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A LATTICE RENORMING THEOREM AND APPLICATIONS TO VECTOR-VALUED PROCESSES

BY

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ABSTRACT. A norm, $\| \|$, on a Banach space E is said to be locally uniformly convex if $\|x_n\| \to \|x\|$ and $\|x_n + x\| \to 2\|x\|$ implies that $x_n \to x$ in norm. It is shown that a Banach lattice has an (order) equivalent locally uniformly convex norm if and only if the lattice is order continuous. This result is used to reduce convergence theorems for (lattice-valued) positive martingales and submartingales to the scalar case.

0. Introduction. A norm, $\| \|$, on a Banach space E is said to have the *Kadec-Klee property* (sometimes property (H)) if whenever $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then $x_n \to x$ strongly. The aim of this paper is to show that if E has an equivalent Kadec-Klee norm then one may obtain convergence theorems for E-valued random processes (X_n) whenever $\|X_n\|$ is a well-known real-valued convergent process and whenever the limit can be identified in the Banach space. For instance, the classical Kadec renorming theorem for separable Banach spaces gives the convergence of vector-valued martingales, uniform amarts and additive processes (ergodic theorem) since their norms are real-valued submartingales, amarts and subadditive processes respectively while the identification of the limit requires the Radon-Nikodym property on the space for the first two processes, the limit of the third process exists in any Banach space.

If now, we consider positive submartingales and subadditive processes valued in a Banach lattice, one needs that the equivalent Kadec-Klee norm be also a lattice norm in order to conclude that the norms of these processes are real-valued submartingales and subadditive processes respectively. §I deals with the existence of such a lattice renorming while in §II we show how this leads to a unified approach for proving the almost sure convergence of the processes mentioned above.

I. Renorming order continuous lattices. A norm, $\| \|$, on a Banach space E is said to be *locally uniformly convex* if $\|x_n\| \to \|x\|$ and $\|x + x_n\| \to 2\|x\|$ imply that $x_n \to x$ strongly. This notion is clearly stronger than the Kadec-Klee property.

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Kadec showed that every separable Banach space has an equivalent locally uniformly convex norm [**2**, p. 176]. Trojanski extended this to all weakly compactly generated spaces [**5**, p. 164]. The difficulty with the known renormings of this type in case E is a lattice is that they use many seminorms of the form $\operatorname{dist}(x, F)$ where F is a finite-dimensional subspace of E. These are almost never lattice seminorms. The purpose of this section is to overcome this difficulty.

Let E be a Banach space which is also a linear lattice with partial order denoted by $<$. That is, for $x, y \in E$, there exists an element $x \vee y$ in E, the least upper bound of x and y. Further, if $0 < x < y$ and $a > 0$, $0 < ax < ay$, and for $a < 0$, $0 > ax > ay$. For any $x \in E$, let $|x| = x \vee 0 - x \wedge 0$, where $x \wedge y$ denotes the greatest lower bound of x and y, The norm on E is a lattice norm if $|x| \leqslant |y|$ implies $\|x\| \leqslant \|y\|$.

Recall that a lattice is said to be *order continuous* if every order-convergent filter is norm convergent. For example, L_1 is order continuous and $C[0, 1]$ is not. In fact, as we shall see, the choice of L_1 as a prototype of an order continuous lattice is good, since our renorming will depend upon the existence of the same type of renorming for L_1. We show here that a Banach lattice is order continuous if and only if it has a lattice-equivalent locally uniformly convex lattice norm.

We need some notation first. Let $(\Omega, \mathfrak{F}, P)$ be a probability space. For any $f \in L_1(\Omega, \mathfrak{F}, P)$, let $\tilde{f}(t) = \sup_{P(A) \leqslant t} \int_A |f| \, dP$. Notice that if f^* denotes the decreasing rearrangement of f, then $\tilde{f}(t) = \int_0^t f^*(s) \, ds$, so the map $f \to \tilde{f}$ is subadditive and maps $L_1(\Omega, P)$ into each $L_p[0, 1]$ space boundedly.

The basic facts about lattices which we use are available in the books [**12**] and [**16**].

