HOMOTOPY GROUPS OF THE SPACE OF
SELF-HOMOTOPY-EQUIVALENCES

BY

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ABSTRACT. Let M be a connected sum of r closed aspherical manifolds of
dimension n > 3, and let EM denote the space of self-homotopy-equivalences of
M, with basepoint the identity map of M. Using obstruction theory, we calculate
πq(EM) for 1 < q < n - 3 and show that πn-3(EM) is not finitely-generated. As
an application, for the case n = 3 and r > 3 we show that infinitely many
generators of π1(EM^3, id_M) can be realized by isotopies, to conclude that
π1(Homeo(M^3), id_M) is not finitely-generated.

0. Introduction. Let EX be the H-space of homotopy equivalences from X to X,
with the identity map of X as basepoint. It contains the basepoint-preserving
self-homotopy-equivalences E0X, the group of homeomorphisms Homeo(X), and,
when X is a smooth manifold, the group of diffeomorphisms Diff(X). The inclu-
sions of these subspaces are H-space homomorphisms. From knowledge of EX,
one hopes to obtain information about these subspaces.

The groups π0(E0X) and π0(EX) have been studied for various classes of spaces.
It was shown by Sullivan [S] and, independently, Wilkerson [W] that when X is a
simply-connected finite complex, π0(EX) is finitely-presented. In contrast, Frank
and Kahn [F-K] showed that for p > 2, π0(E0(S^1 ∨ S^p ∨ S^{2p-1})) is not finitely-
generated. There are examples of finite aspherical 4-complexes K^4 with π0(E0(K^4))
not finitely-generated [M3].

Little is known about the homotopy groups π_i(EX) for i > 1 except for two
important cases. For X an aspherical complex, Gottlieb [G] proved that π_i(EX) ≅
center(π_1(X)) while π_i(EX) = 0 for i > 2. It follows that π_i(E0X) = 0 for j > 1. The
other case is that of the n-sphere S^n, for which π_q(ES^n) = [S^q; Maps(S^n, S^n)]
≅ [S^q ∩ S^n, S^n] ≅ π_{n+q}(S^n).

In this paper, I adapt the obstruction theory of Federer [F] to obtain some
calculations of the homotopy groups of EM, where M is any connected sum of
r > 2 (closed) aspherical (combinatorial) manifolds of dimension n > 3. Specifi-
cally:

(1) For 1 < q < n - 4, π_q EM ≅ ⊕_{i=1}^{r-1} π_{n+q}(S^n_i), hence is finite.
(2) For $n > 4$, $\pi_{n-3}EM$ is a quotient of $\bigoplus_{i=1}^{n-1} \pi_{2n-3}(S^{n-1})$, and is finite.
(3) $\pi_{n-2}EM$ is infinitely-generated as an abelian group.

For the case $n = 3$ and $r > 3$, I show that infinitely many of the generators of $\pi_1(EM)$ can be realized as isotopies (which can be taken to be diffeotopies) of $M$. Therefore:
(4) For $n = 3$ and $r > 3$, $\pi_1(\text{Homeo}(M), \text{id}_M)$ and $\pi_1(\text{Diff}(M), \text{id}_M)$ are infinitely-generated.

The construction of these isotopies is very explicit. The results (1), (2), and (3) appeared in my dissertation, submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University of Michigan. I wish to thank my advisor Professor Frank Raymond for his patient encouragement and helpful suggestions. I also wish to thank the referee for suggesting several significant improvements to the manuscript of this paper.

Here is a description of the program I will use to make these calculations. Let $Y$ be a CW complex and let $Y^k$ denote its $k$-skeleton. If $A$ and $B$ are subcomplexes with $B \subset A \subset Y$, let $(Y; A, B)$ be the space of continuous maps from $A$ to $Y$ which restrict to the inclusion map on $B$. The inclusion map of $A$ is the basepoint of $(Y; A, B)$. Let $Y'[A, B] \subset (Y; A, B)$ be the subspace of maps which extend to all of $Y$. Because $A$ has the Homotopy Extension Property in $Y$, $Y'[A, B]$ consists of path components of $(Y; A, B)$. There are three fibrations in which the projection maps are restriction:

$$Y^{[Y, Y]} \rightarrow Y^{[Y, Y]}, \quad Y^{[Y, Y]} \rightarrow Y^{[Y, Y]}, \quad Y^{[Y, Y]} \rightarrow Y^{[Y, Y]},$$

