DERIVATIONS AND AUTOMORPHISMS OF NONASSOCIATIVE MATRIX ALGEBRAS\textsuperscript{1}

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\textbf{Abstract.} This paper studies the derivation algebra and the automorphism group of $M_n(A)$, $n \times n$ matrices over an arbitrary nonassociative algebra $A$ with multiplicative identity $1$. The investigation also includes results on derivations and automorphisms of the algebras obtained from $M_n(A)$ using the Lie product $[xy] = xy - yx$, and the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$.

1. Introduction. Results concerning automorphisms and derivations of $n \times n$ matrices over a finite dimensional central associative division algebra have been known for some time. The Noether-Skolem theorem combined with a theorem due to Jacobson gives that automorphisms as well as derivations of these algebras are inner. This present paper generalizes these theorems by considering derivations and automorphisms of $M_n(A)$, $n \times n$ matrices over an arbitrary nonassociative algebra $A$ with $1$. The investigation also determines the derivations and automorphisms of the algebras obtained from $M_n(A)$ using the Lie and Jordan products. With one exception the derivation algebras are shown to all follow the same pattern. They consist of inner derivations by matrices with entries in the nucleus $N$ of $A$, and of derivations gotten by applying derivations of $A$ to each matrix entry. Every derivation is the sum of these two kinds of derivations. More variability arises in the automorphism groups. The common feature is a subgroup composed of conjugations by invertible matrices in $M_n(N)$, and of automorphisms obtained by applying automorphisms of $A$ entry by entry. Each element of this subgroup is a product of these two kinds of maps.

Our motivation to investigate such derivation algebras and automorphism groups was due in part to conversations with physicists concerned with building algebraic models in particle theories. Of particular interest to them were the cases when $A$ was taken to be an octonion algebra or a certain 7-dimensional noncommutative Jordan algebra \cite{1}. The results we present here have already been used in a negative sense to exclude some algebras from their considerations.

The main results concerning the group of antiautomorphisms and automorphisms of $M_n(A)$ are contained in Corollary 3.14. Theorem 4.8 and Corollaries 4.9, 4.10 describe the derivation algebras of $M_n(A)$, $L_n(A)$, $L'_n(A)$, and $K_n(A)$ where

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$L_n(A)$ is $M_n(A)$ under the product $[ ]$, $L'_n(A) = [L_n(A), L_n(A)]$ and $K'_n(A)$ is $L'_n(A)$ with its center factored out. We present results for the automorphism groups of $L_n(A)$, $L'_n(A)$, and $K'_n(A)$ in Theorems 4.12 and 4.13, and Corollary 4.14. Finally, if $J_n(A)$ denotes $M_n(A)$ viewed under the Jordan product, then its derivation algebra is obtained in Theorem 5.5, and its automorphism group is described in Theorem 5.6.

2. Basic concepts. Let $A$ be an arbitrary nonassociative algebra with identity $1$ over the field $\Phi$, and let $N$ denote the nucleus of $A$. That is

$$N = \{ a \in A | (a, b, c) = (b, a, c) = (b, c, a) = 0 \text{ for all } b, c \in A \}$$

where $(a, b, c) = (ab)c - (b, a, c) = (b, c, a) = 0$. The algebra $N$ is associative, and it is not difficult to show that if $a \in N$, then $ad_a: A \to A$ given by $ad_a(b) = [ab] = ab - ba$ is a derivation of $A$.

In the same manner we speak of the nucleus of $M_n(A)$. An easy calculation shows that it is $M_n(N)$, and as above for $x \in M_n(N)$, $ad_x$ is a derivation of $M_n(A)$. If $\text{Der } M_n(A)$ denotes the derivation algebra of $M_n(A)$, then $ad_{M_n(N)} = \{ ad_x | x \in M_n(N) \}$ is a subalgebra of $\text{Der } M_n(A)$. Let $e_{ij}$ denote the matrix with 1 in the $(i,j)$ position and 0 elsewhere. Then these matrix units lie in $M_n(N)$, and the derivations $ad_{e_{ij}}$ will be particularly useful in what follows.

In addition to the inner derivations $ad_{M_n(N)}$, one can also obtain a derivation of $M_n(A)$ by beginning with a derivation of $A$ and applying it to each matrix entry. The resulting derivations form a subalgebra of $\text{Der } M_n(A)$, which we denote by $(\text{Der } A)^\#$. The notation is suggestive of the fact that $(\text{Der } A)^\#$ is isomorphic to $\text{Der } A$. Similarly one can obtain an automorphism of $M_n(A)$ by applying an automorphism of $A$ to each entry. The resulting set of automorphisms is a subgroup, called $(\text{Aut } A)^\#$, of the full automorphism group $\text{Aut } M_n(A)$. If $u \in M_n(N)$ is invertible then conjugation by $u$, denoted by $\chi_u$, belongs to $\text{Aut } M_n(A)$. We write $\text{GL}(n, N)$ for the subgroup composed of the mappings $\chi_u$.

One final note, let us observe that if $\varphi \in \text{Aut } M_n(A)$ or $\varphi \in \text{Der } M_n(A)$, then $\varphi$ maps $M_n(N)$ into $M_n(N)$. This is true since $M_n(N)$ is characterized as the nucleus of $M_n(A)$, and derivations and automorphisms preserve the nucleus.

3. Generalized automorphisms and anti-automorphisms of $M_n(A)$. Given any algebra $A$ with nucleus $N$ let $Z(A) = \{ a \in N | [ab] = 0 \text{ for all } b \in A \}$ denote the center of $A$. We consider linear transformations $\varphi$ of $A$ onto $A$ with $\varphi(1) = 1$ and with the following property:

There exist idempotents $f, g \in Z(A)$ such that $f + g = 1, fg = 0$, and if $f' = \varphi(f), g' = \varphi(g)$, then $f', g' \in Z(A)$ and $\varphi: fA \to f'A$ is an algebra isomorphism and $\varphi: gA \to g'A$ is an anti-isomorphism.

Lemma 3.1. The set of all such mappings, $\text{GAut } A$, is a group.

Proof. Given $\varphi \in \text{GAut } A$, then the condition holds for $\varphi^{-1}$ and the idempotents $f', g'$. Suppose now $\varphi_1$ and $\varphi_2$ belong to $\text{GAut } A$ and the corresponding idempotents are $f_1, g_1$ and $f_2, g_2$. Then one can verify the above property is satisfied.
for \( \varphi_2 \varphi_1 \) using the idempotents \( f = f_2 f'_1 + g_2 g'_1 \) and \( g = f_2 g'_1 + g_2 f'_1 \) where \( f'_1 = \varphi_1(f_1) \) and \( g'_1 = \varphi_1(g_1) \). □

The group \( \text{GAut } A \) contains two distinguished subgroups, \( \text{Aut } A \) and \( \text{AAut } A \), where \( \text{AAut } A \) is the group consisting of all automorphisms and anti-automorphisms of \( A \). Corresponding to each \( \varphi \in \text{GAut } A \) is the decomposition of \( A \) into the direct sum of the ideals \( fA \) and \( gA \). So if \( A \) has no proper direct summands (for example if \( A \) is simple), then \( \text{GAut } A = \text{AAut } A \).

A straightforward calculation shows that each \( \varphi \in \text{GAut } A \) maps the nucleus of \( A \) onto itself.

Following the procedure above we define \( \text{GAut } M_n(A) \). The idempotents corresponding to elements in \( \text{GAut } M_n(A) \) lie in \( Z(M_n(A)) \) which is \( Z(A)I \), where \( I \) is the identity matrix. For each \( \varphi \in \text{GAut } M_n(A) \), \( \varphi(M_n(N)) = M_n(N) \) since \( \varphi \) preserves the nucleus, and \( \varphi \) restricted to \( M_n(N) \) lies in \( \text{GAut } M_n(N) \). In this section we investigate the structure of \( \text{GAut } M_n(A) \) under the assumption that \( N \) is Artinian. As the preceding remarks indicate the place to begin is with \( \text{GAut } M_n(N) \), but first some preliminary results brought to our attention by L. Levy [7, Lemma 3.2, p. 282].

**Lemma 3.2.** Let \( R \) be an associative ring with 1 and \( M \) be a right \( R \)-module such that
\[
M = M_1 \oplus \cdots \oplus M_n = M'_1 \oplus \cdots \oplus M'_n
\]
are two direct sum decompositions of \( M \) into submodules such that \( M_i \) is isomorphic to \( M'_i \) for \( 1 \leq i \leq n \). Then there is an invertible \( \sigma \) in the ring of \( R \)-endomorphisms of \( M \) such that \( \sigma^{-1} \pi_i \sigma = \pi'_i \) for each \( 1 \leq i \leq n \), where \( \pi_i \) (\( \pi'_i \)) denotes the projection of \( M \) onto \( M_i \) (\( M'_i \)).

**Proof.** For each \( i \) extend the isomorphism between \( M_i \) and \( M'_i \) to an \( R \)-endomorphism of \( M \) by defining it to be 0 on the other summands. Take \( \sigma \) to be the sum of the resulting \( n \) endomorphisms. □

**Lemma 3.3.** Let \( R \) be an associative ring with 1 and let \( \{e_i\} \) and \( \{f_i\} \) for \( i = 1, \ldots, n \) be two sets of orthogonal idempotents summing to 1. Assume for each \( i \), \( e_i R \) is isomorphic to \( f_i R \) as \( R \)-modules. Then there is a unit \( u \in R \) with \( u^{-1} f_i u = e_i \) for \( i = 1, \ldots, n \).