PROPOSITION I.1. *Let* $\{f_n\}, f$ *be in* $L_1(\Omega, P)$, *so that* $\tilde{f}_n \to \tilde{f}$ *a.s. and* $\widetilde{f + f_n} \to 2\tilde{f}$ *a.s. Then* $f_n \to f$ *in norm.*

PROOF. Since $\tilde{f}_n(t) \to \tilde{f}(t)$ a.s., and since $\tilde{f}(t) \to 0$ as $t \to 0$, we see that the original sequence $\{f, f_n\}$ is equi-integrable. That is, for any $\varepsilon > 0$ there is $\delta > 0$ so that $P(A) < \delta$ implies $\int |f_n| \chi_A \, dP < \varepsilon$ and $\int |f| \chi_A \, dP < \varepsilon$. This allows us to show that $\tilde{f}_n \to \tilde{f}$ uniformly on $[0, 1]$. Let $\delta = \delta(\varepsilon/3)$ from above and choose a finite set $\{x_1, \ldots, x_m\}$ in $[0, 1]$ so that, for $x \in [0, 1]$, $\min|x - x_i| < \delta/3$, and such that $\tilde{f}_n(x_i) \to \tilde{f}(x_i)$ for $i = 1, 2, \ldots, m$. Let N be large enough so that $n \geqslant N$ implies $|\tilde{f}_n(x_i) - \tilde{f}(x_i)| < \varepsilon/3$. We have, then, that for x in $[0, 1]$ and $|x - x_i| < \delta$, $|\tilde{f}_n(x) - \tilde{f}(x)| \leqslant |\tilde{f}_n(x) - \tilde{f}_n(x_i)| + |\tilde{f}_n(x_i) - \tilde{f}(x_i)| + |\tilde{f}(x) - \tilde{f}(x_i)|$. The first and last terms are $< \varepsilon/3$ by the equi-integrability, so the sequence is converging uniformly. Without loss of generality we may assume that $f \geqslant 0$. We first also make the further assumption that $f_n \geqslant 0$ for each n. The distribution function of f, $P[f \geqslant \lambda]$, is left continuous, so that for each λ, $\alpha > 0$, there is $\beta > 0$ so that $P[f \geqslant \lambda - \beta] - P[f \geqslant \lambda] < \alpha$. For notational purposes, let $A_\lambda = [f \geqslant \lambda] = B_\lambda^c$ and $A_{n,\lambda} = [f_n \geqslant \lambda] = B_{n,\lambda}^c$. Fix $\lambda > 0$. Now let $\varepsilon > 0$ and let δ be as before. Let $\eta > 0$ be small enough so that $P(A_{\lambda - \eta} \cap B_\lambda) < \delta/4$ and $\eta < \delta/4$. Next, let $0 < \gamma < \eta^2$. Select n large enough so that $|\widetilde{f_n + f}(P(A_\lambda)) - 2\tilde{f}(P(A_\lambda))| < \gamma/2$ and $|\tilde{f}_n(P(A_\lambda)) - \tilde{f}(P(A_\lambda))| < \gamma/2$. Choose a set A with $P(A) = P(A_\lambda)$ so that $\int_A (f_n + f) \, dP > \widetilde{f_n + f}(P(A_\lambda)) - \gamma/2$. This yields

$$2\int_{A_\lambda} f\,dP - \gamma < \int_A f\,dP + \int_A f_n\,dP \leqslant \int_A f\,dP + \tilde{f}_n(P(A_\lambda))$$
$$\leqslant \int_A f\,dP + \tilde{f}(P(A_\lambda)) + \gamma/2 = \int_A f\,dP + \gamma/2 + \int_{A_\lambda} f\,dP.$$

That is, $\int_A f\,dP > \int_{A_\lambda} f\,dP - 2\gamma$. Next we have

$$\begin{aligned}\int_A f\,dP &= \int_{A\cap A_{\lambda-\eta}} f\,dP + \int_{A\cap B_{\lambda-\eta}} f\,dP\\ &\leqslant \tilde{f}(P(A\cap A_{\lambda-\eta})) + (\lambda-\eta)P(A\cap B_{\lambda-\eta})\\ &= \tilde{f}(P(A) - P(A\cap B_{\lambda-\eta})) + (\lambda-\eta)P(A\cap B_{\lambda-\eta}).\end{aligned}$$

Putting these together, we get

$$\begin{aligned}\lambda P(A\cap B_{\lambda-\eta}) - 2\gamma &\leqslant \tilde{f}(P(A_\lambda)) - \tilde{f}(P(A_\lambda) - P(A\cap B_{\lambda-\eta})) - 2\gamma\\ &\leqslant (\lambda-\eta)P(A\cap B_{\lambda-\eta}).\end{aligned}$$

