

## THE HEWITT REALCOMPACTIFICATION OF PRODUCTS

BY

HARUTO OHTA

**ABSTRACT.** For a completely regular Hausdorff space  $X$ ,  $\nu X$  denotes the Hewitt realcompactification of  $X$ . Given a topological property  $\mathcal{P}$  of spaces, our interest is in characterizing the class  $\mathcal{R}(\mathcal{P})$  of all spaces  $X$  such that  $\nu(X \times Y) = \nu X \times \nu Y$  holds for each  $\mathcal{P}$ -space  $Y$ . In the present paper, we obtain such characterizations in the case that  $\mathcal{P}$  is locally compact and in the case that  $\mathcal{P}$  is metrizable.

**Introduction.** All spaces considered in this paper are assumed to be completely regular Hausdorff and all maps are continuous. The Hewitt realcompactification  $\nu X$  of a space  $X$  is the unique realcompactification of  $X$  to which each real-valued continuous function on  $X$  admits a continuous extension. For details of Hewitt realcompactifications, the reader is referred to [8]. An important problem in the theory concerns when the relation  $\nu(X \times Y) = \nu X \times \nu Y$  is valid. Following [23], [30], we denote by  $\mathcal{R}$  (resp.  $\mathcal{R}(\mathcal{P})$ ) the class of all spaces  $X$  such that  $\nu(X \times Y) = \nu X \times \nu Y$  holds for each space  $Y$  (resp. each  $\mathcal{P}$ -space  $Y$ ), where  $\mathcal{P}$  is a given property of spaces. It is known that: (Comfort [4], [5]) a locally compact, realcompact space of nonmeasurable cardinal belongs to  $\mathcal{R}$ ; (Hušek [12], [14] and McArthur [23]) every member of  $\mathcal{R}$  is realcompact; (Hušek [13], [14]) every member of  $\mathcal{R}$  is of nonmeasurable cardinal; [28] every member of  $\mathcal{R}$  is locally compact. These facts characterize  $\mathcal{R}$  as precisely the class of locally compact, realcompact spaces of nonmeasurable cardinals. Further, in [30], the author has tried to characterize  $\mathcal{R}(\mathcal{P})$  for various properties  $\mathcal{P}$  of spaces, and has proved that  $\mathcal{R} = \mathcal{R}(\text{metacompact}) = \mathcal{R}(\text{subparacompact})$ . It is the purpose of this paper to continue our study along this line, in particular, the following results are established:

(A) Both  $\mathcal{R}(\text{locally compact})$  and  $\mathcal{R}(\text{Moore})$  coincide with the class of all spaces of nonmeasurable cardinals whose Hewitt realcompactifications are locally compact.

(B) The class  $\mathcal{R}(\text{metrizable})$  consists precisely of all weak  $\text{cb}^*$ -spaces, in the sense of Isiwata [20], of nonmeasurable cardinals.

In §1, we present a technical theorem which is useful in finding a pair  $X, Y$  of spaces for which  $\nu(X \times Y) = \nu X \times \nu Y$  fails. Our result (A) is then proved. We also give a positive answer to the following question of Hušek [13, p. 326]: Do there

---

Received by the editors April 12, 1978 and, in revised form, October 24, 1978.

*AMS (MOS) subject classifications* (1970). Primary 54D60, 54B10; Secondary 54D45, 54E35.

*Key words and phrases.* Hewitt realcompactification, product space, locally compact space, metrizable space, weak  $\text{cb}^*$ -space, pseudo- $\text{m}$ -compact space, quotient map, perfect map,  $C$ -embedding, absolute, nonmeasurable cardinal,  $D(\text{m})$ -expandable family.

exist minimal cardinals  $m, n$  for which  $|X| = m, |Y| = n$  and  $v(X \times Y) \neq vX \times vY$ . In §2, we prove the analogue for  $vX$  of the corollary to Glicksberg's theorem [9, Theorem 1]: For onto maps  $f_i: X_i \rightarrow Y_i$  ( $i = 1, 2$ ),  $\beta(Y_1 \times Y_2) = \beta Y_1 \times \beta Y_2$  holds whenever  $\beta(X_1 \times X_2) = \beta X_1 \times \beta X_2$ , where  $\beta X$  is the Stone-Ćech compactification of  $X$ . It is shown that some additional conditions must be imposed in order that the analogous "v" theorem holds. In §3, we apply our theory to prove (B), and also show that  $\mathfrak{R}$  (locally compact, metrizable) is precisely the class of all spaces of nonmeasurable cardinals. When studying the relation  $v(X \times T) = vX \times vT$  with a metrizable factor  $T$ , the central issue is the weak  $cb^*$  property in another factor  $X$ . It is proved that, in case  $X$  satisfies the countable chain condition and  $T$  is metrizable, the relation holds if and only if (i) either  $X$  or  $T$  is of nonmeasurable cardinal and (ii) either  $X$  is a weak  $cb^*$ -space or  $T$  is locally compact. Finally a number of problems are posed in §4.

Throughout the paper,  $m$  and  $n$  denote cardinal numbers, and  $m^+$  denotes the smallest cardinal greater than  $m$ . We let  $w, d, c$  and  $\chi$  denote the following cardinal functions: weight, density, cellularity and character (cf. [7]).  $|A|$  denotes the cardinality of a set  $A$ , and  $m_1$  stands for the first measurable cardinal. Since  $m_1$  (if it exists) is greater than any nonmeasurable cardinal, that  $|A|$  is nonmeasurable is denoted by  $|A| < m_1$ . We also denote by  $C(X)$  the set of real-valued continuous functions on a space  $X$ . For general terminology, see [7] and [8].

**1. Characterizations of  $\mathfrak{R}$  (locally compact) and  $\mathfrak{R}$  (Moore).** Two subsets  $A$  and  $B$  of a space  $X$  are said to be *completely separated in  $X$*  if there is  $f \in C(X)$  such that  $f(A) = \{0\}$  and  $f(B) = \{1\}$ . A family  $\{F_\alpha\}$  of subsets of a space  $X$  is called *expandable* if there is a locally finite family  $\{G_\alpha\}$  of open sets in  $X$  with  $F_\alpha \subset G_\alpha$  for each  $\alpha$ . We introduce a new class of expandable families.

**1.1. DEFINITION.** A family  $\{F_\alpha | \alpha \in A\}$  of subsets of a space  $X$  is  *$D(m)$ -expandable* if there exists a locally finite family  $\{G_\alpha | \alpha \in A\}$  of open sets in  $X$  with  $F_\alpha \subset G_\alpha$  for each  $\alpha \in A$  and each  $F_\alpha$  is the union of at most  $m$  subsets each of which is completely separated from  $X - G_\alpha$ .

If  $n \geq m$ , then a  $D(m)$ -expandable family is  $D(n)$ -expandable. As a space is completely regular Hausdorff, every expandable family in  $X$  is  $D(|X|)$ -expandable, and a uniformly locally finite family defined in [17] is  $D(1)$ -expandable (cf. [27]). Recall from [17] that a space  $X$  is *pseudo- $m$ -compact* if each locally finite family of nonempty open sets in  $X$  has cardinality less than  $m$ . Pseudocompact spaces are known to be precisely pseudo- $\aleph_0$ -compact spaces. The following theorem plays an essential role in our discussions.