**Proof.** Apply the preceding lemma to
\[
R = e_1 R \oplus \cdots \oplus e_n R = f_1 R \oplus \cdots \oplus f_n R.
\]
The projection \( \pi_i \) (\( \pi'_i \)) is just left multiplication by \( e_i \) (\( f_i \)). The \( R \)-isomorphism \( \sigma \) is left multiplication by \( u = \sigma(1) \), and \( \sigma^{-1} \) is left multiplication by \( u^{-1} \). Then
\[
e_i = \pi_i(1) = \sigma^{-1} \pi'_i \sigma(1) = \sigma^{-1} \pi'_i(u) = \sigma^{-1} (f_i u) = u^{-1} f_i u.
\]

Given \( \varphi \in \text{GAut } M_n(A) \) direct verification shows that \( \varphi \) is an automorphism of \( J_n(A) \), where \( J_n(A) \) is \( M_n(A) \) under the Jordan product \( x \circ y = \frac{1}{2}(xy + yx) \) provided \( \text{char } \Phi \neq 2 \). It is convenient for what follows in §5 to work with the larger group \( \text{Aut } J_n(A) \). Though we disallow \( \text{char } \Phi = 2 \) in this process, this restriction is unnecessary in dealing with \( \text{GAut } M_n(A) \).
Jacobson [4, Theorem 7.4, p. 26] has shown that as a consequence of results in either [4] or [6] the following theorem holds:

Let $A$ and $B$ be arbitrary algebras with identities, and let $B_j$ be the algebra obtained from $B$ under the Jordan product. Suppose $\varphi: J_n(A) \to B_j$ is a Jordan homomorphism such that the images of the matrix units $e_{ij}$ lie in the nucleus of the algebra $C$ generated by $J_n(A)$. Then $C = C_1 \oplus C_2$ where the $C_i$ are ideals of $C$, and if $\pi_i$ denotes the projection of $C$ onto $C_i$, then $\pi_i \varphi$ is a homomorphism, and $\pi_2 \varphi$ an antihomomorphism.

We could use this theorem to reduce our considerations to automorphisms and antiautomorphisms. However, we do not follow this line of attack because one virtue of working with $GAut M_n(A)$ is that it allows us to study automorphisms and antiautomorphisms simultaneously, and because we can obtain in the process results concerning automorphisms of $J_n(A)$ with the property that $\varphi(e_{ii}) = e_{ii}$ for all $i$, without assuming from the outset that all the $\varphi(e_{ij})$ lie in the nucleus.

**Lemma 3.4.** Assume $\text{char } \Phi \neq 2$, and let $\varphi \in Aut J_n(N)$, where $N$ is an associative Artinian ring with $1$. Then there is an invertible $u \in M_n(N)$ such that

$$u^{-1} \varphi(e_{ii}) u = e_{ii}.$$  

**Proof.** The elements $\varphi(e_{ii})$ for $i = 1, \ldots, n$ are orthogonal idempotents summing to $1$. Thus,

$$M_n(N) = \varphi(e_{11})M_n(N) \oplus \cdots \oplus \varphi(e_{nn})M_n(N) = e_{11}M_n(N) \oplus \cdots \oplus e_{nn}M_n(N).$$

Jacobson and Rickart [6, Lemma 3, p. 487] have shown that there are matrix units $\{g_{ij}\}$ and $\{h_{ij}\}$ such that $\varphi(e_{ij}) = g_{ij} + h_{ij}$ and $h_{ij}h_{kl} = 0 = h_{kl}g_{ij}$ for all $i, j, k, l$. Therefore, $\varphi(e_{ii})M_n(N)$ is isomorphic as an $M_n(N)$-module to $\varphi(e_{ii})M_n(N)$ via left multiplication by $g_{ii} + h_{ii}$. Thus, $M_n(N)$ is isomorphic to the direct sum of $n$-copies of $\varphi(e_{ii})M_n(N)$, and we write this $M_n(N) \approx \varphi(e_{ii})M_n(N)^{(n)}$. Similarly $e_{ii}M_n(N)$ is isomorphic as a right $M_n(N)$-module to $e_{ii}M_n(N)$ using left multiplication by $e_{ii}$. Hence, $e_{ii}M_n(N)^{(n)} \approx M_n(N) \approx \varphi(e_{ii})M_n(N)^{(n)}$. Breaking $e_{ii}M_n(N)$ into a finite number of indecomposable $M_n(N)$-modules, and collecting isomorphic indecomposables, one can use the Krull-Schmidt theorem to argue that $e_{ii}M_n(N)$ is isomorphic to $\varphi(e_{ii})M_n(N)$. Thus by the preceding lemma, there is an invertible $u \in M_n(N)$ so that $u^{-1} \varphi(e_{ii}) u = e_{ii}$ as desired. □

**Corollary 3.5.** Let $A$ be an algebra with $1$ such that the nucleus $N$ of $A$ is Artinian. Then for each $\varphi \in GAut M_n(A)$, there is an invertible $u \in M_n(N)$ so that $\psi = \chi_u \varphi \in GAut M_n(A)$ has the property that $\psi(e_{ii}) = e_{ii}$ for $i = 1, \ldots, n$.

Automorphisms of $J_n(A)$ with the property that $\psi(e_{ii}) = e_{ii}$ will be the topic of the next lemma, but one additional piece of notation is necessary.

Suppose $\theta \in GAut A$ with idempotents $f, g$. Define $\theta^\#: M_n(A) \to M_n(A)$ by letting $\theta^\#$ on $fI \cdot M_n(A)$ be $\theta$ applied to each entry, and letting $\theta^\#$ on $gI \cdot M_n(A)$ be $\theta$ applied to each entry followed by taking the transpose of the resulting matrix.
One can show $\theta^* \in \text{GAut } M_n(A)$. Let $(\text{GAut } A)^*$ denote the group of these mappings, and observe it is isomorphic to the group $\text{GAut } A$.

**Theorem 3.6.** Assume $\text{char } \Phi \neq 2$, and suppose $\psi \in \text{Aut } J_n(A)$ with $\psi(e_{ii}) = e_{ii}$ for all $i = 1, \ldots, n$. Then there is an invertible $\nu \in M_n(N)$ such that $\chi_\nu \psi \in (\text{GAut } A)^*$.

**Proof.** There are only five types of nonzero products in $J_n(A)$, and for $i, j, k$ distinct they are

\begin{align*}
ae_{ii} \circ be_{ij} &= (a \circ b)e_{ii}, \\
ae_{ii} \circ be_{ij} &= \frac{1}{2} a be_{ij}, \\
be_{ij} \circ be_{jj} &= \frac{1}{2} a be_{ij}, \\
be_{ij} \circ be_{jk} &= \frac{1}{2} a be_{jk}, \\
be_{ij} \circ be_{ji} &= \frac{1}{2} (a be_{ii} + b ae_{ij}).
\end{align*}

We apply $\psi$ to these relations, and make different specializations of $a$ and $b$.

Relation (3.1) with $b = 1$ says $\psi(ae_{ii}) = \psi(ae_{ii}) \circ e_{ii}$. But the first three equations above demonstrate the fact that the only elements of $J_n(A)$ which lie in the 1-eigenspace relative to right multiplication by $e_{ii}$ are of the form $ce_{ii}$. Therefore

\begin{equation}
\psi(ae_{ii}) = \psi(a)e_{ij}.
\end{equation}

Again by using (3.1) we see $\psi_i$ is an automorphism of $A$ under the product $a \circ b = \frac{1}{2}(ab + ba)$.

From (3.2) and (3.3) it follows that $\psi(ae_{ij})$ must lie in $Ae_{ij} + Ae_{ji}$, which is the intersection of the $\frac{1}{2}$-eigenspace relative to multiplication by $e_{ij}$ with the $\frac{1}{2}$-eigenspace relative to multiplication by $e_{ji}$. As a special case of this fact:

\begin{equation}
\psi(e_{ij}) = \alpha_i e_{ij} + \beta_j e_{ji}.
\end{equation}

It follows directly from $e_{ij} \circ e_{ij} = 0$ that

\begin{equation}
\alpha_i \beta_{ji} = 0 = \beta_j \alpha_i.
\end{equation}

Now using (3.4) and (3.5) with $a = b = 1$, we deduce further results concerning the $\alpha$'s and $\beta$'s, namely

\begin{align*}
\alpha_i \alpha_j &= \alpha_{ik}, \\
\beta_k \beta_j &= \beta_{ki}, \\
\alpha_j \alpha_i + \beta_i \beta_j &= 1.
\end{align*}

If $\psi$ belongs to $\text{GAut } M_n(A)$, then it is immediate that $\alpha_{ij}, \beta_{ji} \in N$. We do not make this assumption in order to use Theorem 3.6 in §5. Instead the fact that $\alpha_{ij}, \beta_{ji} \in N$ will be a consequence of the next few steps.

**Lemma 3.7.** $\alpha_i, \beta_j$ lie in the middle nucleus of $A$ for all $i \neq j$.

**Proof.** If $\psi$ is applied to the relation $0 = (ae_{ii} \circ e_{ij}) \circ be_{jj} - ae_{ii} \circ (e_{ij} \circ be_{jj})$, then the $(i, j)$ component of the resulting equation shows that the associator $(\psi_i(a), \alpha_j, \psi_j(b)) = 0$, and the $(j, i)$ component shows that $(\psi_i(b), \beta_{ji}, \psi_i(a)) = 0$. Since $\psi_i, \psi_j$ are onto, the proof is complete. \(\square\)
Let us define \( f_i = \alpha_j \alpha_{ji} \) and \( g_i = \beta_j \beta_{ji} \). Note as a result of Lemma 3.7 and (3.9), if \( n > 3 \) then
\[
f_i = \alpha_j \alpha_{ji} = \alpha_k \alpha_{kj} \alpha_{ji} = \alpha_k \alpha_{ki}.
\]
This shows \( f_i \) does not depend on the \( j \) used to define it as long as \( j \neq i \). Similarly \( g_i \) is also independent of \( j \).

**Lemma 3.8.** For each \( i \), \( f_i \) and \( g_i \) are orthogonal idempotents summing to 1, and they commute with the elements of \( A \). Moreover \( f_i = f_j \), \( g_i = g_j \) for all \( i, j \).

**Proof.** Equation (3.9) says \( f_i + g_i = 1 \). Now
\[
f_i^2 = \alpha_j \alpha_{ji} \alpha_j \alpha_{ji} = \alpha_j (1 - \beta_j \beta_{ji}) \alpha_{ji} = \alpha_j \alpha_{ji} = f_i,
\]
so that \( f_i \) and \( g_i = 1 - f_i \) are idempotents. Multiplying \( f_i + g_i = 1 \) on the right by \( f_i \), then on the left, gives \( f_i^2 + g_i f_i = f_i \) and \( f_i^2 + f_i g_i = f_i \). Thus \( g_i f_i = 0 = f_i g_i \), and they are orthogonal.