Thus $P(A\cap B_{\lambda-\eta}) \leqslant 2\gamma/\eta < 2\eta$. Finally, $A = (A\cap A_\lambda)\cup(A\cap B_\lambda\cap A_{\lambda-\eta})\cup(A\cap B_{\lambda-\eta})$ and $P(A) = P(A_\lambda)$ give $P(A\cap A_\lambda) \leqslant P(A) - \delta$. Similarly, we get that $P(A\cap B_{n,\lambda-\eta}) < \delta$, so that $P(A_\lambda\cap B_{n,\lambda-\eta}) < 2\delta$. What we have shown is that for all $\lambda > 0$ and $\eta > 0$, $P[f \geqslant \lambda, f_n < \lambda - \eta] \to 0$ as $n\to\infty$. It is a routine exercise to show that this implies that $P[f_n < f - \eta] \to 0$ as $n\to\infty$. Once again, fix η, $\eta_1 > 0$. Then $\tilde{f}_n(1) = \int f_n\,dP = \int_{[f_n > f+\eta]} f_n\,dP + \int_{[f_n < f+\eta]} f_n\,dP$ so

$$\begin{aligned}\tilde{f}_n(1) &\geqslant \int_{[f_n > f+\eta]} f_n\,dP + \int_{[f-\eta_1 < f_n < f+\eta]} f_n\,dP + \int_{[f_n < f-\eta_1]} f_n\,dP\\ &\geqslant \eta P[f_n \geqslant f+\eta] + \int_{[f_n > f-\eta_1]} f\,dP - \eta_1 + \int_{[f_n < f-\eta_1]} f_n\,dP.\end{aligned}$$

As $n\to\infty$, the left-hand side tends to $\tilde{f}(1)$, the second term on the right tends to $\tilde{f}(1)$ and the last term on the right tends to zero. Thus, $\limsup P[f_n > f + \eta] \leqslant \eta_1/\eta$. Since η_1 is chosen arbitrarily, this then yields $P[f_n > f+\eta] \to 0$ as $n\to\infty$. Therefore, we have $f_n \to f$ in measure. Since we already knew that $\{f_n\}$ is equi-integrable, this forces $f_n \to f$ in L_1. Without assuming $f_n > 0$, we have shown that $|f_n| \to f$ in L_1 norm. It is easy to see that this together with $\widetilde{f_n + f} \to 2\tilde{f}$ forces $f_n \to f$ in L_1 norm, completing the proof.

Now we are ready to renorm $L_1(\Omega, P)$.

THEOREM I.1. *If (Ω, P) is a probability space, then $L_1(\Omega, P)$ has an equivalent locally uniformly convex lattice norm.*

PROOF. Setting $\|f\|_1 = \|f\|_{L_1(\Omega,P)}$ and $\|f\|_2 = \|\tilde{f}\|_{L_2([0,1])}$, we let

$$\|f\| = \left(\|f\|_1^2 + \|f\|_2^2\right)^{1/2}.$$

It is clear that this is a lattice norm, order equivalent to the $L_1(\Omega, P)$ norm. Let f_n, f be in $L_1(\Omega, P)$ so that $\|f_n\| \to \|f\|$ and $\|f_n + f\| \to 2\|f\|$ as $n\to\infty$. Using the triangle inequality, we see that

$$\left\{(\|f\|_1 + \|f_n\|_1)^2 + (\|f\|_2 + \|f_n\|_2)^2\right\}^{1/2} \to 2\left(\|f\|_1^2 + \|f\|_2^2\right)^{1/2},$$

so that, in particular, we must have $\|f_n\|_i \to \|f\|_i$ and $\|f_n + f\|_i \to 2\|f\|_i$ for $i = 1, 2$ as $n \to \infty$. Since $L_2([0, 1])$ is uniformly convex, these force $\tilde{f}_n \to \tilde{f}$ in $L_2([0, 1])$, and hence in measure. Therefore, an arbitrary subsequence of $\tilde{f}_n$ has a subsequence converging to $\tilde{f}$ almost everywhere. Applying Proposition I.1, we have the desired conclusion.

COROLLARY I.1. *$L_1([0, 1])$ has an order-equivalent, rearrangement-invariant, locally uniformly convex norm.*

PROOF. If $(\Omega, P) = ([0, 1], \mu)$, the above norm is rearrangement invariant.

We have now done most of the work required to prove the renorming theorems for general order-continuous lattices. There is just one ingredient left to be built into the norm—a uniform integrability condition in the lattice norm. Suppose that E is an order-continuous lattice with a weak order unit (that is, $e \in E^+$ such that $e \wedge |f| = 0$ implies $f = 0$). Then we may consider this lattice as a function space on some (Ω, P) with the property that $L_\infty(\Omega, P) \subset E \subset L_1(\Omega, P)$ and so that for $f \in E$, $\|f\|_1 \leqslant \|f\|_E$. See, for example, **[12**, 1.b.14]. The order continuity of the norm forces $\lim_{P(A)\to 0}\|f\chi_A\|_E \to 0$ for each $f \in E$. Thus, if we let $\Psi_n(f) = \sup_{P(A)\leqslant 1/n}\|f\chi_A\|_E$, and if $\Psi_n(f_k) \to \Psi_n(f)$ as $k \to \infty$ for each n, then the sequence $\{f_k\}$ must have the property that for all $\varepsilon > 0$ there is $\delta > 0$ so that $P(A) < \delta$ implies $\|f_k\chi_A\|_E < \varepsilon$, for all k. One should notice that each of the norms Ψ_n is consistent with the ordering in the lattice, and that, when it makes sense, each is rearrangement invariant as well.