These fit into the following diagram in which the row and columns are fibrations:

$$
\begin{array}{ccc}
\Omega(Y^{[Y, Y]}) & \rightarrow & \Omega(Y^{[Y, Y]}) \\
\downarrow & \downarrow & \downarrow \\
\Omega(Y^{[Y, Y]}) & \rightarrow & \Omega(Y^{[Y, Y]}) \\
\downarrow & \downarrow & \downarrow \\
Y^{[Y, Y]} & \rightarrow & Y^{[Y, Y]}
\end{array}
$$

It is easy to produce from this diagram a long exact sequence:

$$\ldots \rightarrow J_q \rightarrow K_q \rightarrow \pi_q(Y^{[Y, Y]}) \rightarrow J_{q-1} \rightarrow K_{q-1} \rightarrow \ldots$$

where

$$J_q = \text{coker}\left(\pi_{q+1}(Y^{[Y, Y]}) \rightarrow \pi_{q+1}(Y^{[Y, Y]})\right),$$

$$K_q = \text{coker}\left(\partial: \pi_{q+1}(Y^{[Y, Y]}) \rightarrow \pi_q(Y^{[Y, Y]})\right),$$

and $D_q$ is induced by $\partial: \pi_{q+1}(Y^{[Y, Y]}) \rightarrow \pi_q(Y^{[Y, Y]})$.

In §1, under certain assumptions on $Y$, we will identify $J_q$ and $K_q$ as cohomology modules of $Y$ and discuss the boundary homomorphism $D_q$. In §2, we list the properties of connected sums of aspherical manifolds which allow explicit calculations of the modules to be made. The results called (1), (2) and (3) above are obtained in §3, and the isotopies of 3-manifolds are constructed in the final section.
1. Obstruction theory preliminaries. We will denote by $I^q$ the $q$-dimensional cube $[0, 1]^q$, by $J^q$ the closure of the complement of $I^q \times \{0\}$ in $\partial I^{q+1}$.

1.A. The boundary homomorphism for fibrations of spaces of mappings. Suppose that $C \subset B \subset A$ are subcomplexes of the CW complex $Y$. For the fibration $Y^{[A, B]} \to Y^{[A, C]} \to Y^{[B, C]}$ the boundary homomorphism $\partial: \pi_{q+1}(Y^{[B, C]}) \to \pi_q(Y^{[A, B]})$ can be described as follows [F, pp. 346–347]. Let $\langle w \rangle \in \pi_{q+1}(Y^{[B, C]})$; then $w$ is defined on the subset $B \times I^q \times \{1\} \subset Y \times I^{q+1}$. Extend $w$ to $A \times J^q \cup B \times I^{q+1}$ using the projection map to $A$. By the Homotopy Extension Property applied to the pair $(A \times I^q \times \{1\}, A \times \partial I^q \times \{1\} \cup B \times I^q \times \{1\})$, we obtain an extension to all of $A \times I^{q+1}$. If $u$ denotes the restriction of this extension to $A \times I^q \times \{0\}$, then $\langle w \rangle = \partial \langle u \rangle$.

1.B. Calculation of $J_q = \text{coker}(\pi_{q+1}(Y^{[Y^2, Y^1]}) \to \pi_q(Y^{[Y^2, Y]}))$. All cochains and cohomology will be with local coefficients. We will denote by $\text{proj}_{y^2}$ the projection map from $Y \times I^{q+1}$ to $Y$, or its restriction to any subspace of $Y \times I^{q+1}$. Let $\ast \in Y^0$ be the basepoint of $Y$.