To obtain the rest of the conclusions we return to (3.2), set \( b = 1 \), and act on the equation with \( \psi \). The result is a formula for \( \psi(\alpha e_j) \)
\[
\psi(\alpha e_j) = \psi_i(\alpha) \alpha_j e_{ji} + \beta_j \psi_i(\alpha) e_{ji}.
\]
Using this formula let us calculate the \((i, i)\)-coefficient of both sides of
\[
2\psi(\alpha e_i + \alpha e_j) \circ \psi(e_j)
\]
to establish that \( \psi_i(\alpha) = \psi_i(\alpha) \alpha_j \alpha_{ji} + \beta_j \beta_{ji} \psi_i(\alpha) = \psi_i(\alpha) f_i + g_i \psi_i(\alpha) \). Since \( \psi_i \) is onto, we can replace \( \psi_i(\alpha) \) with a \( a \) to obtain
\[
a = a f_i + g_i a \quad \text{for all } a \in A. 
\]
Multiplying \( 1 = f_i + g_i \) on the right by \( a \) shows that \( a = f_i a + g_i a \). Comparing this result with (3.12) gives \( a f_i = f_i a \) for all \( a \in A \), and hence \( g_i a = a g_i \) also.

Finally \( f_i = f_i^2 = \alpha_j \alpha_{ji} \alpha_j \alpha_{ji} = \alpha_j f_j \alpha_{ji} = \alpha_j \alpha_f j = f_i f_j \). Since this is true for each \( i \) and \( j \), \( f_j = f_j f_i = f_i f_j = f_i \). Thus \( g_i = 1 - f_i = 1 - f_j = g_j \) and the proof is finished. □

We shall just write \( f' \) for \( f_i \) and \( g' \) for \( g_i \) hereafter.

**Lemma 3.9.** The elements \( \alpha_j \) and \( \beta_j \) lie in \( N \) for every pair \( i, j \) with \( i \neq j \).

**Proof.** In analogy with equation (3.11), if \( \psi \) is applied to (3.3) with \( a = 1 \), we obtain
\[
\psi(b e_j) = \alpha_j \psi_j(b) e_{ji} + \psi_j(b) \beta_j e_{ji}.
\]
We freely use these two equations in the proof of this lemma. First, let us apply \( \psi \) to \( (a e_j \circ b e_j) \circ e_j - a e_j \circ (b e_j \circ e_j) = 0 \). The \((j, j)\) coefficient is
\[
((\beta_j \psi_j(\alpha))(\psi_j(b))) \alpha_{ji} + \beta_j (\psi_j(b) (\psi_j(\alpha) \alpha_{ji}))
- (\beta_j \psi_j(\alpha))(\psi_j(\alpha) \alpha_{ji}) - (\beta_j \psi_j(b))(\psi_j(\alpha) \alpha_{ji}) = 0.
\]
Since \( \psi_j \) is onto, we can replace \( \psi_j(b) \) with any element \( c \) in \( A \), and \( \psi_j(\alpha) \) with an arbitrary element of \( A \), so let us use \( (\beta_j d')g' \). As a consequence of (3.8) and Lemma 3.8, \( g' \alpha_{ji} = \beta_j \beta_j \alpha_{ji} = 0 \), and thus, \( (\beta_j d')g' \alpha_{ji} = 0 \). Now
\[ \beta_{ji}(\beta_{ij}d)g' = \beta_{ji}(g'\beta_{ij}d) = (\beta_{ji}g')(\beta_{ij}d) = (g'\beta_{ij})(\beta_{ij}d) = (g'\beta_{ij}\beta_{ij})d = g'd, \]

so that from this judicious choice of \( \psi_i(a) \), we deduce

\[ (g'd, c, \alpha_{ij}) = 0. \quad (3.14) \]

If in the above equation the substitution of \( f'(da_{ji}) \) for \( \psi_i(a) \) is made, the result is

\[ (\beta_{ji}, c, f'd) = 0. \quad (3.15) \]

To obtain the other half of the proof that \( \alpha_{ij} \) is in the right nucleus and \( \beta_{ji} \) in the left, we apply \( \psi \) to the relation \( (ae_{ji} \circ be_{ij}) \circ e_{ij} - ae_{ji} \circ (be_{ij} \circ e_{ij}) = \frac{1}{4}[ab]e_{ij} \). The \( (j,j) \) coefficient of the left side must be 0, and in the resulting equation for the \( (j,j) \) coefficient we perform the replacements of \( \psi_i(b) \) with \( c \), and \( \psi_i(a) \) with \( (\alpha_{ij}d)f' \) then with \( g'(d\beta_{ji}) \). The effect is

\[ (f'd, c, \alpha_{ij}) = 0. \quad (3.16) \]
\[ (\beta_{ji}, c, g'd) = 0. \quad (3.17) \]

From relations (3.14)–(3.17) we conclude that \( \alpha_{ij} \) is in the right nucleus and \( \beta_{ji} \) in the left. The appropriate equations to use to achieve \( \alpha_{ij} \) in the left nucleus and \( \beta_{ji} \) in the right are

\[ (e_{ij} \circ ae_{ij}) \circ be_{ij} - e_{ij} \circ (ae_{ij} \circ be_{ij}) = 0, \]
\[ (e_{ij} \circ ae_{ij}) \circ be_{ij} - e_{ij} \circ (ae_{ij} \circ be_{ij}) = \frac{1}{4}[ba]e_{ij}. \]

The substitutions needed in the first equation are \( g'(d\beta_{ji}) \) and \( (\alpha_{ij}d)f' \) for \( \psi_i(b) \), and in the second \( f'(da_{ij}) \) and \( (\beta_{ji}d)g' \) for \( \psi_i(b) \).

**Corollary 3.10.** \( f', g' \in Z(A) \).

**Lemma 3.11.** Let \( f = \psi_i^{-1}(f') \) and \( g = \psi_i^{-1}(g') \). Then \( f, g \) are orthogonal idempotents in the center of \( A \) which sum to 1. They are independent of the defining \( i \). Moreover, \( \psi_i \) is an isomorphism from \( fA \) onto \( f'A \), and an anti-isomorphism from \( gA \) onto \( g'A \), so that \( \psi_i \in GAut A \).

**Proof.** That \( f, g \) are idempotents summing to 1 follows simply from the fact that \( \psi_i \) is an automorphism relative to the \( \circ \)-product. Orthogonality is shown as it was for \( f', g' \) above. To prove the remaining statements we observe that from \( \psi_i(ab_{ij}) = 2\psi_i(ae_{ij}) \circ \psi_i(be_{ij}) \) we can establish: \( \psi_i(ab_{ij})\alpha_{ij} = \psi_i(a)\psi_i(b)\alpha_{ij} \) and \( \beta_{ji}\psi_i(ab_{ij}) = \beta_{ji}\psi_i(b)\psi_i(a) \). Multiplication by \( \alpha_{ji} \) and \( \beta_{ji} \) respectively demonstrates

\[ \psi_i(ab_{ij})f' = \psi_i(a)\psi_i(b)f', \quad (3.18) \]
\[ g'\psi_i(ab_{ij}) = g'\psi_i(b)\psi_i(a). \quad (3.19) \]

But then equation (3.19) with \( a = f \) shows \( g'\psi_i(fb) = g'\psi_i(b)f' = g'f'\psi_i(b) = 0. \) Hence, \( \psi_i \) maps \( fA \) into \( f'A \), and it is an isomorphism on \( fA \) according to (3.18). Likewise \( \psi_i \) is an anti-isomorphism of \( gA \) into \( g'A \). We use the fact that \( A = fA + gA \), and the fact that \( \psi_i \) is onto to conclude it is onto the components \( fA \) and \( gA \).

What remains to be shown is that \( f, g \in Z(A) \) and \( f, g \) are independent of \( i \). We note that \( \psi_i(af) = \psi_i(af)f' \), and \( \psi_i(fa) = \psi_i(fa)f' \) since \( f' \) is the identity on \( f'A \).
Therefore (3.18) shows that \( \psi(af) = \psi(a)f' \) and also \( \psi(fa) = f'\psi(a) = \psi(a)f' \). Hence \( \psi(af) = \psi(fa) \) for all \( a \), and \( af = fa \). To prove that \( f \) lies in the nucleus of \( A \), it suffices to show \( f \) acts as the identity on \( fA \) and \( fA \cdot gA = 0 \). But this can be accomplished by using (3.18) with \( af \) instead of \( a \), and \( f \) instead of \( b \), and by using (3.19) with \( af \) in place of \( a \), and \( bg \) in place of \( b \). Thus, \( f \in Z(A) \) and \( g = 1 - f \in Z(A) \) also.

Now relations (3.11) and (3.13) demonstrate

\[
\alpha_j \psi_j(a) = \psi_j(a) \alpha_j, \quad \beta_j \psi_j(a) = \beta_j \psi_j(a).\tag{3.20, 3.21}
\]

These equations show that \( f' \psi_j(a) = \alpha_j \psi_j(a) \alpha_j \) and \( \psi_j(a)g' = \beta_j \psi_j(a) \beta_j \). Substituting \( a = f \) we see \( f' \psi_j(f) = f' \) and \( \psi_j(f)g' = 0 \). Thus \( \psi_j(f) = f' \) and we are done. \( \square \)

**Lemma 3.12.** There is an invertible diagonal matrix \( v \) in \( \mathbb{M}_n(N) \) such that \( \chi_v \psi = \psi_1^\# \).