We denote by $\tilde{c}_0$ the Banach lattice c_0 equipped with its locally uniformly convex lattice norm constructed by Day **[5**, p. 94]. The following is the main result of this section.

THEOREM I.2. *A Banach lattice, E, has an order-continuous norm if and only if it has an equivalent locally uniformly convex lattice norm.*

PROOF. Assume first that E is order continuous. By Kakutani's theorem, $E = \Sigma_{\alpha\in\Gamma} \oplus E_\alpha$ where each E_α is a band with weak order unit e_α **[12]**. In particular for each $f \in E$, $f = \Sigma_{\alpha\in\Gamma} f_\alpha$, and each of the maps $f \to f_\alpha$ is a (positive) band projection. We first renorm each E_α.

As above, for each α, there is a probability space $(\Omega_\alpha, P_\alpha)$ so that $L_1(\Omega_\alpha, P_\alpha) \supset E_\alpha \supset L_\infty(\Omega_\alpha, P_\alpha)$. For this part of the argument, let us ignore the subscripts, α, that occur. Let $\|\cdot\|_E$ denote the original norm on the lattice, $|||\cdot|||_1$, the norm on $L_1(\Omega, P)$ constructed in Theorem I.1 and $\Psi_n(\cdot)$, the norms defined above. Define the new norm on E by $|||f||| = \|(\|f\|_E, |||f|||_1, \Psi_1(f), \Psi_2(f), \ldots)\|_{\tilde{c}_0}$. It is immediate that $|||\cdot|||$ is a lattice norm equivalent to $\|\cdot\|_E$. Also, as before, if $|||f_n||| \to |||f|||$ and $|||f + f_n||| \to 2|||f|||$, then, by the nature of the norm in $\tilde{c}_0$, this forces $\|f_n\|_E \to \|f\|_E$, $|||f_n|||_1 \to |||f|||_1$ and $\Psi_k(f_n) \to \Psi_k(f)$ when $n \to \infty$ for each k. In particular, we get from Theorem I.1 that $f_n \to f$ in the $L_1(\Omega, P)$ norm. This forces $f_n \to f$ in measure. Set $A_n = [|f_n - f| < \varepsilon]$.

$$\|f_n - f\|_E \leqslant \|(f_n - f)\chi_{A_n}\|_E + \|(f - f_n)\chi_{A_n^c}\|_E \leqslant \varepsilon\|\chi_{A_n}\|_E + \|f\chi_{A_n^c}\|_E + \|f_n\chi_{A_n^c}\|_E.$$

The convergence of $\Psi_k(f_n)$ for each k (to $\Psi_k(f)$) has forced the lattice norm uniform integrability of the sequence, so the terms $\|f\chi_{A_n^c}\|$ and $\|f_n\chi_{A_n^c}\|$ approach 0 as $n \to \infty$. This completes the proof for this case.

To extend this to the general case, let $|||\cdot|||_\alpha$ denote the norm just constructed on E_α for each α. The rest of the renorming follows the lines of the proof of Troyanski's renorming of WCG spaces as presented in Day's book [**4**, p. 162]. Let Q_α denote the band projection of E onto E_α. For any finite set $A \subset \Gamma$, let $E_A(f) = \|(I - \Sigma_{\alpha\in A} Q_\alpha)f\|_E$. This is again a lattice seminorm. Let $G_A(f) = (\operatorname{card} A)\Sigma_{\alpha\in A}|||Q_\alpha f|||_\alpha + E_A(f)$, and let $G_n(f) = \sup\{G_A(f)|\operatorname{card} A = n\}$. If we now let S be the index set $\{\{0\}, \Gamma, \mathbf{N}\}$, and define $\Phi_0(f) = \|f\|_E$, $2^n\Phi_n(f) = G_n(f)$ for $n \in \mathbf{N}$ and $\Phi_\alpha(f) = |||Q_\alpha f|||$, for all $\alpha \in \Gamma$, then $f \to \{\Phi_s(f)\}_{s\in S}$ takes E to $\tilde{c}_0(S)$. Thus, the equivalent lattice norm on E is defined to be $|||f||| = \|\{\Phi_s(f)\}\|_{\tilde{c}_0(S)}$. Again, $|||f_j||| \to |||f|||$ and $|||f + f_j||| \to 2|||f|||$ forces $\Phi_s(f_j) \to \Phi_s(f)$ and $\Phi_s(f + f_j) \to 2\Phi_s(f)$ for all s in S. By the preceding paragraph, we see that $Q_\alpha f_j \to Q_\alpha f$ in E_α for each α. By the definition of G_n, for any set A with cardinality k, we have $k\Sigma_{\alpha\in A}|||Q_\alpha f|||_\alpha + E_A(f) \leqslant G_k(f)$. Thus, $E_A(f_j) \leqslant G_k(f_j) - k\Sigma_{\alpha\in A}|||Q_\alpha f_j|||$. Now let $\varepsilon > 0$, and choose A so that $G_k(f) - G_A(f) < \varepsilon$. Since $G_k(f_j) \to G_k(f)$ and $|||Q_\alpha f_j|||_\alpha \to |||Q_\alpha f|||_\alpha$ as $j \to \infty$, we may choose j large enough so that $E_A(f_j) \leqslant E_A(f) + 2\varepsilon$. The goal now is to make $E_A(f)$ small for the proper choice of k.