**Lemma 1.B.1.** If $\pi_{q+1}(Y^{[Y^2, Y^1]}) \to \pi_{q+1}(Y^{[Y^2, Y]})$ is surjective, then

$$J_q = H^1(Y; \pi_{q+2}Y).$$

**Proof.** Let $\langle f \rangle \in \pi_{q+1}(Y^{[Y^2, Y]})$; then $f|_{Y^2 \times \partial I^{q+1}} = \text{proj}_y$. By assumption we may choose $f$ so that $f|_{Y^2 \times I^{q+1}} = \text{proj}_y$. Consider the difference cochain $d_{q+2}(\text{proj}_y f) \in C^{q+2}(Y^2 \times I^{q+1}, Y^2 \times \partial I^{q+1}; \pi_{q+2}Y) \approx C^1(Y; \pi_{q+2}Y)$. We have $\delta d_{q+2}(\text{proj}_y f) = c_{q+3}(\text{proj}_y f) - c_{q+2}(f) = 0$ since both $\text{proj}_y f$ and $f$ admit extensions to $Y^2 \times I^{q+2}$. Thus we may define $d_1 \langle f \rangle = \langle d_{q+2}(\text{proj}_y f) \rangle \in H^1(Y; \pi_{q+2}Y)$. Changing $f$ by a homotopy on $Y^0 \times I^{q+1}$ alters $d_{q+2}(\text{proj}_y f)$ by a coboundary so $d_1$ is well defined, and it is easy to see that $d_1$ is a homomorphism which vanishes on image($\pi_{q+1}(Y^{[Y^2, Y^1]})$). If $d_1 \langle f \rangle = 0$, then $f|_{Y^2 \times I^{q+1}}$ is homotopic to $\text{proj}_y f$, so $\langle f \rangle \in \text{image}(\pi_{q+1}(Y^{[Y^2, Y^1]}))$; thus $d_1: J_q \to H^1(Y; \pi_{q+2}Y)$ is injective. Given $\{c\} \in H^1(Y; \pi_{q+2}Y)$, define $f: Y^1 \times I^{q+1} \to Y$ so that $f|_{Y^2 \times I^{q+1}} = \text{proj}_y$ and $\langle f \rangle = \langle f \rangle|_{o \times I^{q+1}} = c(\sigma)$ for each $\sigma \in Y^1 \setminus Y^0$. Since $\delta c = 0$, $f$ extends to $Y^2 \times I^{q+1}$, and $d_1 \langle f \rangle = \{c\}$. Therefore $d_1$ is surjective. □

1.C. Calculation of $K_q = \text{coker}(\partial: \pi_{q+1}(Y^{[Y^2, Y^1]}) \to \pi_q(Y^{[Y^2, Y]^1}))$.

**Lemma 1.C.1.** If $H^p(Y; \pi_{p+q}Y) = H^{p-1}(Y; \pi_{p+q}Y) = 0$ for $3 < p < n - 1$, then $K_q \approx H^n(Y; \pi_{n+q}Y)$.

**Proof.** Define $d_n: K_q \to H^n(Y; \pi_{n+q}Y)$ as follows. Let $\langle f \rangle$ represent an element of $K_q$. If $n = 3$, let $d_n \langle f \rangle = \{d_{n+q}(\text{proj}_y f)\}$. Suppose $n > 3$. Then $\delta d_{q+3}(\text{proj}_y f) = c_{q+4}(\text{proj}_y f) - c_{q+3}(f) = 0$ and $\{d_{q+3}(\text{proj}_y f)\} \in H^3(Y; \pi_{q+3}Y) = 0$. Hence there is a homotopy $F: f \approx f_1$ (rel $Y^1 \times I^q$) with $f_1|_{Y^2 \times I^{q+1}} = \text{proj}_y$. Let $g: Y^2 \times I^{q+1} \to Y$ be $F|_{Y^2 \times (I^{q+1})}$. Then $g$ represents an element of $\pi_{q+1}(Y^{[Y^2, Y^1]})$ such that $d_{q+3}(\text{proj}_y f) = d_{q+3}(\text{proj}_y f) = 0$. Moreover $\langle f_1 \rangle = \langle f \rangle - \langle g \rangle$ because $\langle f \rangle$ represents the same element of $K_q$ as $\langle f \rangle$ did. Inductively, for $4 < k < n - 1$, assume $f_k|_{Y^2 \times (I^{q-k})} = \text{proj}_y$. We have $\{d_{k+q}(\text{proj}_y f_k)\} \in H^k(Y, \pi_{k+q}Y) = 0$. Therefore $f_k$ is homotopic to a map, again called $f_k$, such that $f_k|_{Y^2 \times I^{q-k}} = \text{proj}_y$. This completes the induction. Let $d_n \langle f \rangle = \{d_{n+q}(\text{proj}_y f)\} \in H^n(Y; \pi_{n+q}Y)$. 