**1.2. THEOREM.** *Let  $X \times Y$  be  $C$ -embedded in  $X \times vY$ . If there exists a  $D(m)$ -expandable family  $\mathfrak{F}$  in  $Y$ , with  $|\mathfrak{F}| = n$ , such that  $\bigcap \{cl_{vY} F | F \in \mathfrak{F}\} \neq \emptyset$ , then each point  $x \in X$ , with  $\chi(x, X) \leq n$ , has a pseudo- $m$ -compact neighborhood.*

**PROOF.** Suppose on the contrary that there exists a point  $x_0 \in X$ , with  $\chi(x_0, X) \leq n$ , which has no pseudo- $m$ -compact neighborhood. Let  $\{G_\lambda | \lambda \in \Lambda\}$  be a neighborhood base at  $x_0$  in  $X$  with  $|\Lambda| = n$ . Then, for each  $\lambda \in \Lambda$ ,  $cl_X G_\lambda$  is not pseudo- $m$ -compact, and thus there is a locally finite family  $\{G'_{\lambda\mu} | \mu \in M_\lambda\}$  of

nonempty open sets in  $\text{cl}_X G_\lambda$  with  $|M_\lambda| = m$ . Setting  $G_{\lambda\mu} = G'_{\lambda\mu} \cap G_\lambda$  for each  $\mu \in M_\lambda$ , we have a locally finite family  $\{G_{\lambda\mu} | \mu \in M_\lambda\}$  of nonempty open sets in  $X$ . It can be assumed without loss of generality that  $x_0 \notin \cup \{G_{\lambda\mu} | \mu \in M_\lambda\}$ . For each  $\mu \in M_\lambda$ , pick  $x_{\lambda\mu} \in G_{\lambda\mu}$ , and choose  $f_{\lambda\mu} \in C(X)$  such that  $f_{\lambda\mu}(x_{\lambda\mu}) = 0$  and  $f_{\lambda\mu}(X - G_{\lambda\mu}) = \{1\}$ . On the other hand, since  $|\mathcal{F}| = n$ , we may write  $\mathcal{F} = \{F_\lambda | \lambda \in \Lambda\}$ . Then there is a locally finite family  $\{H_\lambda | \lambda \in \Lambda\}$  of open sets in  $Y$  with  $F_\lambda \subset H_\lambda$  for each  $\lambda \in \Lambda$ . Each  $F_\lambda$  is a union of  $m$  subsets each of which is completely separated from  $Y - H_\lambda$ , and so we express it by  $F_\lambda = \cup \{F_{\lambda\mu} | \mu \in M_\lambda\}$ , i.e., there is  $g_{\lambda\mu} \in C(Y)$  such that  $g_{\lambda\mu}(F_{\lambda\mu}) = \{0\}$  and  $g_{\lambda\mu}(Y - H_\lambda) = \{1\}$ . For each  $\lambda \in \Lambda$  and each  $\mu \in M_\lambda$ , let us set

$$J_{\lambda\mu} = \{x_{\lambda\mu}\} \times F_{\lambda\mu}, \quad K_{\lambda\mu} = G_{\lambda\mu} \times H_\lambda,$$

$$h_{\lambda\mu}((x, y)) = \min\{1, f_{\lambda\mu}(x) + g_{\lambda\mu}(y)\}, \quad (x, y) \in X \times Y.$$

Then  $h_{\lambda\mu} \in C(X \times Y)$ ,  $h_{\lambda\mu}(J_{\lambda\mu}) = \{0\}$  and  $h_{\lambda\mu}((X \times Y) - K_{\lambda\mu}) = \{1\}$ . It is easily checked that  $\mathcal{K} = \{K_{\lambda\mu} | \mu \in M_\lambda, \lambda \in \Lambda\}$  is locally finite in  $X \times Y$ . Therefore if we define a function  $h$  on  $X \times Y$  by

$$h(p) = \inf\{h_{\lambda\mu}(p) | \mu \in M_\lambda, \lambda \in \Lambda\}, p \in X \times Y,$$

then  $h$  is continuous. Let us choose  $y_0 \in \cap \{\text{cl}_{vY} F_\lambda | \lambda \in \Lambda\}$ ; then  $y_0 \in vY - Y$ , because  $\mathcal{F}$  is locally finite in  $Y$ . Now we show that  $h$  admits no continuous extension to the point  $p_0 = (x_0, y_0) \in X \times vY$ . Let  $U \times V$  be a given basic neighborhood of  $p_0$  in  $X \times vY$ . There is  $\lambda \in \Lambda$  with  $G_\lambda \subset U$ , and  $V \cap F_{\lambda\mu} \neq \emptyset$  for some  $\mu \in M_\lambda$ . Choose  $y \in V \cap F_{\lambda\mu}$ . Then both  $p_1 = (x_{\lambda\mu}, y)$  and  $p_2 = (x_0, y)$  belong to  $U \times V$  and  $h(p_1) = 0$ , while  $h(p_2) = 1$ . This shows that  $h$  does not extend continuously to  $p_0$ , which contradicts the assumption that  $X \times Y$  is  $C$ -embedded in  $X \times vY$ . Hence the proof is complete.

1.3. COROLLARY. *Let  $X \times Y$  be  $C$ -embedded in  $X \times vY$ . If there exists a locally finite family  $\mathcal{K}$  of nonempty open sets in  $Y$ , with  $|\mathcal{K}| = n$ , such that*

$$\bigcap \{\text{cl}_{vY} H | H \in \mathcal{K}\} \neq \emptyset,$$

*then each point  $x \in X$ , with  $\chi(x, X) \leq n$ , has a pseudo- $c(Y)$ -compact neighborhood.*

PROOF. Let  $\mathcal{K} = \{H_\lambda | \lambda \in \Lambda\}$ , and choose  $y_0 \in \cap \{\text{cl}_{vY} H_\lambda | \lambda \in \Lambda\}$ . For each  $\lambda \in \Lambda$ , by Zorn's lemma, there is a maximal disjoint family  $\mathcal{F}_\lambda$  of nonempty open sets in  $H_\lambda$  such that each  $F \in \mathcal{F}_\lambda$  is completely separated from  $Y - H_\lambda$ . Let us set  $F_\lambda = \cup \{F | F \in \mathcal{F}_\lambda\}$ . For each  $\lambda \in \Lambda$ , the maximality of  $\mathcal{F}_\lambda$  implies that  $y_0 \in \text{cl}_{vY} F_\lambda$ . Since  $|\mathcal{F}_\lambda| \leq c(Y)$ ,  $\{F_\lambda | \lambda \in \Lambda\}$  is a  $D(c(Y))$ -expandable family in  $Y$ , with  $|\Lambda| = n$ , such that  $\cap \{\text{cl}_{vY} F_\lambda | \lambda \in \Lambda\} \neq \emptyset$ . Thus the corollary follows from Theorem 1.2.

1.4. REMARK. Let us say that a family  $\mathcal{G}$  of subsets of a space  $X$  converges to  $x \in X$  if each neighborhood of  $x$  contains some member of  $\mathcal{G}$ , and that a subspace  $S$  of  $X$  is relatively pseudo- $m$ -compact in  $X$  if each locally finite family  $\mathcal{U}$  of nonempty open sets in  $X$  such that  $S \cap U \neq \emptyset$  for each  $U \in \mathcal{U}$  has cardinality less than  $m$ . The conclusion of Theorem 1.2 (resp. Corollary 1.3) can be generalized

as follows: Each convergent family  $\mathcal{G}$  of subsets of  $X$ , with  $|\mathcal{G}| < n$ , has a member which is relatively pseudo- $m$ -compact (resp. relatively pseudo- $c(Y)$ -compact) in  $X$ .