**Proof.** It is an easy consequence of (3.8) that \( \beta_j f' A = f' A \beta_j = g' A \alpha_j = \alpha_j g' A \) = 0 for all \( i \neq j \), and from the remarks immediately following equations (3.20) and (3.21), it follows that \( \psi_j(fa) = f' \psi_j(fa) = \alpha_j \psi_j(fa) \alpha_j \) and \( \psi_j(ga) = g' \psi_j(ga) = \beta_j \psi_j(ga) \beta_j \). Therefore writing \( a = fa + ga \), we conclude from these observations that

\[
\psi_j(a) = (\alpha_j + \beta_j) \psi_j(a)(\alpha_j + \beta_j) \quad \text{for all } a. \tag{3.22}
\]

Let \( v \) denote the matrix with 1 in the \((1, 1)\) position, \( \alpha_j + \beta_j \) in the \((j, j)\) slot for \( j > 2 \), and 0 elsewhere. Then \( v^{-1} \) is also diagonal with diagonal entries 1, \( \alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n \). Thus, \( \chi_v \psi(ae_j) = v^{-1} \psi_j(a)e_jv = \psi_j(a)e_j \). Calculation using (3.8), (3.10) and (3.22) shows

\[
\chi_v \psi(ae_j) = (\alpha_j + \beta_j)(\alpha_j + \beta_j) \psi_1(a)(\alpha_j + \beta_j) \psi_1(a)(\alpha_j + \beta_j) e_j + (\alpha_j + \beta_j) \beta_j \alpha_j + \beta_j(\alpha_j + \beta_j) \alpha_j + \beta_j(\alpha_j + \beta_j) e_j = \psi_1(a) \alpha_j e_j + \beta_j \beta_j \psi_1(a) e_j = \psi_1(af)e_j + \psi_1(ga) e_j = \psi_1(fa)e_j + \psi_1(ga)e_j.\]

Hence, \( \chi_v \psi = \psi_1^\# \in (GAut A)^\# \), and this concludes the proof of Lemma 3.12, and Theorem 3.6. \( \square \)

As a consequence of this theorem and Corollary 3.5 we have

**Theorem 3.13.** Let \( A \) be an arbitrary nonassociative algebra with 1 such that the nucleus \( N \) of \( A \) is Artinian. If \( \text{char } \Phi \neq 2 \), then

\[
GAut M_n(A) = GL(n, N) \cdot (GAut A)^\#.
\]

**Remark.** It should be noted that the subgroup \( GL(n, N) \) of conjugations is a normal subgroup of \( GAut M_n(A) \), hence of \( Aut M_n(A) \) and of \( AAut M_n(A) \) as well,
and it equals
\[
\text{GAut}_A M_n(A) = \{ \varphi \in \text{GAut} M_n(A) | \varphi(aI) = aI \text{ for all } a \in A \}
\]
if \( Z(A) = N \).

**Corollary 3.14.** With assumptions as in Theorem 3.13,
\[
\text{Aut}_n(A) = \text{GL}(n, N) \cdot (\text{Aut } A)^\#
\]
and
\[
\text{AAut}_n(A) = \text{GL}(n, N) \cdot (\text{AAut } A)^\#,
\]
so that \( M_n(A) \) has antiautomorphisms if and only if \( A \) does.

**Proposition 3.15.** If \( N \) is Artinian, \( \text{Aut}_n(A) \) has finite index in \( \text{GAut}_n(A) \).

**Proof.** Since \( N \) is Artinian with 1, we can find orthogonal central idempotents
\( e_1, \ldots, e_q \) such that \( 1 = e_1 + \cdots + e_q \) and each \( e_i \) cannot be expressed as the sum of two other central idempotents. Then \( M_n(N) = \bigoplus_{i=1}^q S_i \) where \( S_i = e_i \cdot M_n(N) \) is an indecomposable ideal of \( M_n(N) \). For any ideal \( T \) of \( M_n(N) \), \( T = \bigoplus_{i=1}^q e_i T \), so if \( T \) is indecomposable, \( T = S_j \) for some \( j \). Given \( \varphi \in \text{GAut}_n(A) \) with idempotents \( fI, gI, f'l, g'l \), then \( fe_i + ge_i = e_i = f'e_i + g'e_i \) implies \( fI \cdot M_n(N), gI \cdot M_n(N), f'l \cdot M_n(N), g'l \cdot M_n(N) \) are all just the sums of certain of the \( S_j \). We conclude that each \( \varphi(S_j) \) is an indecomposable ideal of \( M_n(N) \) and so equals some \( S_j \). Thus, \( \varphi \) permutes the summands and the subgroup of generalized automorphisms fixing all the \( S_j \) has index at most \( q! \) in \( \text{GAut}_n(A) \). Since every \( \psi \) in that subgroup acts as an automorphism or antiautomorphism on each \( S_j \), the index of \( \text{Aut}_n(A) \) in \( \text{GAut}_n(A) \) is at most \( q!2^q \). \( \square \)

The next example demonstrates that Corollary 3.14 is false for \( N \) arbitrary.

Let \( \Phi[t] \) denote the ring of polynomials over \( \Phi \), and let \( B \) be the ideal of \( \Phi[t] \) generated by \( t^2 + 1 \). Define \( A \) to be \( \{(a, b) | a, b \in \Phi[t] \text{ and } a \equiv b \text{ mod } B \} \). If \( w \) denotes the matrix
\[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]
then \( w \) lies in \( M_2(\Phi[t] \times \Phi[t]) \) but not in \( M_2(A) \). The matrix \( w \) is invertible, and conjugation by \( w \) leaves \( M_2(A) \) invariant. If \( \chi_w = \chi_\theta^\# \) for \( \theta \in \text{Aut}_n(A) \), and \( \theta^\# \in (\text{Aut } A)^\# \), then for each \( c \in A \), \( cI = \chi_w(cI) = \chi_\theta^\#(cI) = \theta(c)I \), which implies \( \theta = 1 \). Now if \( \chi_w = \chi_v \), then
\[
w = \begin{pmatrix}
(r, s) & 0 \\
0 & (r, s)
\end{pmatrix}
\]
where \( r, s \) are nonzero elements of \( \Phi[t] \). The determinant of \( v \) must be a unit in \( A \), so it has the form \( (\alpha, \alpha) \) where \( \alpha \in \Phi \). From taking determinants we conclude \( (1, -1) = (r^2, s^2)(\alpha, \alpha) \), and hence both \( r \) and \( s \) are in \( \Phi \). We write
\[
v = \begin{pmatrix}
u_1 & 0 \\
u_2 & v_2 \\
u_3 & v_3 \\
u_4 & v_4
\end{pmatrix}
\]
and compute
\[ w = \begin{pmatrix} (r, s) & 0 \\ 0 & (r, s) \end{pmatrix} v. \]

Then \( 1 = ru \) where \( r \in \Phi \), and since \( (u, v_i) \in A \), it follows \( v_i = r^{-1} \) also. But then \( -t = sv_i = sr^{-1} \in \Phi \), a contradiction. Consequently \( \chi_w \in \text{Aut} M_2(A) \), and \( \chi_w \notin \text{GL}(2, A) \cdot (\text{Aut} A)^n \).

**Remark.** We are indebted to L. Levy for his assistance in the creation of this example and for making us aware of work by Rosenberg and Zelinsky [10] on this subject.

4. \( M_n(A) \) under the Lie product. Let \( L_n(A) \) denote the anticommutative algebra obtained by taking \( M_n(A) \) under the product \([xy] = xy - yx\), and let \( L'_n(A) = [L_n(A), L_n(A)]\). The algebra \( L'_n(A) \) is an ideal of \( L_n(A) \) which is invariant under all automorphisms and derivations of \( L_n(A) \). It is spanned by the elements

\[ \begin{align*}
\delta_{ij} &= [a e_{ii}, e_{jj}], \\
\delta_{ij} &= [a e_{ii}, b e_{jj}], \quad i \neq j.
\end{align*} \]

Two noteworthy cases of the second equation occur when \( i = j \) and when \( b = 1 \). The elements obtained are \([a b]e_{ii} \) and \( a e_{ii} - a e_{jj} \). These elements in fact generate all of the ones of the second type since \( a b e_{ii} - b e_{jj} = [a b]e_{ii} + b a e_{jj} - b e_{jj} \). Each element in \( L'_n(A) \) has trace in \([A, A]\). Conversely if \( x = \sum_{i=1}^n a_{ii}e_{ii} \) where \( t = \sum_{i=1}^n a_{ii} \in [A, A] \), then \( x = te_{nn} + \sum_{i=1}^{n-1} a_{ii}e_{ii} - a_{ii}e_{nn} \in L'_n(A) \). Thus \( L'_n(A) \) is precisely the space of all elements with trace in \([A, A]\). Using the above products one can also verify that if \( n > 3 \) when char \( \Phi = 2 \), or if \( n > 2 \) when char \( \Phi \neq 2 \), then \( L''_n(A) = [L'_n(A), L''_n(A)] \).

Let \( Z \) denote the center of \( L_n(A) \); that is \( Z = \{ z \in L_n(A) \mid [z, x] = 0 \text{ for all } x \in L_n(A) \} \). If \( C(A) = \{ c \in A \mid ca = ac \text{ for all } a \in A \} \), then calculation shows that \( Z = \{ c^2 \mid c \in C(A) \} \). The center is an ideal of \( L_n(A) \) invariant under automorphisms and derivations of \( L_n(A) \). Its intersection with \( L'_n(A) \) is the center of \( L'_n(A) \) and is also invariant. We define \( K_n(A) = L'_n(A)/Z \cap L''_n(A) \). Our goal in this section will be to investigate the derivation algebras of \( M_n(A), L_n(A), L'_n(A), \) and \( K_n(A) \), and the automorphism groups of the last three. Each derivation (automorphism) of \( M_n(A) \) induces a derivation (automorphism) of \( L_n(A) \). This assertion remains true for the pairs \( L_n(A), L'_n(A) \) and \( L''_n(A), K_n(A) \).

**Lemma 4.1.** Suppose \( n > 3 \) if char \( \Phi = 2 \), or \( n > 2 \) if char \( \Phi \neq 2 \). Then
(a) \( \sigma, \rho, \) and \( \tau \) are one-to-one, and
(b) the kernel of \( \sigma \) consists of all Lie homomorphisms of \( L_n(A) \) into \( Z \), denoted by \( L\text{-Hom}(L_n(A), Z) \).

**Proof.** If \( \delta \) is in the kernel of \( \tau \), then \( \delta \) maps \( L'_n(A) \) into \( Z \). But then \( L'_n(A) = [L_n(A), L'_n(A)] \) is sent to \( 0 \), which shows \( \delta = 0 \). Since \( L_n(A) = M_n(A) \) as vector spaces, the kernel of \( \rho \) must be \( 0 \). Now any derivation \( \vartheta \) in the kernel of \( \sigma \rho \)
must be 0 on the elements $ae_j$ for $i \neq j$. Since these elements generate $M_n(A)$ under the usual matrix product, it follows that $\partial$ is identically 0. Turning our attention to part (b), we assume $\partial$ is in the kernel of $\sigma$. Then $\partial$ is 0 on the elements in (4.1) so we need only examine the effect of $\partial$ on elements of the form $ae_j$. For $k \neq l$ we have $[ae_j, be_k] = \delta_{jk} a e_{jl} - \delta_{lj} b e_{kj}$, and $\partial$ applied to this relation shows $[\partial(ae_j), be_k] = 0$ for all $b \in A$. From this we deduce $\partial(ae_j) \in Z$. Any Lie homomorphism of $L_n(A)$ into $Z$ is a derivation of $L_n(A)$ which necessarily vanishes on $L_n(A)$. Thus the kernel of $\sigma$ is $\text{LHom}(L_n(A), Z)$ as claimed. □

In a similar fashion we have the natural homomorphisms

$$\text{Aut} M_n(A) \overset{\rho}{\to} \text{Aut} L_n(A) \overset{\sigma}{\to} \text{Aut} L_n(A) \overset{\tau}{\to} \text{Aut} K_n(A).$$

**Lemma 4.2.** Assume $n > 3$ if $\text{char } \Phi = 2$, or $n > 2$ if $\text{char } \Phi \neq 2$. Then

(a) $\sigma \rho$, $\rho$, and $\tau$ are one-to-one, and

(b) the kernel of $\sigma$ consists of all mappings $\varphi$ such that $\varphi$ is the identity map plus an element in $\text{LHom}(L_n(A), Z)$.