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We must show this assignment is well defined. Suppose \( \langle f' \rangle \) is another homotopy class with \( f'_1|_{Y^{-1} \times I^r} = \text{proj}_y \) and \( f'_1 \simeq f \) (rel \( Y^1 \times I^q \)). Then \( f_1 \simeq f \) (rel \( Y^1 \times I^q \)) and we must show \( f_1 \simeq f' \) (rel \( Y^{n-2} \times I^q \)). For \( n = 3 \), this is automatic, so assume \( n > 3 \). Let \( G: Y \times I^q \times I \to Y \) be a homotopy from \( f_1 \) to \( f'_1 \). Inductively, for \( 2 < k < n - 2 \), suppose \( G|_{Y^{k-1} \times I^{q+k}} = \text{proj}_y \). Then \( \{d_{q+1} + k(\text{proj}_y, G)\} \in H^k(Y; \pi_{(q+1)+k} Y) = 0 \) so \( G \) is homotopic (rel \( Y \times \partial I^{q+1} \)) to a new homotopy, also called \( G \), with \( G|_{Y^1 \times I^{q+1}} = \text{proj}_y \). This completes the induction; thus \( f_1 \simeq f' \) (rel \( Y^{n-2} \times I^q \)) so \( d_n \langle f'_1 \rangle = d_n \langle f_1 \rangle \).

Clearly, \( d_n \) is a surjective homomorphism. It remains to show \( d_n \) is injective. Suppose \( d_n \langle f \rangle = 0 \). By the preceding argument, we can find \( \langle f' \rangle \) with \( f'_1|_{Y^{-1} \times I^r} = \text{proj}_y \), \( d_n \langle f'_1 \rangle = 0 \), and \( f \simeq f_1 \) (rel \( Y^1 \times I^q \)). Let \( H: f \simeq f \) \( \text{proj}_y \) (rel \( Y^1 \times I^q \)). Letting \( g = H|_{Y^1 \times I^{q+1}} \), we have \( \langle f \rangle = \partial \langle g \rangle \), so \( \langle f \rangle \) represents the zero element of \( K_q \).

1.D. An important example. The following example illustrates the techniques we use for computing homotopy groups of mapping spaces, and it is pertinent to the manifolds we will be considering. Let \( X = S^{n-1} \times I \). We regard it as a cell complex with six cells: two 0-cells \( * \times \{0\} \) and \( * \times \{1\} \), one 1-cell \( \sigma = * \times I \) connecting the 0-cells, two \((n-1)\)-cells \( S^{n-1} \times \{0\} \) and \( S^{n-1} \times \{1\} \), and one \( n \)-cell \( \tau \). Letting \( C = \partial X, B = \sigma \cup \partial X, \) and \( A = X \), the fibration of \( \Sigma A \) becomes

\[
X^{[X, \sigma \cup \partial X]} \to X^{[X, \partial X]} \to X^{[\sigma \cup A, \partial X]}.
\]

It is not difficult to observe that \( X^{[\sigma \cup A, \partial X]} = X^{[\sigma, \partial A]} \simeq \Omega X \) and, because the attaching map of \( \tau \) is null-homotopic, that \( X^{[X, \sigma \cup \partial X]} \simeq \Omega^n X \). Therefore the homotopy exact sequence for the fibration becomes

\[
\ldots \to \pi_{q+2}(X) \to \pi_{n+q}(X) \to \pi_q(X^{[X, \partial X]}) \to \pi_{q+1}(X) \to \pi_{n+q-1}(X) \to \ldots
\]

It is a lengthy exercise (written out in [M2]) to check that, up to sign, \( D_q(u) \) equals the Whitehead product \([z, u]\) where \( z \) is a generator of \( \pi_{n-1}(X) \simeq \mathbb{Z} \).

For the calculations of §§3 and 4, we should describe the isomorphisms \( d_1: \pi_{q+1}(X^{[\sigma \cup A, \partial X]}) \simeq \pi_{q+2}(X) \) and \( d_1: \pi_q(X^{[X, \sigma \cup \partial X]}) \simeq \pi_{q+1}(X) \) more explicitly. Let \( f: (\sigma \cup \partial X) \times I^{q+1} \to X \) represent an element of \( \pi_{q+1}(X^{[\sigma \cup A, \partial X]}) \); then the restriction of \( f \) to \( \partial X \times I^{q+1} \cup (\sigma \cup \partial X) \times \partial I^{q+1} \) equals the projection map to \( X \). We define \( d_1(f) \) to be the value of the difference cochain \( d_{q+2}(f, \text{proj}_X) \) on the \( (q+2) \)-cell \( \sigma \times I^{q+1} \). The definition of \( d_n \) is similar.