Our next work is to construct spaces  $Y$  which have a  $D(\aleph_0)$ -expandable family  $\mathcal{F}$  such that  $\bigcap \{cl_{vY}F \mid F \in \mathcal{F}\} \neq \emptyset$ . A space is called 0-dimensional if it has a base consisting of open-and-closed sets. For an ordinal  $\alpha$ , we denote by  $W(\alpha)$  the set of all ordinals less than  $\alpha$  topologized with the order topology, and by  $\omega_0$  (resp.  $\omega_1$ ) the first infinite (resp. first uncountable) ordinal.

1.5. FACT. *For every infinite cardinal  $n$ , there exists a 0-dimensional locally compact space  $Y = Y_1(n)$ , with  $|Y| = w(Y) = n \cdot \aleph_1$ , that has a  $D(\aleph_0)$ -expandable family  $\mathcal{F}$  such that  $|\mathcal{F}| = n$  and  $\bigcap \{cl_{vY}F \mid F \in \mathcal{F}\} \neq \emptyset$ .*

PROOF. Let  $T_1 = W(\omega_1 + 1) \times W(\omega_0 + 1)$ , and let  $T_2 = \Lambda \cup \{\infty\}$  be the one point compactification of a discrete space  $\Lambda$  of cardinality  $n$ . We denote a base for the topology on  $T_i$  by  $\mathfrak{B}_i$  for  $i = 1, 2$ . Let  $E = \{(\omega_1, \beta) \mid \beta < \omega_0\}$ , and let  $Z'$  be the quotient space obtained from  $R = T_1 \times T_2$  by collapsing the set  $\{(\omega_1, \beta)\} \times T_2$  to a point  $z(\beta) \in Z'$  for each  $(\omega_1, \beta) \in E$ . Let  $\phi': R \rightarrow Z'$  be the quotient map. Let  $Z_0$  be the set  $Z'$ , retopologized by letting  $\bigcup \{\mathfrak{B}(B) \mid B \in \mathfrak{B}_1\}$  be a base, where

$$\mathfrak{B}(B) = \begin{cases} \{\phi'(B \times T_2)\} & \text{if } B \cap E \neq \emptyset, \\ \{\phi'(B \times B') \mid B' \in \mathfrak{B}_2\} & \text{if } B \cap E = \emptyset. \end{cases}$$

Then the natural map  $\phi: R \rightarrow Z_0$  is continuous, and hence  $Z_0$  is compact. Let us set

$$Z = Z_0 - \phi(\{((\gamma, \omega_0), \infty) \mid \gamma < \omega_1\}).$$

The space  $Z$  is a 0-dimensional locally compact space with  $|Z| = w(Z) = n \cdot \aleph_1$ . Since  $z(\beta)$  is a  $P$ -point for each  $\beta < \omega_0$ , it is easily checked that  $Z$  is  $C$ -embedded in  $Z \cup \{z_0\}$ , where  $z_0 = z(\omega_0)$ , and so  $z_0 \in vZ - Z$  by [8, 8.6]. Setting  $D_\lambda = \phi(\{((\gamma, \omega_0), \lambda) \mid \gamma < \omega_1\})$  for each  $\lambda \in \Lambda$ , we have a discrete family  $\{D_\lambda \mid \lambda \in \Lambda\}$  of closed subsets in  $Z$  such that  $z_0 \in \bigcap \{cl_{vZ}D_\lambda \mid \lambda \in \Lambda\} \neq \emptyset$ . Define a subspace  $Y$  of the product space  $Z \times W(\omega_0 + 1)$  by

$$Y = (Z \times \{\omega_0\}) \cup (\bigcup \{D_\lambda \times W(\omega_0 + 1) \mid \lambda \in \Lambda\}).$$

Then  $Y$  is a 0-dimensional locally compact space with  $|Y| = w(Y) = n \cdot \aleph_1$ , because  $Y$  is a closed subspace of  $Z \times W(\omega_0 + 1)$ . It remains to show the existence of a  $D(\aleph_0)$ -expandable family in  $Y$  satisfying the stated conditions. Since  $Z \times \{\omega_0\}$  is  $C$ -embedded in  $Y$ , it follows from [8, 8.10(a)] that  $vZ = v(Z \times \{\omega_0\}) \subset vY$ , and hence we may consider  $z_0$  as an element of  $vY - Y$ . Setting  $F_\lambda = D_\lambda \times W(\omega_0)$  for each  $\lambda \in \Lambda$ , we have a discrete family  $\mathcal{F} = \{F_\lambda \mid \lambda \in \Lambda\}$  of open sets in  $Y$  such that  $z_0 \in \bigcap \{cl_{vY}F_\lambda \mid \lambda \in \Lambda\}$ . Then, since each  $F_\lambda$  is a union of countably many open-and-closed subsets in  $Y$ ,  $\mathcal{F}$  is a  $D(\aleph_0)$ -expandable family in  $Y$ . Hence  $Y$  is proved to be the desired space.

1.6. FACT. *For every infinite cardinal  $n$ , there exists a 0-dimensional Moore space  $Y = Y_2(n)$ , with  $|Y| = w(Y) = n \cdot \exp \aleph_0$ , that has a  $D(\aleph_0)$ -expandable family  $\mathcal{F}$  such that  $|\mathcal{F}| = n$  and  $\bigcap \{cl_{vY}F \mid F \in \mathcal{F}\} \neq \emptyset$ .*

PROOF. In [30], for every infinite cardinal  $n$ , we constructed a 0-dimensional Moore space  $Z = Z(n)$ , with  $|Z| = w(Z) = n \cdot \exp \aleph_0$ , that has a discrete family  $\mathcal{D}$  of closed subsets such that  $|\mathcal{D}| = n$  and  $\bigcap \{cl_{vZ} D \mid D \in \mathcal{D}\} \neq \emptyset$ . The desired space  $Y_2(n)$  can be made from  $Z(n)$  by the same procedure as in the proof of 1.4.

For later use, we quote a theorem due to Hušek [13]:

1.7. THEOREM (HUŠEK). *Let  $Q$  be a discrete space. Then  $v(P \times Q) = vP \times vQ$  holds if and only if either  $|P| < m_1$  or  $|Q| < m_1$  (i.e., either  $|P|$  or  $|Q|$  is nonmeasurable).*

We are now in a position to prove main theorems of this section. For the notion of locally pseudocompact spaces see [5]. We remark that the assumption  $|X| < m_1$  of Theorem 1.8 is useful only for the implications (a)  $\rightarrow$  (b) and (a)  $\rightarrow$  (c).

1.8. THEOREM. *The following conditions on a space  $X$  with  $|X| < m_1$  are equivalent:*

- (a)  $X$  is locally pseudocompact.
- (b)  $X \times Y$  is  $C$ -embedded in  $X \times vY$  for each 0-dimensional locally compact space  $Y$  with  $w(Y) \leq \chi(X) \cdot \aleph_1$ .
- (c)  $X \times Y$  is  $C$ -embedded in  $X \times vY$  for each 0-dimensional Moore space  $Y$  with  $w(Y) \leq \chi(X) \cdot \exp \aleph_0$ .

