)+2} \right)^{1/2} = \theta_0 \quad \text{as} \quad \epsilon \to 0,
\]

and the proof of the lemma follows.

**Lemma 2.2.** For any \( x^0 \in \mathbb{R}^n \) and \( t \) real,

\[
t^l \left| V_L (|x - x^0|, t + \tau) - V_L (|x|, t) \right| \to 0 \quad \text{as} \quad t \to \infty,
\]

uniformly with respect to \( x \in \mathbb{R}^n \).

The proof is immediate from (1.6).

Choose a point \( y^0 \in \mathbb{R}^n \) such that \( u_0 (y^0) > 0 \); then, for some \( \delta_0 > 0, \eta_0 > 0, \)

\[
u_0 (x) > \delta_0 \quad \text{if} \quad |x - y^0| < \eta_0.
\]

But then we can find \( L_1 > 0, \tau_1 > 0 \) such that

\[
u_0 (x) > V_{L_1} (|x - y^0|, \tau_1) \quad \text{for all} \quad x \in \mathbb{R}^n.
\]

Indeed, we simply have to choose

\[
L_1 \tau_1 = c_1, \quad c_1 \text{ sufficiently small},
\]

and then choose \( L_1 \) sufficiently small (depending on \( c_1 \)).

If we prove that

\[
t^l \left| u(x, t) - V_{L_1} (|x - y^0|, t) \right| \to 0 \quad \text{as} \quad t \to \infty,
\]

uniformly in \( x \in \mathbb{R}^n \) in any set \(|x| < Ct^{l/n}\), then, in view of Lemma 2.2, the assertion of Theorem 1.1 would follow.

For simplicity of notation we fix the origin at the point \( y^0 \). Thus (2.7) becomes

\[
u_0 (x) > V_{L_1} (r, \tau_1), \quad r = |x|,
\]

and the assertion (2.8) reduces to the assertion (1.8). From (2.9) it follows, by comparison, that

\[
u(x, t) > V_{L_1} (r, t + \tau_1) \quad (x \in \mathbb{R}^n, t > 0).
\]

From now on, until the end of §4, we impose the restriction:

\[
u_0 \text{ has compact support}.
\]

We can then find \( L_2 > 0, \tau_2 > 0 \) such that

\[
u_0 (x) < V_{L_2} (r, \tau_2) \quad \text{for all} \quad x \in \mathbb{R}^n.
\]

Indeed, we simply take first \( L_2 \tau_2 = c_2 \) where \( c_2 \) is sufficiently large and then choose \( L_2 \) to be sufficiently large, depending on \( c_2 \).

From (2.12), we deduce that

\[
u(x, t) < V_{L_2} (r, t + \tau_2) \quad (x \in \mathbb{R}^n, t > 0).
\]
Suppose for some \( t_0 > 0 \), \( t \) real, \( t_0 + t > 0 \),
\[
    u(x, t_0) \equiv V_L(r, t_0 + t).
\]
(2.14)

Then, by uniqueness,
\[
    u(x, t) = V_L(r, t + t) \quad \text{if} \quad t > t_0.
\]
Recalling Lemma 2.2, the assertion of Theorem 1.1 then follows. Thus, in order to prove Theorem 1.1 (under the condition (2.11)), we may assume, without loss of generality, that
\[
    \text{for any } L > 0, \quad t \text{ real}, \quad t > 0, \quad t + \tau > 0,
    u(x, t) \geq V_L(r, t + t) \quad (x \in \mathbb{R}^n).
\]
(2.15)

For any fixed \( t > 0 \), denote by \( \Sigma_t \) the set of all points \( (L, \tau) \) such that \( L > 0 \), \( \tau > 0 \), and
\[
    u(x, t) > V_L(r, t + t),
\]
(2.16)
and set
\[
    L(t) = \sup_{(L, \tau) \in \Sigma_t} L.
\]
(2.17)

3. Properties of \( L(t) \).

**Lemma 3.1.** There exists a point \( (L^*, \tau^*) \in \Sigma_t \), such that \( L(t) = L^* \) and
\[
    \tau^* \leq C(t + 1);
\]
(3.1)

\( C \) is a constant independent of \( t \).