2. Connected sums of aspherical manifolds. The letter \( M \) will always denote a connected sum \( M_1 \# M_2 \# \cdots \# M_r \) of \( r \geq 2 \) closed aspherical manifolds of dimension \( n \geq 3 \). We note that \( \pi_1 M = \pi_1 M_1 \star \pi_1 M_2 \star \cdots \star \pi_1 M_r \) is torsion-free, since each \( \pi_1 M_i \) is (being the fundamental group of a finite-dimensional aspherical complex).

2.A. The homotopy groups of \( M \). We will state some results and notation to be used later. Except where otherwise noted, detailed proofs may be found in [M1].

The following theorem extends Bloomberg's [B] description of the universal cover of a connected sum.
Theorem 2.A.1 The universal cover \( \tilde{M} \) of \( M \) is homotopy-equivalent to a 1-point union \( \bigvee_{i=1}^{2n-4} S_g^i \) of \((n-1)\)-spheres. Furthermore, the action of \( \pi_1 M \) on \( \tilde{M} \) corresponds to the left permutation action of \( \pi_1 M \) on the indices. That is, \( g_i \in \pi_1 M \) sends \( S_g^i \) homeomorphically to \( S_g^{i+1} \).

We will use \( e \) to denote the identity element of a group \( \pi \).

Definition. A \( \mathbb{Z} \)-module \( A = A_e \) is called a \( \pi \)-basis for a \( \mathbb{Z}\pi \)-module \( N \) if

1. \( N \cong \bigoplus_{g \in \pi} A_g \)
2. \( g : A_e \to A_g \) is the action of \( \pi \) on \( A_e \subseteq N \).

It follows that \( g : A_h \to A_{gh} \) for all \( g, h \in \pi \), and that any element of \( N \) can be written uniquely (up to order of summands) as \( \sum_{i=1}^{2n-4} g_i a_i \), where \( g_i \in \pi \), \( a_i \in A_e \).

Let \( X \) be a connected simplicial complex with universal cover \( \tilde{X} \). Let \( \pi = \pi_1 X \) and denote by \( H^i_j(X; N) \) the \( \pi \)-cohomology of \( X \) with local coefficients in \( N \) (and finite cochains). The following lemma is standard for the case \( N = \mathbb{Z}_{\pi} \), \( A_e = \mathbb{Z} \cdot e \).

Lemma 2.A.2. (a) \( H^i_j(X; N) \cong H^i_j(\tilde{X}; A_e) \),
(b) \( H^i_j(X; \pi_q M) = 0 \) for \( 2 < j < n - 1 \).
(c) \( H^i_j(M; \pi_q M) \cong A_q \).

The proof parallels the proof of the standard case (for details, see the appendix of [M2]). Using Theorem 2.A.1 and Lemma 2.A.2(a) together with Poincaré Duality in \( \tilde{M} \), one obtains

Lemma 2.A.3. Let \( q \) be a dimension in which \( \pi_q M \) has a \( \pi_1 M \)-basis \( A_q \). Then

(a) \( H^i_j(M; \pi_q M) \cong \bigoplus_{g \in \pi} \pi_{q-1}(S_g^i) \).
(b) \( H^i_j(M; \pi_q M) = 0 \) for \( 2 < j < n - 1 \).
(c) \( H^i_j(M; \pi_q M) \cong A_q \).