Since the proof of 4.2 involves only a slight modification of the proof of 4.1, we omit it.

The Lie nucleus, which we introduce next, will play the role that $M_n(N)$ played in our investigations of $\text{GAut} M_n(A)$.

For each anticommutative algebra $L$ with product $[ \ ]$ the Lie nucleus of $L$ is

$$\nu(L) = \{ x \in L | [ [xy]z] + [ [yz]x] + [ [zx]y] = 0 \text{ for all } y, z \in L \}. $$

If $\text{ad}_x$ denotes the map $\text{ad}_x(y) = [xy]$, then an alternate description of the Lie nucleus is

$$\nu(L) = \{ x \in L | \text{ad}_x \in \text{Der } L \}. $$

The space $\nu(L)$ is invariant under automorphisms and derivations of $L$. As a result, if $x, x' \in \nu(L)$ then $\text{ad}_x(x') \in \nu(L)$, and $\nu(L)$ is a subalgebra of $L$. It follows then that $\nu(L)$ is a Lie algebra.

Let us specialize now to the case $L = L_n(A)$. We define $D(A) = \{ a \in A | \text{ad}_a \in \text{Der } A \}$, and note that this amounts to saying $D(A) = \{ a \in A | [a, bc] = [ab]c + b[ac] \}$.

**Lemma 4.3.** Assume $n > 3$. Then $\nu(L_n(A)) = L_n(N) + D(A)I$.

**Proof.** Using the definition of $D(A)I$ above and the fact that $M_n(N)$ is the nucleus of $M_n(A)$, one can show that $L_n(N) + D(A)I \subseteq \nu(L_n(A))$. For the proof of the reverse containment let

$$\lambda(x, y, z) = [ [xy]z] + [ [yz]x] + [ [zx]y]$$

and assume $x = \sum b_i e_{ij} \in \nu(L_n(A))$. Then since $\nu(L_n(A))$ is a subalgebra containing $L_n(N)$, for $k, l, m$ distinct we have $[ [xe_{ik}]e_{mk}]e_{ml} = b_{kl}e_{ik} \in \nu(L_n(A))$. Dropping the subscript on $b_{kl}$ for simplicity, we have $be_{lk} \in \nu(L_n(A))$ for $l \neq k$. From $\lambda(be_{lk}, ae_{km}, ce_{mk}) = 0$ for all $a, c \in A$ we obtain $(ba)ce_{lk} - b(ac)e_{lk} = 0$, and hence $(ba)c = b(ac)$. Similarly $\lambda(ae_{mi}, be_{lk}, ce_{kl}) = 0$ and $\lambda(ae_{mk}, ce_{kl}, be_{lk}) = 0$.
imply \((ab)c = a(bc)\) and \((ac)b = a(cb)\). We conclude \(b\) is in \(N\), and \(be_{ik} \in L_n(N) \subseteq v(L_n(A))\).

It suffices to suppose \(x = \sum_{i=1}^n b_ie_{ii} \in v(L_n(A))\). For \(k \neq l\) we have

\[ [x, e_{kl}] = (b_k - b_l)e_{kl} \in v(L_n(A)) \]

and by the above \(b_k - b_l \in N\). Now a computation of \(\lambda(x, ae_{qr}, ce_{sr})\) shows

\[
\delta_{rs}(b_qa - ab_r)ce_{qr} - \delta_{rq}(b_qa - ab_r)e_{sr}
\]

This expression can be made more transparent by writing for each \(i\) and \(j\)

\[ b_i = b_i + \beta_i, \ b_j = b_j + \gamma_j \text{ where } \beta_i, \gamma_j \in N. \]

The result is

\[
0 = \delta_{rs}\{(b_q) [c + [ac, b_i] + a[cb_q] + (a\beta_i)c - a(\beta_i)c - b_i]e_{qr} - \delta_{rs}(c[a, b_i] + [ca, b_i] - [c]_b_i + (c\gamma_i)a - c(\gamma_i)a)e_{sr}.\}
\]

This proves that \(b_q \in D(A)\) for each \(q\) and hence, \(x = \sum_{i=1}^n (b_i - b_i)e_{ii} + b_nI \in L_n(N) + D(A)I\). Thus, the calculation shows \(v(L_n(A)) \subseteq L_n(N) + D(A)I\), and it also demonstrates that \(D(A)I \subseteq v(L_n(A))\) as was asserted above. □

The same argument can be used to show

**Lemma 4.4.** For \(n > 3\), \(v(L'_n(A)) = v(L_n(A)) \cap L'_n(A)\) and \(v(K_n(A)) = v(L'_n(A)) + Z_n L'_n(A)\).

Let us adopt the notation \(Z' = Z \cap L'_n(A)\) so that \(K_n(A) = L'_n(A)/Z'\).

**Lemma 4.5.** Assume \(\text{char } \Phi \neq 2, 3\), and let \(T\) be a transformation on \(K_n(A)\) with the property:

\[
[e_{ii} - e_{jj} + Z', T(x + Z')] = T([e_{ii} - e_{jj} + Z', x + Z'])
\]

for all \(x \in L'_n(A)\) and all \(i, j\). Then for each pair \(i \neq j\), \(T\) induces a transformation \(T_{ij}\) on \(A\) such that

\[
T(ae_{ij} + Z') = T_{ij}(a)e_{ij} + Z'.
\]

**Proof.** When \(\text{char } \Phi \neq 2, 3\) the space \(Ae_{ij} + Z'\) can be characterized in the following way:

\[
Ae_{ij} + Z' = \{x + Z' \in K_n(A) | [e_{ii} - e_{jj} + Z', x + Z'] = 2x + Z' \}.
\]

The definition of \(T\) makes this space \(T\)-invariant. Thus, for each \(a \in A\), there is a unique \(b \in A\) such that the coset \(be_{ij} + Z'\) equals \(T(ae_{ij} + Z')\). The \(T_{ij}\) desired is given by \(T_{ij}(a) = b\). □

Lemma 4.5 will be used next to study derivations \(\partial\) with the property that \(\partial(e_{ii} - e_{jj} + Z') = 0\), and in the future to investigate automorphisms \(\varphi\) such that \(\varphi(e_{ii} - e_{jj} + Z') = e_{ii} - e_{jj} + Z'\).

**Theorem 4.6.** Assume \(n > 3\) and \(\text{char } \Phi \neq 2, 3\). Let \(\partial \in \text{Der } K_n(A)\) be such that \(\partial(e_{ii} - e_{jj} + Z') = 0\) for all \(i, j\). Then there is a \(\Delta \in (\text{Der } A)^*\) and a diagonal matrix \(y \in M_n(N)\) such that \(\partial = \Delta + \text{ad}_y\).
Proof. The derivation \( \partial \) satisfies the hypotheses of Lemma 4.5, and so
\[
\partial(e^i_j + Z') = \alpha^i_j e^i_j + Z' \quad \text{for } i \neq j.
\]
The elements \( \alpha^i_j \) belong to \( N \) since \( e^i_j + Z \) is in \( \nu(K_n(A)) \) which is derivation-invariant, and off-diagonal entries of \( \nu(K_n(A)) \) lie in \( N \).

Applying \( \partial \) to the following relations
\[
\begin{align*}
[e^i_{jk} + Z', e^j_{ik} + Z'] &= e^i_j + Z' \quad \text{for } i, j, k \text{ distinct}, \\
[e^i_j + Z', e^i_j + Z'] &= e^i_i - e^i_j + Z'
\end{align*}
\]
shows that
\[
\alpha^i_k + \alpha^k_j = \alpha^i_j, \quad \alpha^i_j + \alpha^j_i = 0.
\]
Let \( y \) be the diagonal matrix with 0, \( \alpha_2, \ldots, \alpha_n \) down the diagonal. Then\[\text{ad}_y(e^i_j + Z') = (\alpha_1 - \alpha_i)e^i_j + Z' = \alpha^i_y e^i_j + Z'.\]Thus \( \Delta = \partial - \text{ad}_y \in \text{Der } K_n(A) \) has the property that \( \Delta(e^i_j + Z') = 0 \) as well as \( \Delta(e^i_j - e^i_j + Z') = 0 \) for each pair \( i, j \) with \( i \neq j \). According to Lemma 4.5, \( \Delta \) induces transformations \( \Delta_y \) of \( A \) such that \( \Delta(\alpha^i_j + Z') = \Delta_y(\alpha^i_j)e^i_j + Z' \) for \( i \neq j \). Since
\[
\Delta(\alpha^i_j + Z') = [\Delta(\alpha e^i_k + Z'), e^i_k + Z'] = \Delta_{ik}(\alpha)e^i_k + Z',
\]
it follows that \( \Delta_y = \Delta_{ik} \) for \( k \neq j \). An analogous argument shows the first subscript can also be altered. Thus, all the induced transformations are equal and we call the common map \( \eta \).