**Proof.** The inequality (2.16) implies
\[
    \int_{\mathbb{R}^n} V_L(r, t + t)dx \leq \int_{\mathbb{R}^n} u(x, t)dx = I
\]
where (1.5) was used. Recalling (1.7) we conclude that \( L \leq I^{m-1} \). From (2.10) we also deduce that \( L \geq L_1 \). Thus, in seeking to find \( \sup L \) in (2.17) we may restrict the \( L \) to lie in the interval
\[
    L_1 \leq L \leq I^{m-1}.
\]
(3.2)

In view of (2.16) and (3.2),
\[
    \text{support of } x \rightarrow u(x, t) \text{ contains the set } r < C_1(t + \tau)^{1/n}
\]
and in view of (2.13),
\[
    \text{support of } x \rightarrow u(x, t) \text{ is contained in the set } r < C_2(t + \tau_2)^{1/n};
\]
both constants \( C_1, C_2 \) are positive and independent of \( t \). It follows that
\[
    t + \tau \leq (C_2/C_1)^{n/(n+2)}(t + \tau_2), \quad \text{i.e.,} \quad \tau \leq C(t + 1) \quad (C > 0).
\]
Thus, in computing \( \sup L \) in (2.17) we may restrict ourselves to \( (L, \tau) \in \hat{\Sigma}_t \), where
\[
    \hat{\Sigma}_t = \Sigma_t \cap \{ L_1 < L < I^{m-1}, 0 < \tau < C(t + 1) \}.
\]
Since \( \hat{\Sigma}_t \) is a compact set, it follows that
\[
    \sup_{\hat{\Sigma}_t} L = \max_{\hat{\Sigma}_t} L = L^*.
\]
with \((L^*, \tau^*) \in \hat{\Sigma}\), rendering the maximum; this completes the proof.

**Lemma 3.2.** \(L(t)\) is monotone decreasing in \(t\), and there exists a sequence \(t_j \uparrow \infty\) such that

\[
L(t_j) < L(t_{j+1}) \quad \text{for all } j. \tag{3.3}
\]

**Proof.** Since \((2.13)\) for a fixed \(t\) implies the same inequality for \(t\) replaced by \(t'\), \(t' > t\), it follows that \(L(t') < L(t)\). To prove \((3.3)\), we fix point \(t = t_0\) and construct a point \(t_1 > t_0\) such that

\[
L(t_0) < L(t_1); \tag{3.4}
\]

this would establish \((3.3)\).

We have

\[
u(x, t_0) > V_L(r, t_0 + t) \quad \text{where } L = L(t_0). \tag{3.5}
\]

We claim: there exists a \(\xi > 0\) such that

\[
u(x, t_0 + \xi) \equiv V_L(r, t_0 + \xi + t) \quad \text{for } r < (\beta/c)L^{1/n}(t_0 + \tau + \xi)^{1/n}. \tag{3.6}
\]

Indeed, otherwise

\[
u(x, t) \equiv V_L(r, t + \tau) \quad \text{whenever } t > t_0, \ V_L > 0. \tag{3.7}
\]

Recalling \((2.15)\) we conclude that, for some \(x^0 \in \mathbb{R}^n, \delta > 0,\)

\[
u(x, t_0) > 0, \quad V_L(r, t_0 + \tau) = 0 \quad \text{if } |x - x^0| < \delta.
\]

But then

\[
u(x, t_0) > V_{L_0^*}(|x - x^0|, t_0 + \tau_*) \tag{3.8}
\]

for some \(L_0^* > 0, \tau_0 > 0\).

Choose now a point \((\delta, \hat{r})\) such that

\[
|\delta| = (\beta/c)L^{1/n}(\hat{r} + \tau)^{1/n}
\]

(i.e., \(\delta\) lies on the boundary of the support \(x \rightarrow V_L(r, \hat{r} + \tau)\)) and

\[
|\delta - x^0| < (\beta/c)L^{1/n}(\hat{r} + \tau_*)^{1/n}
\]

(i.e., \(V_{L_0^*}(|\delta - x^0|, \hat{r} + \tau_*) > 0\)). By \((3.8)\) and comparison we then deduce that \(u(\delta, \hat{r}) > 0\), whereas from \((3.7)\) we get that \(u(\delta, \hat{r}) = 0\), a contradiction.