We will first describe \( A_q \) for \( 2 < q < 2n - 4 \). Order the elements of \( \pi \) arbitrarily as \( g_1, g_2, \ldots \). For \( k > 1 \) let \( T_k = \bigvee_{j=1}^{k-1} S_g^i \subset \bigvee_{g \in \pi} \bigvee_{i=1}^{2n-4} S_g^i = T \). Then for all \( m > 2 \), \( \pi_m(M; *) \cong \pi_m(T) \cong \operatorname{ind}\lim_{k \to \infty} \pi_m(T_k) \). According to Hilton [H2], \( \pi_q(T_k) \cong \bigoplus_{g \in \pi} \pi_{q-1}(S_g^i) \) and thus \( \pi_q(T) \cong \bigoplus_{g \in \pi} \bigoplus_{i=1}^{2n-4} \pi_{q-1}(S_g^i) \). Since \( g_1 S_g^i = S_g^{i+1} \), it is clear that \( A_q = \bigoplus_{i=1}^{2n-4} \pi_q(S_g^i) \) is a \( \pi \)-basis for \( \pi_q T \cong \pi_q M \).

In dimensions \( 2n - 3 < q < 3n - 6 \) the first Whitehead products appear. As above, we have \( \pi_m M \cong \bigoplus_{g \in \pi} \pi_{m-1}(S_g^i) \), and we may choose generators \( z^i_g \) of \( \pi_{m-1}(S_g^i) \) so that \( g_1 z^i_g = z^i_{g_1 g} \). For each \( (\alpha, \beta) \in \pi \times \pi \) and \( 1 < i < j < n - 1 \), let \( z_{a, \beta} \) be a generator of \( \pi_{2n-3}(S^i_{a, \beta}) \), where \( S^i_{a, \beta} \) is a copy of the \((2n-3)\)-sphere mapped to \( T \) in such a way that the induced homomorphism sends \( z^i_{a, \beta} \) to the Whitehead product \( [z^i_{a, \beta}] \). We will always exclude the case of both \( i = j \) and \( a = \beta \). In all the remaining cases, according to Hilton [H2], the image of \( \pi_m(S^i_{a, \beta}) \) is a direct summand of \( \pi_m T \) for all \( m \), and it will be regarded as a subgroup. Moreover, using direct limits again, there is a direct sum decomposition when \( 2n - 3 < q < 3n - 6 \):

\[
\pi_q(T) \cong \bigoplus_{g \in \pi} \bigoplus_{i=1}^{2n-4} \pi_q(S_g^i) \bigoplus \bigoplus_{1 \leq i < j \leq 2n-3} \pi_q(S^i_{a, \beta}) \bigoplus \bigoplus_{(\alpha, \beta) \in \pi \times \pi} \bigoplus_{1 \leq k < l \leq 2n-3} \pi_q(S^i_{a, \beta}).
\]
Since $[z_a^i, z_{\beta}^i] = (-1)^{n-1}[z_{\beta}^i, z_a^i]$ there are commutative diagrams for all $m$:

$$\pi_m(S_{a,\beta}) \rightarrow \pi_m(\tilde{M})$$

$$(-1)^{n-1} \rightarrow$$

$$\pi_m(S_{\beta,a}^i)$$

The action of $\pi$ on $\pi_{2n-3}(\tilde{M})$ satisfies $g \cdot z_{a,\beta}^m = z_{g \cdot a,\beta}^m$. Now let $\Gamma$ be a subset of $\pi$ having the following properties:

1. $e \notin \Gamma$.
2. For every $g \in \pi$ with $g \neq e$, exactly one of $g$ and $g^{-1}$ is contained in $\Gamma$.

Since $\pi$ is torsion-free, the second condition makes sense. In [M1] the following was proved.

**Lemma 2.A.4.** For $2 < q < 3n - 6$, $\pi_q M$ has a $\pi$-basis $A_q$ given in the following table:

<table>
<thead>
<tr>
<th>range of $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 &lt; q &lt; n - 2$</td>
</tr>
<tr>
<td>$n - 1 &lt; q &lt; 2n - 4$</td>
</tr>
<tr>
<td>$2n - 3 &lt; q &lt; 3n - 6$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r &gt; 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi_q(S_e)$</td>
</tr>
<tr>
<td>$\pi_q(S_{g_e})$</td>
</tr>
<tr>
<td>$\bigoplus_{g \in \Gamma} \pi_q(S_{e,g})$</td>
</tr>
</tbody>
</table>

We will also need the following observation, immediate from Theorem 2.A.1 and the fact that $\pi_1 M$ is infinite.

**Lemma 2.A.5.** Let $q > 2$. For every nonzero $x$ in $\pi_q M$, there is a $g$ in $\pi_1 M$ such that $gx \neq x$.

**2.B. The relation between $E_0 M$ and $EM$.** $M^k$ will denote the $k$-skeleton of $M$. The evaluation map $ev: f \rightarrow f(*)$ gives a surjection from $EM$ to $M$ which is a fibration with fiber $E_0 M$.