PROOF. We proved in [28] that if  $X$  is a locally pseudocompact space of nonmeasurable cardinal, then  $X \times Y$  is  $C$ -embedded in  $X \times vY$  for each  $k$ -space  $Y$ . Since both locally compact spaces and Moore spaces are  $k$ -spaces, (a)  $\rightarrow$  (b) and (a)  $\rightarrow$  (c) follow from this result. To prove (b)  $\rightarrow$  (a) ((c)  $\rightarrow$  (a)) suppose on the contrary that  $X$  is not locally pseudocompact at  $x_0 \in X$ . Let  $Y$  be the space  $Y_1(n)$  ( $Y_2(n)$ ) constructed in 1.5 (1.6), where  $n = \chi(x_0, X)$ . Then it follows from Theorem 1.2, that  $X \times Y$  is not  $C$ -embedded in  $X \times vY$ . This contradiction completes the proof.

1.9. THEOREM. *The following conditions on a space  $X$  are equivalent:*

- (a)  $vX$  is locally compact and  $|X| < m_1$ .
- (b)  $v(X \times Y) = vX \times vY$  holds for each 0-dimensional locally compact space  $Y$  with  $w(Y) \leq \chi(vX) \cdot \aleph_1$ .
- (c)  $v(X \times Y) = vX \times vY$  holds for each 0-dimensional Moore space  $Y$  with  $w(Y) \leq \chi(vX) \cdot \exp \aleph_0$ .

PROOF. Since (a)  $\rightarrow$  (b) and (a)  $\rightarrow$  (c) follow from Hušek [14, Corollary (a), p. 177] (cf. also [28]), we prove only (b)  $\rightarrow$  (a) and (c)  $\rightarrow$  (a). By Theorem 1.8,  $vX$  is locally pseudocompact, and hence it is locally compact, because every pseudocompact realcompact space is compact [7, 3.11.1]. Suppose that  $|X| \geq m_1$ ; then  $\chi(vX) \geq m_1$  by [21, Theorem 2]. If we take for  $Y$  a discrete space of cardinality  $m_1$ , then it follows from Theorem 1.7 that  $v(X \times Y) \neq vX \times vY$ . This contradicts (b) and (c) simultaneously. Hence the proof is complete.

In [14], Hušek proved that if  $X$  satisfies 1.9(a), then  $v(X \times Y) = vX \times vY$  holds for each  $k$ -space  $Y$ . Therefore Theorem 1.9 tells us that both  $\mathcal{R}$  (locally compact)

and  $\mathfrak{R}$  (Moore) coincide with the class of spaces  $X$  such that  $\nu X$  is locally compact and  $|X| < m_1$ .

1.10. REMARKS. (1) Let  $\Psi$  be the space described in [8, 5I, p. 79];  $\Psi$  is constructed as follows: Let  $\mathfrak{E}$  be a maximal infinite almost-disjoint family of infinite subsets of the set  $N$  of integers. Then  $|\mathfrak{E}| = \exp \aleph_0$ . The space  $\Psi$  is the union of  $N$  with a new set  $D = \{\omega_E | E \in \mathfrak{E}\}$  of distinct points endowed with the following topology: Each point of  $N$  is isolated, and a neighborhood of  $\omega_E$  is any set containing  $\omega_E$  and all but a finite number of points of  $E$ . It is well known that  $\Psi$  is a 0-dimensional pseudocompact (and hence  $\beta\Psi = \nu\Psi$  by [8, 8A4, p. 125]) locally compact Moore space. In [25], Mrówka showed that  $\mathfrak{E}$  can be chosen so that  $\beta\Psi$  is the one point compactification. Then, dividing  $D$  into a disjoint family of countable infinite subsets, we have a discrete family  $\mathfrak{D}$  of closed subsets in  $\Psi$  such that  $|\mathfrak{D}| = \exp \aleph_0$  and  $\bigcap \{\text{cl}_{\nu\Psi} D' | D' \in \mathfrak{D}\} \neq \emptyset$ . Thus, by the same method as in the proof of 1.5, we can make a 0-dimensional locally compact Moore space  $Y$ , with  $w(Y) = \exp \aleph_0$ , that has a  $D(\aleph_0)$ -expandable family  $\mathfrak{F}$  such that  $|\mathfrak{F}| = \exp \aleph_0$  and  $\bigcap \{\text{cl}_{\nu Y} F | F \in \mathfrak{F}\} \neq \emptyset$ . This fact combined with Theorem 1.2 implies that the following condition (d) is also equivalent to 1.9(a) under the assumption that  $\chi(\nu X) \leq \exp \aleph_0$ .

(d)  $\nu(X \times Y) = \nu X \times \nu Y$  holds for each 0-dimensional locally compact Moore space  $Y$  with  $w(Y) \leq \chi(\nu X) \cdot \exp \aleph_0$ .

We do not know whether (d) implies 1.9(a) or not in general.

(2) We can apply our theory to answer the following question of Hušek [13, p. 326]: Do there exist spaces  $X, Y$  of cardinalities  $\aleph_0$  and  $\aleph_1$ , respectively, such that  $\nu(X \times Y) \neq \nu X \times \nu Y$ ? Let  $X$  be the space of rational numbers with the usual topology. We take for  $Y$  the space  $Y_1(\aleph_0)$  constructed in 1.4. Then  $|X| = w(X) = \aleph_0$  and  $|Y| = w(Y) = \aleph_1$ . Since  $X$  is not locally pseudocompact, it follows from Theorem 1.2 that  $\nu(X \times Y) \neq \nu X \times \nu Y$ .

**2. Mapping theorems.** In this section, we give mapping theorems which will be used in the next section. As is well known, for a map  $f: X \rightarrow Y$ , there exists a continuous extension  $\nu f: \nu X \rightarrow \nu Y$  of  $f$  [8, 8.7]. If  $f_i: X_i \rightarrow Y_i$  is a map for  $i = 1, 2$ , then the product map  $f = f_1 \times f_2$  from  $X_1 \times X_2$  to  $Y_1 \times Y_2$  is defined by  $f((x_1, x_2)) = (f_1(x_1), f_2(x_2))$  for  $(x_1, x_2) \in X_1 \times X_2$ .

2.1. THEOREM. Let  $f_i: X_i \rightarrow Y_i$  ( $i = 1, 2$ ) be onto maps. If  $\nu f_1 \times \nu f_2$  is a quotient map from  $\nu X_1 \times \nu X_2$  onto  $\nu Y_1 \times \nu Y_2$ , then  $\nu(X_1 \times X_2) = \nu X_1 \times \nu X_2$  implies  $\nu(Y_1 \times Y_2) = \nu Y_1 \times \nu Y_2$ .

More precisely, we have the following theorem:

2.2. THEOREM. Let  $F_i: X_i^* \rightarrow Y_i^*$  ( $i = 1, 2$ ) be onto maps such that  $F = F_1 \times F_2$  is a quotient map, and let  $X_i$  (resp.  $Y_i = F_i(X_i)$ ) be a dense  $C$ -embedded subspace of  $X_i^*$  (resp.  $Y_i^*$ ). If  $X_1 \times X_2$  is  $C$ -embedded in  $X_1^* \times X_2^*$ , then  $Y_1 \times Y_2$  is  $C$ -embedded in  $Y_1^* \times Y_2^*$ .