Choose a set A_1 so that $E_{A_1}(f) < \varepsilon$ and so that $\min_{\alpha\in A_1}|||Q_\alpha f|||_\alpha = a > 0$. Let $A_2 = A_1 \cup \{\beta|\ |||Q_\beta f|||_\beta \geqslant a\}$. Since $\{|||Q_\alpha f|||_\alpha\} \in c_0(\Gamma)$, there is $\Delta > 0$ so that for $\beta \notin A_2$, $|||Q_\beta f|||_\beta < a - \Delta$. Let $k \geqslant \{\operatorname{card} A_2, (\varepsilon + \|f\|)/\Delta\}$, and select A with cardinality k so that $G_k(f) - G_A(f) < \varepsilon$. Suppose, $A_2 \not\subseteq A$ and let $\beta \in A_2 \setminus A$, $\gamma \in A \setminus A_2$. Let $B = A \cup \{\beta\} \setminus \{\gamma\}$. Then

$$\begin{aligned} G_k(f) - E_A(f) - k \sum_{\alpha\in A} |||Q_\alpha f|||_\alpha & \\ &\geqslant E_B(f) + k \sum_{\alpha\in B} |||Q_\alpha f|||_\alpha - E_A(f) - k \sum_{\alpha\in A} |||Q_\alpha f||| \\ &\geqslant -E_A(f) + k\Delta \geqslant -\|f\|_E + k\Delta > \varepsilon, \end{aligned}$$

which is a contradiction. Thus, $A \supset A_2 \supset A_1$ and so $E_A(f) \geqslant E_{A_1}(f) < \varepsilon$. We have, then, that for this choice of k and A, and for j sufficiently large, $E_A(f_j) < 3\varepsilon$. Finally, $\|f - f_j\|_E \leqslant E_A(f) + E_A(f_j) + k\Sigma_{\alpha\in A}|||Q_\alpha(f - f_j)|||_\alpha$. Each term of this finite sum tends to zero as $j \to \infty$, so we get for arbitrary ε, a j_0 so that $j > j_0$ implies $\|f - f_j\|_E < 4\varepsilon$, and we are done with the renorming.

If E is not order continuous, then E contains positive, norm one vectors $\{x_n\}$ converging weakly to zero, and x such that $x_n \leqslant x$ for all n. Thus, $0 \leqslant x - x_n < x$ and we have $\|x - x_n\| \to \|x\|$, but $x - x_n \not\to x$ so the norm cannot be locally uniformly convex and cannot even be Kadec-Klee.

We now write a more general version of Corollary I.1. Since the proofs are nearly identical with those of above, we do not include them here.

COROLLARY I.2. *If E is a separable, rearrangement invariant function space on* [0, 1], *it has an equivalent, rearrangement invariant locally uniformly convex lattice norm.*

COROLLARY I.3. *Suppose that E is a complemented subspace of a Banach lattice, F, and suppose that E does not contain a subspace isomorphic to c_0. Then E has an equivalent locally uniformly convex norm.*

PROOF. By **[9]**, E embeds into a lattice F_1 which does not contain c_0. Any such F_1 has order-continuous norm, and so may be renormed as above. This renorming is as desired on E.

II. Almost sure convergence of vector-valued processes. In this section, we will use the following version of the renorming theorem:

If E is a separable order continuous Banach lattice, then it admits an equivalent lattice norm which is Kadec-Klee with respect to a countable and norming subset D of the positive cone of the dual (i.e. $\|x\| = \sup\{x^*(|x|);\ x^* \in D\}$ for every $x \in E$).

Now let $(\Omega, \mathfrak{F}, P)$ be a probability space and (S_n) a sequence of E-valued Bochner integrable random variables. Since the S_n's are almost separably valued, we may and shall assume without loss of generality that E is separable. If E has also an order continuous norm, it will be equipped with the norm $\|\ \|$ and the set D mentioned above.