**Theorem 2.B.1.** The exact homotopy sequence for the fibration $E_0 M \rightarrow EM \rightarrow M$ decomposes into short exact sequences for every $q > 1$:

$0 \rightarrow \pi_{q+1} M \rightarrow \pi_q E_0 M \rightarrow \pi_q EM \rightarrow 0$.

**Remark.** This holds for $q = 0$ also, since $\pi_1 M$ is centerless.

**Proof of the theorem.** This will be a consequence of

**Lemma 2.B.2.** Suppose $g: M^1 \times I^q \rightarrow M$ and $g|_{M^1 \times \partial I^q} = \text{proj}_M$. Then $g$ is homotopic (rel $M^1 \times \partial I^q$) to a map $g_1$ with $g_1|_{M^q \times \partial I^q} = \text{proj}_M$.

Deferring the proof of the lemma for a moment, we consider an element $<f> \in \pi_q EM$. Then $f: M \times I^q \rightarrow M$ with $f|_{M \times \partial I^q} = \text{proj}_M$. Applying the lemma
to $f|_{M^{1} \times I^{r}}$, we can homotop $f|_{M^{1} \times I^{r}}$ and hence $f$ (rel $M \times \partial I^{q}$) so that $f^{*} \times I^{r} = \text{proj}_{M}$. Thus $\text{ev}_{q}(f) = \langle f^{*} \times I^{q} \rangle = 0 \in \pi_{q} M$. □

**Proof of Lemma 2.B.2.** For $q = 1$, $\langle f^{*} \times I \rangle$ is central in $\pi_{1}(M, *)$, which has trivial center, and the result follows easily. Assume $q > 2$. Consider $d_{q} = d_{q}^{*} (\text{proj}_{M}, g) \in C^{q}(M \times I^{q}; \pi_{q}(M))$. We have $\delta d_{q} = c_{q}^{+1}(\text{proj}_{M}) - c_{q+1}(g) = 0$ since both extend to the $(q + 1)$-skeleton. We will show that $\delta d_{q} = 0$ only if $d_{q}([* \times I^{q}]) = 0$.

We may assume that the paths used to define the local coefficient system are the unique paths in some maximal tree in the 1-skeleton of $M$. Let $\sigma$ be a 1-simplex in the tree with $\partial \sigma = \tau - \tau$. Then $0 = \delta d_{q}([\sigma \times I^{q}]) = d_{q}^{*}[\sigma(1) \times I^{q}] - d_{q}^{*}[0 \times I^{q}]$; hence $d_{q}^{*}[\tau \times I^{q}] = d_{q}^{*}[\sigma \times I^{q}]$. By induction on the distance of $\tau$ from $\tau$ in the maximal tree, we have $d_{q}^{*}[\tau \times I^{q}] = d_{q}^{*}[\sigma \times I^{q}]$ for every 0-simplex $\tau$ of $M$.

Now suppose $\sigma$ is any 1-simplex not in the maximal tree, representing an element $g_{\sigma} \in \pi_{1} M$. Then $0 = \delta d_{q}^{*}[\sigma \times I^{q}] = g_{\sigma} d_{q}^{*}([1 \times I^{q}]) - d_{q}^{*}[0 \times I^{q}]$ so $d_{q}^{*}[\sigma \times I^{q}] = g_{\sigma} d_{q}^{*}[\tau \times I^{q}]$. Therefore $g d_{q}^{*}[\tau \times I^{q}] = d_{q}^{*}[\sigma \times I^{q}]$ for every $g \in \pi_{1} M$. By Lemma 2.A.5, this implies $d_{q}^{*}[\sigma \times I^{q}] = d_{q}^{*}[\tau \times I^{q}] = 0$ for every $\tau \in M^{0}$. Therefore the image of $\tau \times I^{q}$ is a null-homotopic $q$-sphere based at $\tau$, so we can homotop $f$ (rel $M \times \partial I^{q}$) so that $f|_{M^{0} \times I^{q}} = \text{proj}_{M}$, which was to be proved. □