By applying \( \Delta \) to the relation
\[
[\alpha e^i_{jk} + Z', b e^j_{ki} + Z'] = a b e^i_{ij} + Z' \quad \text{for } i, k, j \text{ distinct}
\]
we obtain \( \eta \in \text{Der } A \). Thus, at least on elements of the form \( \alpha e^i_j + Z' \), \( \Delta = \eta^\# \), the derivation obtained by acting by \( \eta \) on each entry. But these elements generate \( K_n(A) \) as the next two relations show:
\[
\begin{align*}
[\alpha e^i_j + Z', e^i_j + Z'] &= a e^i_i - a e^i_j + Z', \\
[a e^i_i - a e^j_j + Z', b e^i_i - b e^j_j + Z'] &= [ab] e^i_i + Z'.
\end{align*}
\]
And these equations can be used to verify that \( \Delta = \eta^\# \in (\text{Der } A)^\# \). □

Theorem 4.7. Let \( \partial \) be an arbitrary element of \( \text{Der } K_n(A) \) where \( \text{char } \Phi \neq 2, 3 \) and \( n > 3 \). Then there is a \( w \in M_n(\mathbb{N}) \) such that \( \partial' = \partial - \text{ad}_w \) has the property that \( \partial'(e^i_i - e^i_j + Z') = 0 \) for all \( i, j \).

Proof. When \( i \neq j \),
\[
\partial[e^i_i - e^i_j + Z'] = [\partial(e^i_j + Z'), e^i_j + Z'] + [e^i_j + Z', \partial(e^i_j + Z')],
\]
and from this relation it is apparent that \( \partial(e^i_i - e^i_j + Z') \) has a unique coset representative with nonzero entries only in the \( i \)th and \( j \)th rows and columns. When \( j = n \) let us denote this coset in the following fashion:
\[
\partial(e^i_i - e^i_n + Z') = \sum_{k \neq i} \beta^i_k e^i_k + \sum_j \beta^i_j e^i_j
\]
\[
+ \sum_{k \neq i, k \neq n} \beta^i_{kn} e^i_{kn} + \sum_{l \neq i} \beta^i_{nl} e^i_{nl} + Z'.
\]
Since \( \partial (e_{ij} - e_{ij} + Z') = \partial (e_{ii} - e_{nn} + Z') - \partial (e_{ij} - e_{mn} + Z') \), we have

\[
\beta_{kn}^{i} = \beta_{kn}, \quad \beta_{nk}^{i} = \beta_{nk}^{i}, \quad \text{for } k \text{ distinct from } i, j, \text{ and } n. \quad (4.8)
\]

The elements \( \beta_{kl}^{i} \) lie in \( N \) for \( k \neq l \) since the Lie nucleus is preserved by derivations. Showing that the diagonal \( \beta \)'s are 0, and deriving relationships between the various \( \beta \)'s for different \( i \)'s, such as \( (4.8) \) above, will enable us to construct the matrix \( w \).

Let \( \partial (e_{ij} + Z') = \sum \alpha_{kl} e_{kl} + Z' \), and compare coefficients of the \( (i, i) \) and \( (i, j) \) entries in

\[
[ \partial (e_{ii} - e_{nn} + Z'), e_{ij} + Z'] + [e_{ii} - e_{nn} + Z', \partial (e_{ij} + Z')] = \partial (e_{ij} + Z').
\]

The resulting relations are

\[
-\beta_{ij}^{i} = \alpha_{ii}, \quad -\beta_{ii}^{i} + \alpha_{ij} = \alpha_{ij}. \quad (4.10)
\]

Now if \( e_{ii} - e_{nn} \) is replaced by \( e_{jj} - e_{mn} \) in \( (4.9) \) an analogous argument shows

\[
-\beta_{jj}^{i} = -\alpha_{ii}. \quad (4.11)
\]

Therefore it follows that

\[
\beta_{ii}^{i} = 0 \quad \text{and} \quad \beta_{jj}^{i} = -\beta_{jj}^{i}. \quad (4.12)
\]

Finally, if \( e_{nn} \) is used in place of \( e_{ij} \) in \( (4.9) \), one can show \( \beta_{nn}^{i} = 0 \) also. We define

\[
\gamma_{ij}^{i} = \beta_{ij}^{i} \quad \text{for } j, i \neq n,
\]

\[
\gamma_{kn} = -\beta_{kn}^{i} \quad \text{for } j \neq k, n,
\]

\[
\gamma_{nk} = \beta_{nk}^{i}.
\]

The equations in \( (4.8) \) show that \( \gamma_{kn} \) and \( \gamma_{nk} \) are independent of the defining \( j \). Let

\[
w = \sum \gamma_{kl} e_{kl}
\]

and note that \( w \in M_{n}(N) \), so that \( \text{ad}_w \) is a derivation. The image of each \( e_{ij} - e_{nn} + Z' \) under \( \text{ad}_w \) is the same as under \( \partial \) except for the \( (i, n) \) and \( (n, i) \) entries. Thus, if \( \partial' = \partial - \text{ad}_w \), then

\[
\partial'(e_{ii} - e_{nn} + Z') = \xi_{ii} e_{in} + \xi_{mi} e_{ni} + Z'.
\]

Comparison of the \( (n, j) \) and \( (i, n) \) coefficients in \( (4.9) \) now shows

\[
\xi_{ii} - \alpha_{ii} = \alpha_{mi}, \quad 2\alpha_{in} = \alpha_{in}. \quad (4.13)
\]

Similar equations resulting from the use of \( e_{jj} - e_{nn} \) are

\[
-\xi_{ii} + \alpha_{in} = -\alpha_{in}, \quad -2\alpha_{nj} = -\alpha_{nj}. \quad (4.14)
\]

Thus, \( \xi_{nn} = 0 \) and \( \xi_{mi} = 0 \) for each \( i \) and \( j \), and \( \partial' = \partial - \text{ad}_w \) is the desired derivation. \( \square \)

Combining Theorems 4.6 and 4.7 gives

**Theorem 4.8.** Assume \( \text{char } \Phi \neq 2, 3 \) and \( n > 3 \). Then for any algebra \( A \) with \( 1, \)
\[
\text{Der } K_n(A) = (\text{Der } A)^* + \text{ad}_{M_{n}(N)}. \]

**Corollary 4.9.** With assumptions as in the previous theorem,

\[
\text{Der } M_{n}(A) = \text{Der } L_{n}(A) = \text{Der } K_{n}(A) = (\text{Der } A)^* + \text{ad}_{M_{n}(N)}. \]
Proof. Each element of $\text{Der } K_n(A)$ is a derivation of $M_n(A)$ and $L'_n(A)$. However the mappings $\tau_\sigma$ and $\tau$ in Lemma 4.1 are the canonical injections of $\text{Der } M_n(A)$ and $\text{Der } L'_n(A)$ into $\text{Der } K_n(A)$. So these algebras can be no larger than $\text{Der } K_n(A)$, and the above equalities hold. \(\square\)

Using 4.1 and this result we have

**Corollary 4.10.** $\text{Der } L_n(A) = (\text{Der } A)^* + \text{ad}_{M_n(N)} + \text{LHom}(L_n(A), Z)$.

**Remarks.** Martindale [8] has obtained a result in a similar vein as Corollary 4.10. He showed that if $R$ is a primitive associative ring with a nontrivial idempotent, then any Lie derivation of $R$ is the sum of a derivation of $R$ plus a Lie homomorphism of $R$ into $Z$.

It should be commented that in showing $\text{Der } M_n(A) = (\text{Der } A)^* + \text{ad}_{M_n(N)}$ no restrictions on char $A$ or on $n$ are necessary. In this case $Ae_{ij} = \{x \in M_n(A)|e_{ui}x = xe_{uj} = x\}$ so that any derivation $\partial$ with the property that $\partial(e_{ii}) = 0$ induces a transformation $\partial_j$ on $Ae_{ij}$ regardless of char $A$. The additional hypotheses are needed in the computation of $\text{Der } K_n(A)$ to determine the structure of the Lie nucleus. But in the calculation of $\text{Der } M_n(A), M_n(N)$ is mapped into itself, so no such contortions are necessary. Rather than make a separate case for $M_n(A)$, we chose to treat $M_n(A), L_n(A), L'_n(A)$, and $K_n(A)$ simultaneously at this small expense.

**Corollary 4.11 (Jacobson [3, Theorem 8, p. 215]).** Let $S$ be a finite dimensional central simple associative algebra over $\Phi$. Then every derivation of $S$ is inner.

**Proof.** Let $F$ be a splitting field for $S$ over $\Phi$. Then $S \otimes_\Phi F \cong M_n(F)$. Thus by Corollary 4.9 and the preceding remarks, $\text{Der}_F(S \otimes_\Phi F) \cong \text{Der}_F(M_n(F)) \cong \text{ad}_{M_n(F)}$. Now if $\partial \in \text{Der } S$, then $\partial \otimes 1 \in \text{Der}_F(S \otimes_\Phi F)$. In particular, for each $x \in S, \text{ad}_x \in \text{Der } S$, and since $S$ is central it follows that $\dim_\Phi \text{ad } S = n^2 - 1$. Thus,

$$n^2 - 1 = \dim_F(\text{Der}_F M_n(F)) \geq \dim_\Phi \text{Der } S > \dim_\Phi \text{ad } S = n^2 - 1.$$  

Hence, $\text{Der } S = \text{ad}_S$ as claimed. \(\square\)

**Theorem 4.12.** Assume $n > 3$ and char $A \neq 2, 3$. Let $\varphi \in \text{Aut } K_n(A)$ be such that $\varphi(e_{ii} - e_{ij} + Z') = e_{ii} - e_{ij} + Z'$ for all $i, j$. Then there is a $\psi \in (\text{Aut } A)^*$ and an invertible diagonal matrix $u \in M_n(N)$ such that $\varphi = \chi_u \psi$ where $\chi_u$ is conjugation by $u$.

**Proof.** Just as in the proof of Theorem 4.6, by Lemma 4.5 there exist elements $\alpha_{ij} \in N$ for $i \neq j$ such that $\varphi(e_{ij} + Z') = \alpha_{ij}e_{ij} + Z'$. Only this time applying $\varphi$ to relations (22) and (23) yields

$$\begin{align*}
\alpha_{ik} \alpha_{kj} &= \alpha_{ij} & \text{for } i, k, j \text{ distinct}, \\
\alpha_{ii} \alpha_{ji} &= 1 & \text{for } i \neq j.
\end{align*}$$  

(4.15)

Let $u$ be the diagonal matrix with $1, \alpha_{12}, \ldots, \alpha_{1n}$ down the diagonal. Note that $u^{-1}$ is also diagonal with $1, \alpha_{21}, \ldots, \alpha_{n1}$ as its diagonal entries. Then $\chi_u(e_{ij} + Z') = u^{-1} e_{ij} u + Z' = \alpha_{ij} e_{ij} + Z' = \alpha_{ij} e_{ij} + Z'$. Thus $\psi = \chi_u^{-1} \varphi \in \text{Aut } K_n(A)$ has the property that it fixes each $e_{ij} + Z'$ as well as each $e_{ii} - e_{ij} + Z'$ for $i \neq j$. 