If $(\mathfrak{F}_n)_n$ is a sequence of sub-σ-algebras increasing to $\mathfrak{F}$, we say that $(S_n, \mathfrak{F}_n)$ is a submartingale if (S_n) is $(\mathfrak{F}_n)$-adapted and if $E^{\mathfrak{F}_n}[X_{n+1}] \geqslant X_n$ a.e. for each $n \geqslant 0$. The basic facts about vector measures which we use are available in the book **[6]**.

In the following, we will make constant use of the following lemma proved in **[13]**, p. 103].

LEMMA II.1. *Let I be a denumerable set, and for each $i \in I$, let $(S_n^i)_n$ be a real submartingale. If* $\sup_n \int \sup_i (S_n^i)^+ < \infty$, *then*

(i) *for every $i \in I$, (S_n^i) converges a.e. to a limit S_∞^i,*

(ii) *the submartingale* $(\sup_i S_n^i)$ *converges a.e. to* $\sup_i S_\infty^i$.

THEOREM II. 1. **[8]**. *If E is a Banach lattice with the Radon-Nikodym property, then every E-valued L^1-bounded positive submartingale converges strongly a.e.*

PROOF. For every $A \in \bigcup_n \mathfrak{F}_n$, $(\int_A S_n)$ is an increasing sequence (after a certain rank) which is norm bounded; hence it converges to $\mu(A)$. The vector measure μ is of bounded variation, hence it is strongly additive. By the Caratheodory-Hahn-Kluvanek extension theorem **[6]**, p. 27], μ can be extended to a countably additive measure on $\mathfrak{F}$, absolutely continuous with respect to P and of finite variation. Thus, by the Radon-Nikodym property, there exists an E-valued Bochner integrable random variable S_∞ such that $\lim_n \int_A S_n = \int_A S_\infty$ for every $A \in \mathfrak{F}$.

For each $f \in D$, $f(S_n)$ is a real-valued submartingale which converges necessarily to $f(S_\infty)$ outside Ω_f with $P(\Omega_f) = 0$. On the other hand, we have for every $x \in E_+$, $\|x\| = \sup_{f \in D} f(x)$, hence Lemma II.1 yields that $\|S_n\| = \sup_{f\in D} f(S_n)$ converges to $\sup_{f \in D} f(S_\infty) = \|S_\infty\|$ outside Ω_0 with $P(\Omega_0) = 0$. The Kadec-Klee property of the norm gives the strong convergence of (S_n) outside $\Omega_0 \cup \bigcup_{f \in D} \Omega_f$.

REMARKS. (1) The classical Kadec renorming theorem (mentioned in the introduction) gives a relatively easy proof of the well-known convergence theorem for vector-valued martingales [3]. If E is a Banach space with the Radon-Nikodym property and (S_n) is an E-valued L^1-bounded martingale, then the same reasoning as in Theorem II.1 gives a random variable S_∞ such that $\lim_n \int_A S_n = \int_A S_\infty$ for every $A \in F$. Lemma II.1 applies to give that $\|S_n\| = \sup_{f\in D}|f(S_n)|$ converges a.e. to $\|S_\infty\|$. The convergence follows from the Kadec-Klee property of the norm.

(2) A similar proof can be given for uniform amarts, since it may be shown [1] that $\|S_n\|$ is then a real amart and if the space has the R.N.P., it converges necessarily a.e. to $\|S_\infty\|$.

THEOREM II.2. *If E has an order-continuous norm, and (S_n) is a submartingale such that $0 \leqslant S_n \leqslant E^{\mathfrak{F}_n}[S]$ for some random variable S, then (S_n) converges strongly a.e.*

PROOF. Let Ω_0 be the null set where $E^{\mathfrak{F}_n}[S]$ does not converge. Without loss of generality, we may assume that the limit is S. (Otherwise, replace S by $E^{\mathfrak{F}}(S)$ where $\mathfrak{F} = \bigvee \mathfrak{F}_n$.) Clearly $S \leqslant S_n \vee S \leqslant E^{\mathfrak{F}_n}[S] \vee S$, and hence $\lim S_n \vee S = S$ outside Ω_0. For each $f \in D$, there is an Ω_f with $P(\Omega_f) = 0$ such that $f(S_n)$ converges on $\Omega \setminus \Omega_f$. Since $S_n \vee S + S_n \wedge S = S_n + S$, we get that off the set $\Omega_1 = \Omega_0 \cup \bigcup_D \Omega_f$, $\lim f(S_n \wedge S)$ exists for every $f \in D$. Since order intervals in an order-continuous lattice are weakly compact, we deduce that $S_n \wedge S$ converges weakly off Ω_1. Therefore, S_n converges weakly to some random variable, S_∞.