PROOF. Let us set  $f_i = F_i|X_i$  ( $i = 1, 2$ ) and  $f = f_1 \times f_2$ . To show that  $Y_1 \times Y_2$  is  $C$ -embedded in  $Y_1^* \times Y_2^*$ , let  $g \in C(Y_1 \times Y_2)$ . Since  $h = g \circ f \in C(X_1 \times X_2)$ , by

our assumption, there exists  $H \in C(X_1^* \times X_2^*)$  such that  $H|(X_1 \times X_2) = h$ . We shall show that (\*)  $H$  takes on the constant value  $t_p$  on  $F^{-1}(p)$  for each  $p \in Y_1^* \times Y_2^*$ . Let  $x \in X_1$ ; then  $h(x, \cdot) = g(f_1(x), \cdot) \circ f_2$ , where  $h(x, \cdot) = h|(\{x\} \times X_2)$ . Since  $g(f_1(x), \cdot) \in C(Y_2)$ , it has a continuous extension  $G_x$  over  $Y_2^*$ . Then,  $X_2$  being dense in  $X_2^*$ ,  $H(x, \cdot) = G_x \circ F_2$ . Hence it follows that  $H(x, \cdot)$  is constant on  $\{x\} \times F_2^{-1}(y)$  for each  $y \in Y_2^*$ . This implies that  $H$  is constant on  $f_1^{-1}(y_1) \times F_2^{-1}(y_2)$  for each  $(y_1, y_2) \in Y_1 \times Y_2^*$ . Similarly,  $H$  is constant on  $F_1^{-1}(y_1) \times f_2^{-1}(y_2)$  for each  $(y_1, y_2) \in Y_1^* \times Y_2$ . To see (\*), let  $p = (y_1, y_2) \in Y_1^* \times Y_2^*$ . Then it follows from these facts that

$$\begin{aligned}
 H(x, \cdot) &= H(x', \cdot) \quad \text{for each } x, x' \in F_1^{-1}(y_1), \\
 H(\cdot, x) &= H(\cdot, x') \quad \text{for each } x, x' \in F_2^{-1}(y_2),
 \end{aligned}$$

and from which (\*) is proved. Define a function  $G$  on  $Y_1^* \times Y_2^*$  by  $G(p) = t_p$  for each  $p \in Y_1^* \times Y_2^*$ . Then  $H = G \circ F$  and  $G|(Y_1 \times Y_2) = g$ . Since  $F$  is a quotient map and  $H$  is continuous, it follows from [7, 2.4.2] that  $G$  is continuous. Hence our proof is complete.

Theorem 2.2 remains true if “ $C$ -embedded” is replaced by “ $C^*$ -embedded”. Ishii proved in [18] that if  $f: X \rightarrow Y$  is an open perfect onto map, then so is  $\nu f$ . This leads to the following corollary of Theorem 2.1.

2.3. COROLLARY. *If  $f_i: X_i \rightarrow Y_i$  is an open perfect map onto  $Y_i$  for  $i = 1, 2$ , then  $\nu(X_1 \times X_2) = \nu X_1 \times \nu X_2$  implies  $\nu(Y_1 \times Y_2) = \nu Y_1 \times \nu Y_2$ .*

The following theorem shows that in Theorem 2.1 the assumption that  $\nu f_1 \times \nu f_2$  is quotient onto cannot be dropped, even when  $f_1$  is an identity and  $f_2$  is a perfect map. Recall from [13] that a space  $X$  is *pseudo- $m_1$ -compact* if the cardinality of each locally finite family of nonempty open sets in  $X$  is nonmeasurable.

2.4. THEOREM. *Among the following conditions on a space  $X$ , (a)  $\rightarrow$  (b)  $\rightarrow$  (c) is valid. Conversely, (c)  $\rightarrow$  (a) holds if  $|X| < m_1$ .*

(a)  $\nu X$  is locally compact.

(b) For each space  $Y$  satisfying  $\nu(X \times Y) = \nu X \times \nu Y$  and each quotient image  $Z$  of  $Y$ ,  $\nu(X \times Z) = \nu X \times \nu Z$  holds.

(c) As in (b), with “perfect” instead of “quotient”.

PROOF. (a)  $\rightarrow$  (b). Let  $Y$  be a space satisfying  $\nu(X \times Y) = \nu X \times \nu Y$ , and let  $Z$  be the image of  $Y$  under a quotient map  $f$ . Since  $\nu X$  is locally compact, by Whitehead’s theorem [7, 3.3.17],  $\text{id}_{\nu X} \times f$  is a quotient map, where  $\text{id}_{\nu X}$  is the identity map of  $\nu X$ . It follows from Theorem 2.2 that  $X \times Z$  is  $C$ -embedded in  $\nu X \times Z$ . Hušek proved in [13] that if  $P$  is a locally compact, realcompact space, then  $\nu(P \times Q) = \nu P \times \nu Q$  if and only if either  $|P| < m_1$  or  $Q$  is pseudo- $m_1$ -compact. If we apply this theorem to our case, then  $|X| < m_1$  or  $Y$  is pseudo- $m_1$ -compact. If  $Y$  is pseudo- $m_1$ -compact, so is  $Z$ . Hence it follows that  $\nu(\nu X \times Z) = \nu X \times \nu Z$ . Thus we have  $\nu(X \times Z) = \nu X \times \nu Z$ .

(b)  $\rightarrow$  (c). Obvious.

(c)  $\rightarrow$  (a). Suppose that  $|X| < m_1$  and  $\nu X$  is not locally compact at  $x_0 \in \nu X$ .

Then, by [7, 3.11.1],  $x_0$  has no pseudocompact neighborhood in  $\nu X$ . Let  $n = \max\{|\nu X|, \chi(x_0, \nu X)\}$ ; then  $n < m_1$ . Let  $\omega_\alpha$  be the initial ordinal of  $n^+$ , and let  $T = W(\omega_\alpha + 1) \times W(\omega_0 + 1)$ . Let  $\Lambda$  be a discrete space of cardinality  $n$ , and let  $S_0$  be the quotient space obtained from  $R_0 = T \times \Lambda$  by collapsing the set  $\{(\omega_\alpha, \beta)\} \times \Lambda$  to a point  $s(\beta)$  for each  $\beta \in E$ , where  $E = \{2n | n < \omega_0\} \cup \{\omega_0\}$ . Let  $g: R_0 \rightarrow S_0$  be the quotient map. Let us set  $S = S_0 - \{s_0\}$ , where  $s_0 = s(\omega_0)$ , and let  $R = R_0 - g^{-1}(s_0)$ . Then it is easily checked that  $\nu S = S_0$  and  $\nu R = R_0$ . If we set

$$G = g(\{(\gamma, 2n) | \gamma < \omega_\alpha, n < \omega_0\} \times \Lambda),$$

then  $G$  is a cozero-set of  $S_0$ , and hence  $G = G^* \cap S_0$  for some cozero-set  $G^*$  of  $\beta S_0 (= \beta S)$ . Let us set  $Z = S \cup G^*$ . We now need the following lemma:

2.5. LEMMA. *Let  $X \supset X_1 \supset X_2$ . Suppose that  $X_2$  is dense in  $X$  and is  $C$ -embedded in  $X_1$ . Then, for each open set  $H$  of  $X$ ,  $X_2 \cup H$  is  $C$ -embedded in  $X_1 \cup H$ .*