Having proved \((3.6)\) we can now write

\[
u(x, t_0 + \xi) \equiv V_L(r, t_0 + \xi + \tau) \quad \text{if } r < (\tilde{\beta}\beta/c)L^{1/n}(t_0 + \tau + \xi)^{1/n} \equiv r_1 \tag{3.9}
\]

if \(0 < \tilde{\beta} < 1, 1 - \beta\) sufficiently small.

We shall compare \(u(x, t + \xi)\) with \(V_L(r, t + \tau + \xi)\) for \(r < r_1, t > t_0\). Notice that

\[
V_L(r, t + \tau) > 0 \quad \text{if } r < r_1;
\]

hence also \(u\) is positive. We therefore have, in the classical sense,

\[
u_t = \Delta u^{m}, \quad (V_L)_t = \Delta (V_L)^m.
\]

It is easily seen that the function \(w = u - V_L\) then satisfies in the cylinder, \(r < r_1, t > t_0,\)

\[
w_t = a \Delta w + \sum b_i w \chi + cw \tag{3.10}
\]
for some smooth coefficients \( a, b, c; a > 0 \). Since \( w > 0 \) in the cylinder, the strong maximum principle implies that either

\[ w(x, t) > 0 \quad \text{if} \quad |x| < r_1, t > t_0, \]

or else \( w \equiv 0 \); the second possibility is ruled out by (3.9). Thus, for any \( 0 < \theta_1 < 1 \) and for any \( \eta > \xi (\eta - \xi \text{ small enough}) \)

\[ u(x, t_0 + \eta) > V_L(r, t_0 + \tau + \eta) \quad \text{if} \quad r < \theta_1 \cdot (\beta/c)L^{1/n}(t_0 + \tau + \eta)^{1/n}. \]  
\[ (3.11) \]

By Lemma 2.1,

\[ V_L(r, t_0 + \tau + \eta) > V_L(r, t_0 + \tau + \eta - \varepsilon) \quad \text{if} \quad r > \theta_0, 0 < \theta_0 < 1. \]

On the other hand,

\[ |V_L(r, t_0 + \tau + \eta) - V_L(r, t_0 + \tau + \eta - \varepsilon)| = o(1) \to 0 \quad \text{if} \quad \varepsilon \to 0, \]

uniformly in \( r \). Taking \( \theta_1 > \theta_0 \) and choosing \( \varepsilon \) sufficiently small (depending on \( \xi, \eta \)) we conclude, using (3.11), that

\[ u(x, t_0 + \eta) > V_L(x, t_0 + \tau + \eta - \varepsilon) \]
in a region \( r < R \) containing the support of the function

\[ r \to V_L(r, t_0 + \tau + \eta - \varepsilon). \]

But then, if \( L' > L \) and \( L' - L \) is sufficiently small,

\[ u(x, t_0 + \eta) > V_L(r, t_0 + \tau + \eta - \varepsilon). \]

It follows that

\[ L(t_0 + \eta) \geq L' > L = L(t_0). \]

Setting \( t_1 = t_0 + \eta \), (3.4) follows.

4. Proof of Theorem 1.1 for \( u_0 \) having compact support. Consider the one-parameter family of functions

\[ u_k(x, t) = k^n u(kx, k^{n/\ell}t), \quad k > 1. \]  
\[ (4.1) \]

From (2.13) we deduce that

\[ u_k(x, t) < k^n V_{L_x}(kx, k^{n/\ell}t + \tau_2) = V_{L_x}(r, t + \tau_2/k^{n/\ell}). \]  
\[ (4.2) \]

Therefore, for any \( \delta > 0, \)

\[ u_k(x, t) < C_\delta \quad (x \in \mathbb{R}^n, t > \delta, k > 1) \]  
\[ (4.3) \]

where \( C_\delta \) is a constant depending on \( \delta \).

Applying the continuity result of Caffarelli and Friedman [5], [6] we deduce that the \( u_k(k, t) \) are equicontinuous in \( (x, t) \in \mathbb{R}^n \times [\delta, \infty) \), for \( k > 1. \)  
\[ (4.4) \]

Hence from any sequence \( k^* \uparrow \infty \) we can extract a subsequence \( k_\ell \) such that, for any \( \delta > 0, \)

\[ u_k(x, t) \to w(x, t) \text{ uniformly in} \ (x, t) \text{ in compact subsets of} \ \mathbb{R}^n \times [\delta, \infty); \]  
\[ (4.5) \]