Again, Lemma II.1 applies and gives that $\|S_n\|$ converges $\|S_\infty\|$ a.e. and the theorem follows by the Kadec-Klee property.

COROLLARY II.1. *If E has an order-continuous norm, let μ be an E-valued vector measure order bounded by two differentiable vector measures; that is there exists X and Y in $L^1[E]$ such that for every $A \in \mathfrak{F}$*

$$\int_A Y\,dP \leqslant \mu(A) \leqslant \int_A X\,dP.$$

Then, there exists Z in $L^1[E]$ such that $Y \leqslant Z \leqslant X$ a.e. and $\mu(A) = \int_A Z\,dP$ for every $A \in \mathfrak{F}$.

PROOF. Clearly, we may assume that $0 \leqslant \mu(A) \leqslant \int_A X\,dP$ for every $A \in \mathfrak{F}$. We associate now to μ, a positive martingale as follows. Assuming the σ-field separable, we may construct an increasing sequence of σ-fields $(\mathfrak{F}_n)_n$ generated by finite partitions $\{B_n^p;\ 1 \leqslant p \leqslant p_n\}$ and generating the σ-field $\mathfrak{F}$. Set

$$Z_n = \sum_{1 \leqslant p \leqslant p_n} \frac{\mu(B_n^p)}{P(B_n^p)} \chi_{B_n^p} \quad \text{with } \frac{\mu(B_n^p)}{P(B_n^p)} = 0 \text{ if } P(B_n^p) = 0.$$

The sequence (Z_n) is an $\mathfrak{F}_n$-martingale verifying $0 \leqslant Z_n \leqslant E^{\mathfrak{F}_n}[X]$. By the preceding theorem, Z_n converges to Z. Clearly, $0 \leqslant Z \leqslant X$ and $\mu(A) = \int_A Z\,dP$ for every $A \in \mathfrak{F}$.

In case the field is not separable, we note that, by the first part of the proof, for every separable sub-σ-field, $\mathfrak{F}^1$ of $\mathfrak{F}$, $\{\mu(A) | A \in \mathfrak{F}^1\}$ is totally bounded. Hence, $\{\mu(A) | A \in \mathfrak{F}\}$ is also totally bounded, and thus, separable. We may therefore assume that E is separable. Let $D \subset E^*$ be a countable set as usual. The result follows by virtue of the fact that it is valid for the separable sub-σ-field with respect to which all $d(f \circ \mu)/dP$ are measurable for all $f \in D$.

The following corollary can be alternately proved as a corollary of Theorem III.2.18 in [**6**].

COROLLARY II.2. *If E has an order-continuous norm and X is an E-valued, Bochner integrable random variable then*

$$\int_A X^+ \, dP = \sup_{B \in \mathcal{F}} \int_{A \cap B} X \, dP \quad \text{for every } A \in \mathcal{F}.$$

PROOF. Since E has an order-continuous norm, $\mu^+(A) = \sup_{B \in \mathcal{F}} \int_{A \cap B} X \, dP$ exists and is a countably additive measure verifying for every $A \in \mathcal{F}$, $0 \leqslant \mu^+(A) \leqslant \int_A X^+ \, dP$. Corollary II.1 applies to give Z in $L^1[E]$ such that $0 \leqslant Z \leqslant X^+$ and $\mu^+(A) = \int_A Z \, dP$ for all $A \in \mathcal{F}$. The same reasoning applied to μ^- shows that there exists Y, $0 \leqslant Y \leqslant X^-$ and $\mu^-(A) = \int_A Y \, dP$. We get that $Z - Y = X^+ - X^-$ a.e. and since both of them are orthogonal, we get that $Z = X^+$ and $Y = X^-$.

COROLLARY II.3 (VECTOR-VALUED STRASSEN THEOREM). *If F is a vector space, E an order-continuous Banach lattice, Ψ: $F \to L^1[E]$ is sublinear and φ: $F \to E$ is linear such that $\varphi(x) \leqslant \int \Psi(x) \, dP$ for all $x \in F$, then there exists a linear operator T: $F \to L^1[E]$ such that $T(x) \leqslant \Psi(x)$ and $\varphi(x) = \int T(x) \, dP$ for all x in F.*

PROOF. Let M be the space of all F-valued measurable simple functions. The vector space F can be identified to the subspace of all constant functions from Ω to F. Obviously, the mapping θ: $M \to E$ defined by $\theta(f) = \int \Psi(f(w))(w) \, dP$ is sublinear, thus, the generalized Hahn-Banach theorem [**14**, p. 109] gives a linear mapping Φ: $M \to F$ such that $\Phi \leqslant \theta$ on M and $\Phi = \varphi$ on F. Therefore, for every $x \in F$, we have $-\int_A \Psi(-x) \, dP \leqslant \Phi(x\chi_A) \leqslant \int_A \Psi(x) \, dP$ for all $A \in F$. Applying Corollary II.1 to the measures μ_x: $(\Omega, \mathcal{F}, P) \to E$ defined by $\mu_x(A) = \Phi(x\chi_A)$ gives a Bochner integrable random variable f_x such that

$$\mu_x(A) = \int_A f_x \, dP \quad \text{for every } A \in \mathcal{F}.$$

It is immediate to check that the operator T: $F \to L^1[E]$ defined by $T(x) = f_x$ verifies the claimed properties.