The proof is left to the reader, since it requires only routine verification. We continue the proof of Theorem 2.4. By Lemma 2.5 and [7, 3.11.10],  $\nu Z = S_0 \cup G^*$ , and hence  $s_0 \in \nu Z - Z$ . Setting

$$F_\lambda = g(\{(\gamma, 2n + 1) | \gamma \leq \omega_\alpha, n < \omega_0\} \times \{\lambda\})$$

for each  $\lambda \in \Lambda$ , we obtain a locally finite family  $\{F_\lambda | \lambda \in \Lambda\}$  of open sets in  $Z$ . Since each  $F_\lambda$  is a countable union of open-and-closed subsets of  $Z$ ,  $\{F_\lambda | \lambda \in \Lambda\}$  is a  $D(\aleph_0)$ -expandable family in  $Z$  such that  $\bigcap \{cl_{\nu Z} F_\lambda | \lambda \in \Lambda\} \ni s_0$ . Since  $\chi(x_0, \nu X) < |\Lambda|$ , it follows from Theorem 1.2 that  $\nu(X \times Z) \neq \nu X \times \nu Z$ . For our end, it suffices to show that  $Z$  is the perfect image of a space  $Y$  satisfying  $\nu(X \times Y) = \nu X \times \nu Y$ . There exists the extension  $\beta g: \beta R_0 \rightarrow \beta S_0$  of  $g$ . Let us set  $Y = R \cup H^*$ , where  $H^* = (\beta g)^{-1}(G^*)$ , and set  $f = (\beta g)|_Y$ . Then, since  $H^*$  is a cozero-set of  $\beta R_0 (= \beta R)$ ,  $\nu Y = R_0 \cup H^*$  by Lemma 2.5 and [7, 3.11.10]. Further it is easily checked that  $f$  is a perfect map from  $Y$  onto  $Z$  and  $Y$  is locally compact. Since  $|Y| < m_1$ , it follows from [5, Theorem 2.1] that  $X \times Y$  is  $C$ -embedded in  $\nu X \times Y$ . It remains to show that  $\nu X \times Y$  is  $C$ -embedded in  $\nu X \times \nu Y$ . Since  $|\nu X| < n$ , a similar argument to that of [8, 8.20] shows that  $\nu X \times W(\omega_\alpha) \times W(\omega_0 + 1)$  is  $C$ -embedded in  $\nu X \times W(\omega_\alpha + 1) \times W(\omega_0 + 1)$ . Thus  $\nu X \times R$  is  $C$ -embedded in  $\nu X \times R_0$ . Since  $\nu X \times H^*$  is an open set of  $\nu X \times \beta Y$ , it follows from Lemma 2.5 that

$$(\nu X \times R) \cup (\nu X \times H^*) \quad (= \nu X \times Y)$$

is  $C$ -embedded in

$$(\nu X \times R_0) \cup (\nu X \times H^*) \quad (= \nu X \times \nu Y).$$

Hence the proof is complete.

2.6. REMARK. In case  $|X| \geq m_1$ , (c)  $\rightarrow$  (a) of Theorem 2.4 need not be true. If  $D$  is a discrete space of cardinality  $m_1$ , then, by Theorem 1.6,  $D$  satisfies 2.4(c). But it is known [5, p. 115] that  $\nu D$  is not even a  $k$ -space.

The following corollary is proved by using Theorem 2.4 repeatedly.

2.7. COROLLARY. *Let  $f_i: X_i \rightarrow Y_i$  ( $i = 1, 2$ ) be quotient onto maps. If both  $\nu X_1$  and  $\nu Y_2$  are locally compact, then  $\nu(X_1 \times X_2) = \nu X_1 \times \nu X_2$  implies  $\nu(Y_1 \times Y_2) = \nu Y_1 \times \nu Y_2$ .*

**3. Characterizations of  $\mathfrak{R}$  (metrizable).** A space  $X$  is called a *weak cb\*-space* if for each decreasing sequence  $\{F_n | n < \omega_0\}$  of regular closed sets in  $X$  with empty intersection,  $\bigcap \{cl_{vX} F_n | n < \omega_0\} = \emptyset$  holds, where a regular closed set is the closure of an open set. This notion was introduced by Isiwata [20] as a common generalization of realcompact spaces and weak cb-spaces in the sense of Mack and Johnson [22]. Since normal countably paracompact spaces, extremally disconnected spaces [8, 1H, p. 22] and pseudocompact spaces (or more generally,  $M'$ -spaces in the sense of Isiwata [19]) are weak cb-spaces, they are weak cb\*-spaces. In this section, we prove the following theorem:

**3.1. THEOREM.** *The following conditions on a space  $X$  are equivalent:*

- (a)  $X$  is a weak cb\*-space and  $|X| < m_1$ .
- (b)  $v(X \times T) = vX \times vT$  holds for each metrizable space  $T$ .
- (c)  $v(X \times D(d(X))^\omega) = vX \times vD(d(X))^\omega$ .

Here,  $D(d(X))^\omega$  denotes the product of countably many copies of a discrete space of cardinality  $d(X)$ . Associated with each space  $X$ , there exist an extremally disconnected space  $E(X)$  and a perfect irreducible map (i.e., a perfect map which takes proper closed subsets onto proper subsets)  $e_X$  from  $E(X)$  onto  $X$ . The space  $E(X)$  is unique up to homeomorphism and is called the *absolute* of  $X$  (cf. [16], [31]). To prove Theorem 3.1, we make use of the following lemmas. The next lemma follows immediately from [10, Theorem 2.4] and [11, Proposition 1.2]; the first part also appears in [15].

**3.2. LEMMA.** *A space  $X$  is a weak cb\*-space if and only if  $vE(X) = E(vX)$  holds. Moreover, in case  $vE(X) = E(vX)$ , then  $e_{vX}$  is the extension of  $e_X$  over  $vE(X)$ .*

**3.3. LEMMA [29].** *Let  $X$  be a space and  $T$  a metrizable space. If either  $X$  is extremally disconnected or  $T$  is locally compact, then  $X \times T$  is  $z$ -embedded in  $\beta X \times T$  (i.e., each zero-set of  $X \times T$  is the restriction to  $X \times T$  of a zero-set of  $\beta X \times T$ ).*

The next lemma is a corollary of Blair [1, Theorem 7.6]:

**3.4. LEMMA (BLAIR).** *Let  $X \times Y$  be  $z$ -embedded in  $\beta X \times Y$ . If either  $|X| < m_1$  or  $Y$  is pseudo- $m_1$ -compact, then  $v(X \times Y) = vX \times vY$  holds.*

**PROOF OF THEOREM 3.1.** (a)  $\rightarrow$  (b). Let  $X$  be a weak cb\*-space with  $|X| < m_1$  and  $T$  a metrizable space. Since  $|E(X)| < m_1$ , it follows from Lemmas 3.3 and 3.4 that  $v(E(X) \times T) = vE(X) \times vT$ . By Lemma 3.2  $v e_X (= e_{vX})$  is a perfect map from  $vE(X)$  onto  $vX$ , and so  $v e_X \times id_{vT}$  is perfect. Hence it follows from Theorem 2.1 that  $v(X \times T) = vX \times vT$ .

(b)  $\rightarrow$  (c). Obvious.