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Beyond this stage the proof is exactly parallel to the proof of Theorem 4.6. One shows the transformations induced from \( \psi \) are all equal, and then deduces from (4.5) that the common map, call it \( \theta \), belongs to \( \text{Aut} A \). Finally then, using (4.6) and (4.7) one verifies \( \psi = \theta^* \in (\text{Aut} A)^* \). □

These conclusions remain valid for any \( \varphi \in \text{Aut} L_n(A) \) (\( \text{Aut} L'_n(A) \)) with the property that \( \varphi(e_{ij}) = e_{ij} \) (\( \varphi(e_{ij} - e_{ji}) = e_{ij} - e_{ji} \)).

Given any antiautomorphism \( \varphi \) of \( M_n(A) \), \( -\varphi \) is an automorphism of \( L_n(A) \), thus of \( L'_n(A) \) and \( K_n(A) \) as well. The collection of all such mappings together with \( \text{Aut} M_n(A) \) forms a subgroup of the automorphism group of each of these algebras. It is isomorphic to \( \text{AAut} M_n(A) \), so we will use that notation to denote it.

**Theorem 4.13.** If \( A \) is finite dimensional over \( \Phi \) where \( \text{char} \ \Phi \neq 2, 3 \), and if \( n > 3 \), then \( \text{Aut} M_n(A) \) (hence \( \text{AAut} M_n(A) \), also) is of finite index in \( \text{Aut} L_n(A) \) and in \( \text{Aut} K_n(A) \).

**Proof.** The group \( G = \text{Aut} L_n(A) \) (or \( \text{Aut} K_n(A) \)) is an algebraic group (see Seligman [11, p. 34]), and \( H = \text{Aut} M_n(A) \) is a closed subgroup of \( G \). The Lie algebra of \( G \) is \( \text{Der} K_n(A) \) which equals the Lie algebra of \( H \), \( \text{Der} M_n(A) \). Therefore, by Proposition 4, p. 33 of Seligman, \( H \) is of finite index in \( G \). □

Now apply Theorem 4.13 and Lemma 4.2 to obtain

**Corollary 4.14.** With hypotheses as in Theorem 4.13, \( \text{Aut} M_n(A) + \text{LHom}(L_n(A), \mathbb{Z}) \) has finite index in \( \text{Aut} L_n(A) \).

**Remark.** In the associative case it follows from Martindale [9] that the Lie automorphisms of \( R \), a prime ring with a nontrivial idempotent, are precisely \( \text{AAut} R + \text{LHom}(R, \mathbb{Z}) \).

5. \( M_n(A) \) under the Jordan product. Throughout this section we assume \( \text{char} \ \Phi \neq 2 \). Let \( J_n(A) \) denote the commutative algebra obtained by taking \( M_n(A) \) under the Jordan product \( x \circ y = \frac{1}{2}(xy + yx) \). In this part we investigate the derivation algebra and automorphism group of \( J_n(A) \). The study of derivations of \( J_n(A) \) is facilitated by the use of results on transformations in triality. The notion of triality comes up in the study of the Lie algebras of types \( D_4 \) and \( F_4 \) (see Jacobson [5] for example). There the transformations are defined on an octonion algebra, but here we extend the concept to arbitrary algebras.

Let \( A \) be an arbitrary algebra with 1, and assume \( \xi, \eta, \theta \) are linear transformations on \( A \) with the property that

\[
\xi(ab) = \eta(a)b + a\theta(b) \quad \text{for all} \ a, b \in A.
\]

Then we say \( \xi, \eta, \theta \) are in local triality (or for short, triality). For example if \( \xi = \eta = \theta \), and \( \xi \) is a derivation of \( A \), then \( \xi, \eta, \theta \) are in triality.

**Lemma 5.1.** If \( \xi, \eta, \eta', \theta, \theta' \) are transformations on \( A \) such that \( \xi, \eta, \theta \) and \( \xi, \eta', \theta' \) are in triality then

1. \( \eta' = \eta - R_s \) where \( R_s(a) = as \),
2. \( \theta' = \theta + L_s \) where \( L_s(a) = sa \),

and \( s = \eta(1) - \eta'(1) = \theta'(1) - \theta(1) \) and \( s \) is in the middle nucleus of \( A \).
**Proof.** Observe that if \( \xi, \eta, \theta \) and \( \xi', \eta', \theta' \) are in triality then \( \alpha \xi + \beta \xi', \alpha \eta + \beta \eta', \alpha \theta + \beta \theta' \) are in triality for every pair of scalars \( \alpha, \beta \). Thus, if \( \eta'' = \eta - \eta', \theta'' = \theta - \theta' \), then 0, \( \eta'', \theta'' \) are in triality. This says that \( \eta''(a)b + a\theta''(b) = 0 \) for all \( a, b \in A \). Specializing \( b \) to equal 1 shows that
\[
\eta''(a) = -a\theta''(1) \quad \text{for all } a \in A. \tag{5.1}
\]
Likewise \( a = 1 \) gives
\[
\theta''(b) = -\eta''(1)b \quad \text{for all } b \in B. \tag{5.2}
\]
If both \( a = 1 \) and \( b = 1 \), then \( \eta''(1) = -\theta''(1) \). Let \( s \) be this common value. Then (5.1) and (5.2) shows that \( \eta - \eta' = \eta'' = R_s \) and \( \theta - \theta' = \theta'' = -L_s \). Finally, the relation \( \eta''(a)b + a\theta''(b) = 0 \) implies \( (as)b - a(sb) = 0 \) as desired. □

**Lemma 5.2.** Let \( \partial \in \text{Der}(J_\Phi(A)) \) have the property that \( \partial(e_i) = 0 \) for \( i = 1, \ldots, n \). Then \( \partial \) induces \( n^2 \) transformations \( \partial_y \) on \( A \) such that \( \partial(\alpha e_j) = \partial_y(\alpha)e_j \).

**Proof.** The proof begins just as Lemma 3.5 does. The same argument using the five types of nonzero products and the eigenspaces of the elements \( e_i \) demonstrates that \( \partial(\alpha e_i) = \alpha \partial_y(\alpha)e_i \) and \( \partial(\alpha e_j) = \partial_y(\alpha)e_j + \partial_y(\alpha)e_j \). In particular, \( \partial(e_j) = \alpha_j e_j + \beta_j e_j \). But since \( e_j^2 = 0 \), we have \( 2e_j \circ \partial(e_j) = 0 \). Therefore \( \beta_j = 0 \) for \( j \neq i \). Moreover, \( e_j \circ \alpha e_j = 0 \) implies \( \alpha_j e_j \circ \alpha e_j + e_j \circ (\partial_y(\alpha)e_j + \partial_y(\alpha)e_j) = 0 \), and from this we see \( \partial_y(a) = 0 \) for all \( a \). Hence, \( \partial(\alpha e_j) = \partial_y(\alpha)e_j \) in all cases. □

**Lemma 5.3.** Let \( \text{char } \Phi \neq 2, \) and \( n > 2 \). Assume \( \partial \in \text{Der} J_\Phi(A) \) has the property that \( \partial(e_i) = 0 \) for \( i = 1, \ldots, n \). Then there is a diagonal matrix \( \gamma \in M_n(N) \) such that \( \partial = \text{ad}_\gamma \in (\text{Der } A)\gamma \).

**Proof.** By the preceding lemma \( \partial \) induces the transformations \( \partial_y \), and for \( i, j \) distinct we see from relations (3.2), (3.3), (3.5) that the following transformations are in triality on \( A \):
\[
\partial_{ij}, \quad \partial_{ii}, \quad \partial_{ji}, \tag{5.3}
\]
\[
\partial_{ij}, \quad \partial_{ji}, \quad \partial_{ij}, \tag{5.4}
\]
\[
\partial_{ii}, \quad \partial_{ji}, \quad \partial_{ji}. \tag{5.5}
\]

Let \( \alpha_{ij} = \partial_y(1) \). Since relation (3.1) implies \( \partial_{ii} \) is a derivation on \( A \) under the Jordan product, it follows that \( \alpha_{ii} = 0 \). From (5.5) we conclude that \( \alpha_{ij} = -\alpha_{ji} \) for \( j \neq i \), so that by Lemma 5.1 we have
\[
\partial_{ij} = \partial_{ii} + R_{\alpha_{ij}}, \tag{5.6}
\]
\[
\partial_{ij} = \partial_{ji} - L_{\alpha_{ij}}, \tag{5.7}
\]
and also the fact that \( \alpha_{ij} \) is in the middle nucleus of \( A \). Subtracting (5.4) from (5.3) and then adding (5.5) shows that \( \partial_{ii}, \partial_{ji}, \partial_{ij} - \partial_{ji} + \partial_{ji} \) are in triality. However, using (5.6) and (5.7) we may express each of the last three transformations in terms of \( \partial_{ii} \). Note that
\[
\partial_{ji} = \partial_{ii} + R_{\alpha_{ij}} - L_{\alpha_{ij}} \tag{5.8}
\]
and interchanging \( j \) and \( i \) in (5.7) gives: \( \partial ii - \partial ji = L_{\alpha ii} - \partial ii + L_{\alpha ji}. \) Therefore, \( \partial ii - \partial ji + \partial ji = \partial ii + R_{\alpha ii} - \partial ii - R_{\alpha ji} + L_{\alpha ii} + \partial ii - L_{\alpha ii} = \partial ii. \) Hence \( \partial ii, \partial ji, \partial ji \) are in triality and \( \partial ii \) is a derivation on \( A. \)