\( w \), the limit function, may a priori depend on the sequence.
By the energy inequality [14], [9] for (1.1),
\[ \int_\delta^T \int_{R^n} |\nabla (u_k)|^2 \, dx \, dt \leq C \quad (C \text{ constant}). \]
Hence,
\[ \nabla u_k \rightharpoonup \nabla w \text{ weakly in } L^2_{\text{loc}}(R^n \times (0, \infty)). \]
Since each $u_k$ is a generalized solution, we deduce that also $w$ is a generalized solution. (4.6)
Define
\[ L_0 = \lim_{t \to \infty} L(t). \quad (4.7) \]
We have
\[ u_k(x, t) = k^n u(kx, k^{n/4}t) \]
and
\[ u(x, t) \geq V_{L(0)}(r, t + \tau), \quad \tau < Ct \quad (t > 1). \]
Hence
\[ u_k(x, t) \geq k^n V_{L(k^{n/4})}(kr, k^{n/4}t + \tau, \tau) \quad (4.8) \]
where
\[ \tau_{k, t} \leq \frac{1}{k^{n/4}t} \tau < Ct. \]
For each $t$ there is a subsequence $k_{i, t}$ of $k_i$ for which
\[ \tau_{k_{i, t}} \to \tilde{\tau}, \quad \tilde{\tau} < C. \]
Taking $k = k_{i, t} \to \infty$ in (4.8) and using (4.5), (4.7), we get
\[ w(x, t) > V_{L_0}(r, t + \tilde{\tau}). \quad (4.9) \]
**Lemma 4.1.** For any $t > 0$,
\[ w(x, t) = V_{L_0}(r, t + \tilde{\tau}). \quad (4.10) \]
**Proof.** Suppose otherwise; then for some $t > 0$,
\[ w(x, t) \not\equiv V_{L_0}(r, t + \tilde{\tau}) \quad \text{for } x \in R^n. \]
Proceeding as in Lemma 3.2 (recall that $w$ is a solution of (1.1)) we find that for some $\eta > 0$ and for sufficiently small $\epsilon, 0 < \epsilon < \eta$,
\[ w(x, t + \eta) > V_{\hat{L}}(r, t + \tilde{\tau} + \eta - \epsilon) \quad \text{on supp } V_{\hat{L}}, \]
for some $\hat{L} > L_0$. Recalling (4.5) we deduce that
\[ u_k(x, t + \eta) > V_{\hat{L}}(r, t + \tilde{\tau} + \eta - \epsilon) \]
if $k_i$ is sufficiently large. Hence
\[ L(k_i^{n/4}(t + \eta)) \geq \hat{L} \]
and, consequently, also $L_0 > \hat{L}$, a contradiction.
Lemma 4.2. \( \tilde{\tau}_t = 0 \).

Proof. Since \( w \) is a generalized solution, the same is true of the function
\[
(x, t) \to V_{L_0}(r, t + \tilde{\tau}_t).
\]

Since also
\[
(x, t) \to V_{L_0}(r, t + \tau_\delta) \quad (\delta > 0)
\]
is a solution, and both solutions agree on \( t = \delta \), it follows that they agree for all \( t > \delta \). Hence \( \tilde{\tau}_t = \tilde{\tau}_\delta \) if \( t > \delta \); thus \( \tilde{\tau}_t = \text{const} = \tau^* \).

Next, by (2.10),
\[
\begin{align*}
\frac{\partial u_k(x, t)}{\partial x} &= k^n u(kx, k^n/t) \\
&= k^n V_{L_1}(kr, k^n/t + \tau_1) \\
&= V_{L_1}(r, t + \tau_1/k^n).
\end{align*}
\]

Taking \( x = 0, t = \delta \) we get
\[
\frac{\partial u_k(0, \delta)}{\partial x} > V_{L_1}(0, \delta + \tau_1/k^n) > \beta^{2/(m-1)}/2\delta
\]
if \( k \) is sufficiently large. On the other hand
\[
u_k(0, \delta) \to w(0, \delta) = V_{L_0}(0, \delta + \tau^*).
\]

Therefore
\[
V_{L_0}(0, \delta + \tau^*) > \beta^{2/(m-1)}/2\delta.
\]

Taking \( \delta \to 0 \) we deduce that \( \tau^* \) must be equal to zero.