Corollary II.3 was proved by M. Neumann in the case that E has the Radon-Nikodym property [**14**].

Now let θ be a measure-preserving point transformation on $(\Omega, \mathcal{F}, P)$ and I be the σ-field of θ-invariant sets. If E is a Banach space and S an E-valued Bochner integrable random variable, the partial sums $S_n = \sum_{i=1}^n S \circ \theta^{i-1}$ form an additive process, that is we have for every $n, k \geqslant 0$, $S_{n+k} = S_n + S_k \circ \theta^n$. If E is a Banach lattice and (S_n) is a sequence of random variables satisfying for every $n, k \geqslant 0$, $S_{n+k} \leqslant S_n + S_k \circ \theta^n$, then (S_n) is said to be a subadditive process.

The vector-valued ergodic theorem proved by Mourier in [**13**] asserts that $S_n/n = (1/n)\sum_{i=1}^n S \circ \theta^{i-1}$ converges a.e. to $E^I[S]$. An alternate proof using the Kadec renorming theorem goes as follows: $\|S_n\|$ is a real-valued subadditive process, hence by Kingman's theorem [**10**], $\|S_n\|/n$ converges a.e. A standard argument shows that S_n is Cauchy in $L^1[E]$ which permits us to identify the limit of $\|S_n\|/n$ as $\|E^I[S]\|$ and the theorem follows. A proof of this result using the Kadec-Klee renorming theorem also appears in **[11]**.

THEOREM II.3 **[7]**. *If E is an order-continuous lattice, then for every positive E-valued subadditive process (S_n) we have that S_n/n converges strongly a.e.*

PROOF. Note first that if (x_n) is a positive subadditive sequence $(x_{n+k} \leq x_n + x_k)$ then $(x_{2^k})/2^k$ is decreasing and $\inf(x_n/n) = \inf(x_{2^k}/2^k)$. $\|x_n\|$ is a real subadditive sequence, thus $\lim(\|x_n\|/n) = \lim(\|x_{2^k}\|/2^k) = \|\inf(x_n/n)\|$. Also, $f(x_n)$ is subadditive for every $f \in E_+$ hence $f(x_n/n) \to f(\inf(x_n/n))$. The norm convergence follows from the Kadec property of the norm.

Suppose now (S_n) a positive subadditive process and I the σ-field of θ-invariant sets. Clearly, $(E^I[S_n](w))$ is a subadditive sequence for almost every w, thus, if divided by n, it converges to $z(w) = \inf E^I([S_n]/n)(w)$.

For every $f \in D$, we apply Kingman's theorem **[10]** to the real process $f(S_n)$ to obtain that $\lim f(S_n/n) = f(\inf E^I([S_n]/n)) = f(Z)$ outside Ω_f with $P(\Omega_f) = 0$. Therefore, $\lim \inf(\|S_n\|/n) \geq \|Z\|$ outside $\Omega_0 = \bigcup_{f\in D} \Omega_f$.

Now fix k and suppose $n \geq k$. Denote by $N(n)$ the integral part of n/k and apply subadditivity to obtain

$$S_n \leq \sum_{r=1}^{N(n)} S_k \circ \theta^{(r-1)k} + S_{n-Nk} \circ \theta^{NR} \leq \sum_{r=1}^{N(n)} Y_r + W_N$$

where $Y_r = S_k \circ \theta^{(r-1)k}$ and $W_N = \sum_{j=1}^{k-1} S_j \circ \theta^{Nk}$. It follows that

$$\limsup \frac{\|S_n\|}{n} \leq \limsup \frac{1}{Nk}\left\| \sum_{r=1}^{N} Y_r \right\| + \limsup \frac{1}{Nk}\|W_N\|.$$

But $\sum_{r=1}^{N} Y_r$ is an additive process, hence, by Mourier's theorem, $(1/Nk)\sum_{r=1}^{N} Y_r$ converges to $E^I[S_k/k]$. On the other hand, $\|W_N\|$ has the same distribution for every N, thus by Borel-Cantelli, $\lim(\|W_N\|/N) = 0$.

We get finally, $\lim\sup(\|S_n\|/n) \leq \|\inf E^I[S_k/k]\|$ a.e. and the renorming theorem gives the convergence.

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