(c)  $\rightarrow$  (a). Suppose on the contrary that  $X$  is not a weak cb\*-space. Then there is a locally finite sequence  $\{G_n | n < \omega_0\}$  of open sets in  $X$  with  $\bigcap \{cl_{vX} G_n | n < \omega_0\} \neq \emptyset$ . Since  $c(X) \leq d(X)$ , each point of  $D(d(X))^\omega$  has no pseudo- $c(X)$ -compact neighborhood, and  $\chi(D(d(X))^\omega) = \aleph_0$ . Hence it follows from Corollary 1.3 that  $X \times D(d(X))^\omega$  is not  $C$ -embedded in  $vX \times D(d(X))^\omega$ . This contradicts (c). To

prove that  $|X| < m_1$ , find a discrete family  $\{G_\alpha | \alpha \in A\}$  of nonempty open sets in  $D(d(X))^\omega$  with  $|A| = d(X)$ . Pick  $t_\alpha \in G_\alpha$  for each  $\alpha \in A$ , and set  $D = \{t_\alpha | \alpha \in A\}$ . Then it is easily checked that  $X \times D$  is  $C$ -embedded in  $X \times D(d(X))^\omega$ . Since  $\nu D \subset \nu D(d(X))^\omega$ , (c) implies  $\nu(X \times D) = \nu X \times \nu D$ . Hence it follows from Theorem 1.6 that  $|X| < m_1$  or  $|D| (= d(X)) < m_1$ . If  $|D| < m_1$ , then  $|X| < m_1$  by [8, 12.5]. Thus the proof is complete.

3.5. REMARKS. (1) If  $E(X) \times T$  is  $z$ -embedded in  $\beta E(X) \times T$ , then it follows from [6, Proposition 5.1] and [2, Corollary 3.6] that  $E(X) \times T$  is  $C$ -embedded in  $\nu E(X) \times T$ . Therefore the proof of Theorem 3.1 shows that, more generally, a space  $X$  is a weak  $cb^*$ -space if and only if  $X \times T$  is  $C$ -embedded in  $\nu X \times T$  for each metrizable space  $T$ .

(2) The product  $X \times T$  of a weak  $cb^*$ -space  $X$  with a metrizable space  $T$  need not be  $z$ -embedded in  $\beta X \times T$ . In fact, it was remarked in [29] that  $\dim(X \times T) \leq \dim X + \dim T$  whenever  $X \times T$  is  $z$ -embedded in  $\beta X \times T$ , while Wage showed in [32] that there exist a Lindelöf space (hence a weak  $cb^*$ -space)  $X$  and a metrizable space  $T$  such that  $\dim(X \times T) > \dim X + \dim T$ .

(3) Lemmas 3.3 and 3.4 can be combined with Theorem 1.6 to yield the following result:  $\nu(X \times T) = \nu X \times \nu T$  holds for each locally compact, metrizable space  $T$  if and only if  $|X| < m_1$ .

We conclude this section with a theorem, which gives conditions on  $X$  and  $Y$  necessary and sufficient that the relation  $\nu(X \times Y) = \nu X \times \nu Y$  be valid in a restrictive situation.

3.6. THEOREM. *Let  $X$  be a space satisfying the countable chain condition (i.e.,  $c(X) \leq \aleph_0$ ) and  $T$  a metrizable space. Then  $\nu(X \times T) = \nu X \times \nu T$  holds if and only if (i) either  $|X| < m_1$  or  $|T| < m_1$  and (ii) either  $X$  is weak  $cb^*$  or  $T$  is locally compact.*

PROOF. *Necessity:* (i) is proved just like (c)  $\rightarrow$  (a) in Theorem 3.1. If  $T$  is not locally compact, then  $T$  is not locally pseudocompact by [7, 3.10.21 and 4.1.17]. Thus it follows from Corollary 1.3 that  $X$  is a weak  $cb^*$ -space.

*Sufficiency:* In case  $|X| < m_1$ , then the proof follows from Theorem 3.1 and Lemmas 3.3 and 3.4. In case  $|T| < m_1$ , then  $T$  is realcompact by [8, 15.20]. It follows from [5, Corollary 2.2] and 3.5(1) that  $\nu(X \times T) = \nu X \times \nu T$ . Hence the proof is complete.

3.7. REMARK. Theorem 3.6 fails to be valid if we drop the assumption  $c(X) \leq \aleph_0$ . To see this, we utilize the space  $Q$  of all rational numbers and the space  $Y_0$  due to Comfort [4, p. 99]. The space  $Y_0$  was constructed as the quotient space obtained from the product space

$$Z = W(\omega_0) \times W(\omega_1 + 1) \times W(\omega_1 + 1)$$

by identifying, for each  $n < \omega_0$  and each  $\gamma < \omega_1$ , the two points  $(n, \omega_1, \gamma)$  and  $(n + 1, \gamma, \omega_1)$ . Let  $f: Z \rightarrow Y_0$  be the quotient map, and let us set  $X = Y_0 - \{y_0\}$ , where  $y_0$  is the center point  $f((0, \omega_1, \omega_1)) (= f((n, \omega_1, \omega_1)))$ . Then he proved that  $\nu X = Y_0$ , and a similar argument assures us that  $\nu(X \times Q) = \nu X \times Q$ . Obviously  $Q$  is metrizable but not locally compact. It remains to show that  $X$  is not a weak

cb\*-space. Setting

$$F_n = f(\{i | i \geq n\} \times W(\omega_1 + 1) \times W(\omega_1 + 1)) \cap X$$

for each  $n < \omega_0$ , we obtain a decreasing sequence  $\{F_n | n < \omega_0\}$  of regular closed sets in  $X$  with empty intersection. Then  $y_0 \in \bigcap \{cl_{vX} G_n | n < \omega_0\}$ , and hence  $X$  is not a weak cb\*-space.

**4. Problems and remarks.** Many interesting problems related to our results remain unsolved. Following [20], we say that a space  $X$  is *v-locally compact* if  $vX$  is locally compact.

4.1. Characterize  $\mathfrak{R}(v\text{-locally compact})$ . It is easy to see that

$$\mathfrak{R}(v\text{-locally compact}) = \mathfrak{R}(\text{pseudocompact}).$$

4.2. Characterize  $\mathfrak{R}(\text{realcompact})$ . Hušek [12], [14] and McArthur [23] proved that each member  $X$  of this class, with  $|X| < m_1$ , is realcompact; however, the characterization is not yet known in complete form.

4.3. Characterize  $\mathfrak{R}(\text{weak cb}^*)$ . We note that it follows from Lemma 3.2 and Theorem 2.1 that  $\mathfrak{R}(\text{weak cb}^*) = \mathfrak{R}(\text{extremally disconnected})$ . Moreover, since the space  $Y$  constructed in the proof of [23, Theorem 5.2] is a weak cb\*-space, every member of  $\mathfrak{R}(\text{weak cb}^*)$  is realcompact.

4.4. Find conditions on  $X$  and  $T$  necessary and sufficient that  $v(X \times T) = vX \times vT$  be valid in the case where  $T$  is a metrizable space.

4.5. Let  $f_i: X_i \rightarrow Y_i$  ( $i = 1, 2$ ) be onto maps. When does  $v(Y_1 \times Y_2) = vY_1 \times vY_2$  imply  $v(X_1 \times X_2) = vX_1 \times vX_2$ ?