We have proved so far that
\[
u_k(x, t) \to V_{L_0}(r, t) \quad (4.11)
\]
uniformly in \( (x, t) \) in compact subsets of \( \mathbb{R}^n \times (0, \infty) \). Since the supports of the functions
\[
x \to u_k(x, t)
\]
are uniformly bounded, by (2.13), we conclude that
\[
\int_{\mathbb{R}^n} u_k(x, t) \, dx \to \int_{\mathbb{R}^n} V_{L_0}(r, t) \, dx.
\]

Observing that
\[
\int_{\mathbb{R}^n} u_k(x, t) \, dx = k^n \int_{\mathbb{R}^n} u(kx, k^n/t) \, dx = I,
\]
the assertion (1.9) follows. Thus the limit in (4.11) is independent of the sequence \( k^*_n \) that we have started with (just before (4.5)). It follows that
\[
u_k(x, t) \to V_{L_0}(r, t) \quad \text{as} \; k \to \infty, \quad (4.12)
\]
where the convergence is uniform in \( (x, t) \) in compact subsets of \( \mathbb{R}^n \times (0, \infty) \); in view of (2.13), the convergence is in fact uniform in \( (x, t) \in \mathbb{R}^n \times [\delta, \infty) \), for any \( \delta > 0 \).

Taking \( t = 1 \) in (4.12) we deduce that, as \( k \to \infty \),
\[
k^n u(kx, k^n) - V_{L_0}(r, 1) \to 0 \quad \text{uniformly in} \; x \in \mathbb{R}^n,
\]
or

\[ k^n \left[ u(kx, k^{n/2}) - V_{L_0}(kr, k^{n/2}) \right] \to 0 \]

uniformly in \( x \in \mathbb{R}^n \). Replacing \( kx \) by \( x \) and setting \( t = k^{n/2} \), the assertion (1.8) follows. This completes the proof of Theorem 1.1 in case \( u_0 \) has compact support.

5. Proof of Theorem 1.1. We shall now remove the restriction (2.11). The following estimate due to Benilan [3] and Veron [15] will be needed:

**Lemma 5.1.** There exists a positive constant \( C \) such that, for any nonnegative initial data \( u_0 \) in \( L^1(\mathbb{R}^n) \),

\[ \sup_{x \in \mathbb{R}^n} u(x, t) \leq \frac{C}{t^{1/n}} \left[ \int_{\mathbb{R}^n} u_0(x) \, dx \right]^{2/n}. \]  

(5.1)

For any \( N = 1, 2, \ldots \), let \( u^N_0(x) \) be an initial data satisfying (1.3), such that

\[ u^N_0(x) = u_0(x) \quad \text{if } |x| < N, \]

\[ u^N_0(x) = 0 \quad \text{if } |x| > N + 1, \]

\[ u^N_0(x) < u_0(x) \quad \text{if } N < x < N + 1. \]

Denote by \( u^N(x, t) \) the solution of (1.1), (1.2) corresponding to \( u^N_0 \) and set

\[ u^N_k(x, t) = k^n u^N(kx, k^{n/2}t), \]

\[ I_N = \int_{\mathbb{R}^n} u^N_0(x) \, dx. \]

Then \( I_N \to I \).

By Lemma 5.1,

\[ u_k(x, t) \leq (C/t^{1/n}) I^{2/n}. \]  

(5.2)

Hence, by [5], [6], the \( u_k \) are equicontinuous in compact subsets of \( \mathbb{R}^n \times (0, \infty) \). It follows that for any given sequence \( k^* \uparrow \infty \) there exists a subsequence \( k_i \) such that

\[ u_{k_i}(x, t) \to w(x, t) \quad \text{uniformly in compact subsets of } \mathbb{R}^n \times (0, \infty); \]  

(5.3)

the function \( w \) may a priori depend on the sequence.