As a consequence of equations (5.3) and (5.6) we have
\[
\partial ii(ab) + (ab)\alpha ii = \partial ii(a)b + a\partial ii(b) + a(b\alpha ii)
\]
which implies that \( \alpha ii \) is in the right nucleus of \( A. \) Similarly using (5.4) and (5.7) we deduce that \( \alpha ii \) is in the left nucleus of \( A. \) Let \( y \) be the diagonal matrix with 0, \( \alpha 2, \ldots, \alpha n \) down the diagonal. From equations (5.6), (5.7), (5.8) it follows that \( \partial ji(a) = \partial 11(a) - a\alpha ji, \partial ii(a) = \partial 11(a) + \alpha ii a \) and \( \partial ji(a) = \partial 11(a) + [\alpha ji, a]. \) If \( n > 3, \) then for \( i, j \) distinct from 1, \( \partial ji, \partial ji, \partial ji \) are in triality and \( \partial ji(a) = \partial 11(a) + \alpha ji a - a\alpha ji. \) We conclude from these relations that \( \partial = (\partial 11)^{\#} + \text{ad}_y \) where \( \partial 11 \in \text{Der} A. \)

**Lemma 5.4.** Let \( \text{char } \Phi \neq 2, \) and \( n \geq 2. \) Suppose \( \partial \in \text{Der } J_n(A). \) Then there is a \( w \in M_n(\Phi) \) such that \( \partial' = \partial - \text{ad}_w \) has the property that \( \partial(eii) = 0 \) for all \( i. \)

**Proof.** Apply \( \partial \) to the relation \( eii \circ eii = eii \) to conclude that \( \partial(eii) \) lies in the \( \frac{1}{2} \)-eigenspace relative to multiplication by \( eii. \) Therefore we may write
\[
\partial(eii) = \sum_{k \neq i} \beta_{ki}^i e_{ki} + \sum_{l \neq i} \beta_{il}^i e_{il}.
\]
Now if \( j \neq i, \) then \( \partial(eii) \circ eij + eii \circ \partial(eij) = 0, \) and from this we calculate that
\[
\beta_{ji}^i = -\beta_{ij}^i.
\]
Thus, if \( \gamma_{kl} = \beta_{ki}^i \) for \( k \neq i \) and if \( \gamma_{kk} = 0, \) then \( w = \sum \gamma_{kl} e_{kl} \) has the property that
\[
\text{ad}_w(eii) = \sum_k \gamma_{ki} e_{ki} - \sum_l \gamma_{ii} e_{il} = \sum_k \beta_{ki}^i e_{ki} - \sum_{l \neq i} \beta_{il}^i e_{il} = \sum_k \beta_{ki}^i e_{ki} + \sum_{l \neq i} \beta_{il}^i e_{il} = \partial(eii).
\]
The proof will be complete once it is proven that each \( \gamma_{kl} \in \text{N}, \) and hence \( \text{ad}_w \in \text{Der } J_n(A). \) A computation of \( \partial \) applied to \( aeii \circ eii = aeii \) shows that
\[
\partial(aeii) = \sum_{k \neq i} \beta_{ki}^i ae_{ki} + \sum_{l \neq i} a\beta_{il}^i e_{il} + \Delta(a, i)eii.
\]
Using this relation, we conclude from \( \partial \) acting on \( aeii \circ beij = 0 \) that
\[
0 = \frac{1}{2} \{ (a\beta_{ij}) b + a(\beta_{ij} b) \} e_{ij} + \frac{1}{2} \{ (b \beta_{ij} a) + (b\beta_{ij} a) \} e_{ij}.
\]
Since \( \beta_{ij}^i = -\beta_{ji}^i, \) this implies \( \gamma_{ij} = \beta_{ij} \) in the middle nucleus of \( A. \)

Let us apply \( \partial \) to both sides of \( eii \circ beij = \frac{1}{2} beij. \) Now
\[
\partial(eii) \circ beij = \frac{1}{2} \sum_{k \neq i} \beta_{ki}^i be_{kj} + \beta_{ji}^i e_{ii},
\]
and \( eii \circ \partial(beij) \) has nonzero entries just in the \( i \)th row and column. Thus,
\[
\partial(beij) = \sum_{k \neq i} \beta_{ki}^i be_{kj} + \sum_{l \neq i} c_{il} e_{il} + \sum_l c_{il} e_{il}.
\]
The next step is to equate coefficients after applying $\partial$ to $e_{ij} \circ b_{ij} = \frac{1}{2} b_{ij}$. The result obtained is

$$\partial (b_{ij}) = \sum_{k \neq i} \beta_k^i b_{kj} + \sum_{l \neq j} b_{jl} \beta_l^j e_{ij} + \Delta(b, i, j)e_{ij} + \Delta'(b, j, i)e_{ij}.$$  

Then $\partial (ae_{ii}) \circ b_{ij} + ae_{ii} \circ \partial (b_{ij}) = \frac{1}{2} \partial (abe_{ij})$ implies $(\beta_{kl}) a b = \beta_{kl}^i (ab)$ and $a (b \beta_{ij}) = (ab) \beta_{ij}$. Thus, the $y$'s lie in $N$, $w = \sum \gamma_{kl} e_{kl}$ in $M_n(N)$, and $\text{ad}_w$ in $\text{Der} M_n(A)$. As noted above, then for $\partial' = \partial - \text{ad}_w$, we have $\partial'(e_{ll}) = 0$ for all $i$. □

The net effect of Lemmas 5.3 and 5.4 is a proof of

**Theorem 5.5.** Let $\text{char } \Phi \neq 2$. Then

$$\text{Der } J_n(A) = (\text{Der } A)^\# + \text{ad}_{M_n(N)} = \text{Der } M_n(A).$$

**Remarks.** Recall that the last equality follows from Corollary 4.9 and the comments preceding Corollary 4.11.

Previous results analogous to Theorem 5.5 have been known to hold in the associative case. Jacobson and Rickart showed in [6, Theorems 7 and 22] that $\text{Der } M_n(A) = \text{Der } J_n(A)$ when $A$ is associative, while Herstein [2, p. 55] proved that every Jordan derivation of a prime associative ring $R$ is a derivation of $R$, except when $R$ is a commutative integral domain of characteristic 2. These results would have been useful had we known ab initio that $\partial \in \text{Der } J_n(A)$ implies $\partial (M_n(N)) \subseteq M_n(N)$. Indeed the major effort of Lemma 5.4 is in proving the image of each $e_{ll}$ lies in $M_n(N)$.

The work on the next theorem concerning $\text{Aut } J_n(A)$ has largely been done already, but we collect the results for the convenience of the reader.

**Theorem 5.6.** Assume $\text{char } \Phi \neq 2$. Let $G = \{ \psi \in \text{Aut } J_n(A) | \psi(e_{ii}) = e_{ii} \text{ for all } i \}$ and $H = \{ \psi \in \text{Aut } J_n(A) | \psi(J_n(N)) \subseteq J_n(N) \}$. Then

$$G \subseteq \text{GL}(n, N) \cdot (\text{GAut } A)^\# \subseteq \text{GAut } M_n(A) \subseteq H \subseteq \text{Aut } J_n(A).$$

(i) $G = \text{X} \cdot (\text{GAut } A)^\#$, where $X$ is the subgroup consisting of conjugations by diagonal matrices with entries in $N$.

(ii) If $N$ is Artinian, then $\text{GL}(n, N) \cdot (\text{GAut } A)^\# = \text{GAut } M_n(A) = H$.

(iv) If $A$ is finite dimensional, then $\text{Aut } M_n(A)$, hence $\text{GAut } M_n(A)$ also, is of finite index in $\text{Aut } J_n(A)$.

**Proof.** Assertions (i) and (ii) are consequences of results in §3, notably Theorem 3.6 and Lemma 3.12. Part (iii) follows from Lemma 3.4 and Theorems 3.6 and 3.13. Since $\text{Der } J_n(A) = \text{Der } M_n(A)$, statement (iv) can be concluded by the same algebraic group argument used in §4. □

**Remarks.** For associative rings Herstein [2, p. 50] has shown that if $\varphi$ is a Jordan homomorphism of $R$ onto a prime ring $R'$, then $\varphi$ is a homomorphism or an antihomomorphism. This result is false in the nonassociative case as the next example illustrates.
Choose $\lambda, \mu \in \Phi$ such that $\lambda + \mu = 1$, and let $B$ have basis $\{e_{ij}\}$ for $i, j = 1, \ldots, n$ where the product in $B$ is given by $e_{ij} * e_{kl} = \delta_{jk} \lambda e_{il} + \delta_{il} \mu e_{kj}$. If $B_j$ denotes $B$ under the Jordan product then $B_j$ is Jordan isomorphic to $M_n(\Phi)$ under the correspondence $e_{ij} \rightarrow e_{ii}$. Since $B$ is nonassociative when $\lambda \neq 1, 0$, this map is neither an isomorphism nor an anti-isomorphism of $B$ onto $M_n(\Phi)$.

Now if $A$ is any algebra then $M_n(A) \cong A \otimes M_n(\Phi)$. If in addition $A$ is assumed to be commutative then $J_n(A) \cong A \otimes J_n(\Phi)$. If $A$ is any commutative algebra isomorphic to $J_n(\Phi)$, such as $B_j$ above, then there is an automorphism of $J_n(A)$ mapping $A \otimes 1$ to $1 \otimes J_n(\Phi)$. If the nucleus of $A$ is $\Phi 1$, as with $B_j$, then such an automorphism is not in $\text{Aut} M_n(A)$ or $\text{GAut} M_n(A)$ since it fails to preserve the nucleus which is $1 \otimes M_n(\Phi)$. By taking $A$ to be the tensor product of any number of algebras isomorphic to $J_n(\Phi)$, we see that the index of $\text{Aut} M_n(A)$ in $\text{Aut} J_n(A)$ can be arbitrarily large.

We conclude our remarks with one additional example. Suppose $B$ is the algebra above, and form the algebra $J_n(B)$. The product is given by

$$2e_{ij}e_{qr}^* e_{kl}e_{st} = \delta_{rs}\delta_{jk}\lambda e_{il}e_{qt} + \delta_{ts}\delta_{il}\mu e_{kj}e_{qt} + \delta_{iq}\delta_{jk}\lambda e_{sr}e_{qt} + \delta_{iq}\delta_{jk}\lambda e_{kj}e_{sr}.$$ 

Using this formula one can show that the map $\varphi$ given by $\varphi(e_{ij}e_{qr}) = e_{qr}e_{ij}$ belongs to $\text{Aut} J_n(B)$. If $\lambda \neq 1, 0$ then $\varphi$ interchanges the nonassociative algebra $B$ with $M_n(\Phi)$; hence it fails to belong to $\text{Aut} M_n(B)$. This example illustrates that the coefficient algebra $B$ need not be commutative to make interchanges of this sort possible.

**REFERENCES**