4.6. **REMARK.** Let  $f: Y \rightarrow Z$  be a perfect onto map. Then  $v(X \times Z) = vX \times vZ$  does not necessarily imply  $v(X \times Y) = vX \times vY$ , even when  $id_{vX} \times vf$  is a quotient onto map and  $vX$  is compact. To see this, let us set  $X = W(\omega_1)$ ; then by [23, Theorem 5.5] there exists a realcompact space  $Y_1$  such that  $v(X \times Y_1) \neq vX \times vY_1$ . By [26, Corollary 2.3],  $Y_1$  can be embedded as a closed subspace of a pseudocompact space  $Y_2$ . Let  $i: Y_1 \rightarrow Y_2$  be the embedding. Let us set  $Y = Y_1 \oplus Y_2$  and  $Z = Y_2$ , where  $\oplus$  means the topological sum. Define a map  $f: Y \rightarrow Z$  by  $f(y) = i(y)$  if  $y \in Y_1$  and  $f(y) = y$  if  $y \in Y_2$ . Then  $f$  is a perfect onto map and  $vf$  is a quotient map from  $vY (= Y_1 \oplus vY_2)$  onto  $vZ (= vY_2)$ . Since  $X$  is locally compact, it follows from [7, 3.10.26] that  $X \times Z$  is pseudocompact, and hence

$$v(X \times Z) = vX \times vZ$$

holds by Glicksberg's theorem [9]. On the other hand,  $v(X \times Y) \neq vX \times vY$  obviously. Further,  $vX$  being compact, it follows from [7, 3.3.17] that  $id_{vX} \times vf$  is a quotient onto map.

4.7. Find characterizations of an onto map  $f: X \rightarrow Y$  for which  $vf: vX \rightarrow vY$  is an onto biquotient map in the sense of Michael [24]. We are interested in this problem in view of 4.8(3) below.

4.8. **REMARK.** It seems that the classes  $\mathfrak{R}(\mathcal{P})$  considered above have several common properties. Finally, we list some of these below. Each assertion follows from the results in the bracket.

(1)  $\mathfrak{R}(\mathcal{P})$  includes all locally compact, realcompact spaces of nonmeasurable cardinals [5, Corollary 2.2].

(2)  $\mathfrak{R}(\mathcal{P})$  is closed under cozero-subspaces [3, 3.2].

(3)  $\mathfrak{R}(\mathcal{P})$  is closed under open perfect images (Corollary 2.3); more generally, if  $\nu f: \nu X \rightarrow \nu Y$  is an onto biquotient map, then  $Y \in \mathfrak{R}(\mathcal{P})$  whenever  $X \in \mathfrak{R}(\mathcal{P})$  ([24, Theorem 1.2] and Theorem 2.1).

(4) If each  $\mathcal{P}$ -space is  $\nu$ -locally compact, then  $\mathfrak{R}(\mathcal{P})$  is closed under quotient images (Theorem 2.4).

(5) If  $X \in \mathfrak{R}(\mathcal{P})$  and  $Y$  is a locally compact, realcompact space with  $|Y| < m_1$ , then  $X \times Y \in \mathfrak{R}(\mathcal{P})$  [5, Corollary 2.2].

#### REFERENCES

1. R. L. Blair, *On  $\nu$ -embedded sets in topological spaces*, TOPO-72, Lecture Notes in Math., vol. 378, Springer-Verlag, Berlin and New York, 1974, pp. 46–79.
2. R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Z. **136** (1974), 41–52.
3. ———, *Notes on the Hewitt realcompactification of a product*, General Topology Appl. **5** (1975), 1–8.
4. W. W. Comfort, *Locally compact realcompactifications*, General Topology and its Relations to Modern Analysis and Algebra. II, Proc. Second Prague Topology Sympos., 1966, pp. 95–100.
5. ———, *On the Hewitt realcompactification of a product space*, Trans. Amer. Math. Soc. **131** (1968), 107–118.
6. W. W. Comfort and S. Negreponis, *Extending continuous functions on  $X \times Y$  to subsets of  $\beta X \times \beta Y$* , Fund. Math. **59** (1966), 1–12.
7. R. Engelking, *General topology*, Polish Scientific Publishers, Warsaw, 1977.
8. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.
9. I. Glicksberg, *Stone-Cech compactifications of products*, Trans. Amer. Math. Soc. **90** (1959), 369–382.
10. K. Hardy and I. Juhász, *Normality and weak cb property*, Pacific J. Math. **64** (1976), 167–172.
11. K. Hardy and R. G. Woods, *On  $c$ -realcompact spaces and locally bounded normal functions*, Pacific J. Math. **43** (1972), 647–656.
12. M. Hušek, *The Hewitt realcompactification of a product*, Comment. Math. Univ. Carolin. **11** (1970), 393–395.
13. ———, *Pseudo- $m$ -compactness and  $\nu(P \times Q)$* , Indag. Math. **33** (1971), 320–326.
14. ———, *Realcompactness of function spaces and  $\nu(P \times Q)$* , General Topology Appl. **2** (1972), 165–179.
15. Y. Ikeda, *Mappings and  $c$ -realcompact spaces*, Bull. Tokyo Gakugei Univ. (4) **28** (1976), 12–16.
16. S. Iliadis and S. Fomin, *The methods of centered systems in the theory of topological spaces*, Russian Math. Surveys **21** (1966), 37–62.
17. J. R. Isbell, *Uniform spaces*, Math. Surveys, no. 12, Amer. Math. Soc., Providence, R. I., 1964.
18. T. Ishii, *On the completions of maps*, Proc. Japan Acad. Ser. A Math. Sci. **50** (1974), 39–43.
19. T. Isiwata, *Generalizations of  $M$ -spaces*. I, II, Proc. Japan Acad. Ser. A Math. Sci. **45** (1969), 359–363, 364–367.
20. ———,  *$d$ -,  $d^*$ -maps and  $cb^*$ -spaces*, Bull. Tokyo Gakugei Univ. (4) **29** (1977), 19–52.
21. I. Juhász, *On closed discrete subspaces of product spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **17** (1969), 219–223.
22. J. Mack and D. G. Johnson, *The Dedekind completion of  $C(X)$* , Pacific J. Math. **20** (1967), 231–243.
23. W. G. McArthur, *Hewitt realcompactifications of products*, Canad. J. Math. **22** (1970), 645–656.
24. E. Michael, *Bi-quotient maps and cartesian products of quotient maps*, Ann. Inst. Fourier (Grenoble) **18** (1968), 287–302.
25. S. G. Mrówka, *Set theoretic constructions in topology*, Fund. Math. **94** (1977), 83–92.
26. N. Noble, *Countably compact and pseudocompact products*, Czechoslovak Math. J. **19** (1969), 390–397.

27. H. Ohta, *Topologically complete spaces and perfect maps*, Tsukuba J. Math. **1** (1977), 77–89.
28. \_\_\_\_\_, *Local compactness and Hewitt realcompactifications of products*, Proc. Amer. Math. Soc. **69** (1978), 339–343.
29. \_\_\_\_\_, *Some new characterizations of metrizable spaces* (to appear).
30. \_\_\_\_\_, *Local compactness and Hewitt realcompactifications of products. II* (to appear).
31. D. P. Strauss, *Extremally disconnected spaces*, Proc. Amer. Math. Soc. **18** (1967), 305–309.
32. M. Wage, *An easy counterexample to the inequality  $\dim(X \times Y) < \dim X + \dim Y$*  (to appear).

INSTITUTE OF MATHEMATICS, UNIVERSITY OF TSUKUBA, SAKURAMURA IBARAKI, 300-31 JAPAN

*Current address:* Faculty of Education, Shizuoka University, Ohya, Shizuoka, 422 Japan