By what we have proved in §4, for any \( N \),

\[ u^N_k(x, t) \to V_{L_N}(r, t) \]

uniformly in compact subsets of \( \mathbb{R}^n \times (0, \infty) \), where

\[ L_N = (I^N)^{m-1}. \]

Since \( u \geq u^N \), we also have \( u_k \geq u^N_k \), so that

\[ w(x, t) \geq V_{L_N}(r, t). \]

Taking \( N \to \infty \) we find that

\[ w(x, t) \geq V_{L_0}(r, t) \]  

(5.4)

where \( L_0 \) is defined by (1.9).
From (5.3) we obtain, using Fatou’s lemma,
\[
\int_{\mathbb{R}^n} w(x, t) \, dx < \lim_{k \to \infty} \int_{\mathbb{R}^n} u_k(x, t) \, dx = I;
\]
consequently,
\[
\int_{\mathbb{R}^n} [w(x, t) - V_{L_0}(r, t)] \, dx < I - L_0^{m-1} = 0.
\]
Comparing with (5.4) it follows that
\[
w(x, t) = V_{L_0}(r, t).
\]
We deduce that the entire family \(u_k\) is convergent to the same limit function, namely \(V_{L_0}\).

We have thus proved that
\[
u_k(x, t) = k^n u(kx, k^{n/2}t) \to V_{L_0}(r, t)
\]
uniformly in compact subsets of \(\mathbb{R}^n \times (0, \infty)\).

Choosing \(t = 1\) and replacing \(kx\) by \(x\), the assertion (1.8) follows.

Remark 1. The method of proof of Theorem 1.1 can be used to prove similar results for other equations. We illustrate this in the case of the heat equation
\[
u_t = \Delta u \text{ in } \mathbb{R}^n \times (0, \infty)
\]
with the Cauchy data
\[
u(x, 0) = \nu_0(x) \quad (x \in \mathbb{R}^n),
\]
assuming that
\[
\nu_0(x) \text{ is continuous, } \nu_0(x) \equiv 0,
\]
\[
0 < \nu_0(x) < C_0 e^{-\alpha |x|^2} \quad \text{(for some } C_0 > 0, \alpha > 0).
\]
We shall show that, as \(t \to \infty\),
\[
\frac{t^{n/2}}{(4\pi t)^{n/2}} \left[ \nu(x, t) - \frac{L_0}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \right] \to 0 \quad \text{(5.5)}
\]
uniformly in \(x, |x|^2 < Ct\) (for any \(C > 0\)), where
\[
L_0 = \int_{\mathbb{R}^n} \nu_0(x) \, dx.
\]
(This result, for any \(\nu_0 \in L^1(\mathbb{R}^n)\), can of course be proved directly from the formula
\[
\nu(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} \nu_0(y) \, dy.
\]
We define
\[
V_L(r, t) = \frac{L}{(4\pi t)^{n/2}} e^{-r^2/(4t)}
\]
and set
\[
L(t) = \inf_{(L, r) \in \Sigma} L(t).
\]
where $\Sigma_t$ consists of all pairs $(L, \tau)$ such that $L > 0$, $\tau > 0$,

$$u(x, t) < V_L(r, t + \tau) \quad (x \in \mathbb{R}^n).$$

Then $L(t)$ is strictly increasing (assuming (2.15)); we use here an analog of Lemma 2.1, namely, if

$$V_L(r, \tau + \epsilon) = V_L(r, \tau)$$

then

$$r = r_0 \to (2\pi\tau)^{1/2} \quad \text{as } \epsilon \to 0.$$

Representing $u(x, t)$ in terms of the fundamental solution and the initial data, we find that

$$u(x, t) > \left(\frac{c}{t}\right)e^{-\beta|x|^2} \quad \text{if } t > 1 \quad (c > 0, \beta > 0).$$

We now define

$$u_k(x, t) = k^n u(kx, k^2 t)$$

and continue as in the preceding proof.

**Remark 2.** For the equation (1.1) with $(n - 2)/n < m < 1$ the assertion (1.5) is still valid. Furthermore, P. Benilan (oral communication) proved a uniform Lipschitz continuity in any interval $(\delta, \infty)$, $\delta > 0$, with coefficient depending on $C$, where $||u_0||_{L_1} < C$. Defining $V_L(r, t)$ as in (1.6), but with $c^2$ replaced by $-c^2$, we see that the support of $x \to V_L(r, t)$ is all of $\mathbb{R}^n$. The method of Remark 1 extends with minor changes to the present case, showing that (1.8) is valid.

The same procedure can be applied to nonlinear parabolic equations, such as

$$u_t = \alpha u_{xx} + \beta|u_{xx}| \quad (\alpha > |\beta| > 0)$$

studied in [8], [2]; similarity solutions are constructed in [2].

**References**

15. L. Veron, *Coercivité et propriétés regularisantes des semi-groupes non linéaires dans les espaces de Banach* (to appear).

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