**2.C. A cell structure for $M$.** We describe a cell structure for $M$ that will facilitate our calculations. For $1 < i < r$ let $M_{i} = M_{i}^{*}$-open ball, and let $S_{i} = \partial M_{r}$. For $1 < i < r - 1$ let $X_{i} = S^{n-1} \times I$ be a collar neighborhood of $S_{i}$ in $M_{i}$, so that $S_{i} = S^{n-1} \times \{0\}$. Give each $X_{i}$ a cell structure as in §1.D. Let $a_{i}$, $1 < i < r - 1$, be the 1-cell in $X_{i}$, and assume that $a_{i} \cap S_{i}$ is the basepoint of $M_{i}$. Give the rest of $M_{i}$ any triangulation, for $1 < i < r - 1$, and give $M_{r}$ any triangulation. Form the 1-point union of the $M_{i}$ for $1 < i < r - 1$, and glue $S_{i}$ to its boundary $\bigvee_{r-1}^{i} S_{i}$ to form $M$.

The convenience of this construction stems from the following observation. From the proof of Theorem 2.A.1, the inclusion $\bigvee_{r-1}^{i} S_{i} \hookrightarrow M$ sends $\pi_{n-1}(\bigvee_{r-1}^{i} S_{i})$ isomorphically to $A_{n-1} = \bigoplus_{i=1}^{r-1} \pi_{n-1}(S_{i}^{r-1})$, a $\pi$-basis for $\pi_{n-1} M$. If $\pi_{*} M$ has a $\pi$-basis, then an element of $H_{n-1}(M; \pi_{q} M) = H_{n-1}(\tilde{M}; \pi_{q})$ (by Lemma 2.A.2) can be represented as $\Sigma_{j=1}^{r-1} \Sigma_{j=1}^{N} a_{j} g_{j} \pi_{n-1}(S_{i}^{r-1}) = \Sigma_{j=1}^{r-1} \Sigma_{j=1}^{N} a_{j} g_{j} \pi_{n-1}(S_{i}^{r-1}) = \Sigma_{j=1}^{r-1} g_{j} \pi_{n-1}(S_{i}^{r-1})$, where $x_{i} \in \pi_{q} M$.

**3. Calculations of $\pi_{q}(EM)$.** All cohomology will be with local coefficients.

**3.A. An exact sequence for $\pi_{q}(EM)$.**

**Theorem 3.A.1.** For $1 < q < 2n - 5$ there is an exact sequence

$$
\ldots \rightarrow H^{1}(M; \pi_{q+2} M) \rightarrow H^{n}(M; \pi_{n+q} M) \rightarrow \pi_{q}(EM) \rightarrow H^{1}(M; \pi_{q+1} M) \rightarrow H^{n}(M; \pi_{n+q+1} M) \rightarrow \ldots
$$

**Proof.** Using the diagram of fibrations discussed in the introduction with $Y = M$, and noting that for $q > 1$, $\pi_{q}(M^{1}(\partial M)) = \pi_{q}(EM)$, we obtain an exact sequence for each $q > 1$:

$$
J_{q} \rightarrow K_{q} \rightarrow \pi_{q}(EM) \rightarrow J_{q-1} \rightarrow K_{q-1}.
$$

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In this sequence,

\[ J_q = \text{coker}(\pi_{q+1}(M^{[M^2,M^3]}(M))[M^{[M^2,M^3]}][M^{[M^2,M^3]}]) \]

\[ K_q = \text{coker}(\partial: \pi_{q+1}(M^{[M^2,M^3]}(M)) \rightarrow \pi_q(M^{[M^2,M^3]})), \]

and \( D_q \) is induced by \( \partial: \pi_{q+1}(M^{[M^2,M^3]}(M)) \rightarrow \pi_q(M^{[M^2,M^3]}(M)). \) The theorem is immediate from the following two lemmas.

**Lemma 3.A.2.** For \( q > 0, J_q \approx H^1(M; \pi_{q+2}M). \)

**Proof.** By Lemma 2.B.2, \( \pi_{q+1}(M^{[M^2,-1]}(M)) \rightarrow \pi_{q+1}(M^{[M^2,M^3]})) \) is surjective. Therefore Lemma 1.B.1 applies. □

**Lemma 3.A.3.** For \( 0 < q < 2n - 5, K_q \approx H^n(M; \pi_{n+q}M). \)

**Proof.** By Lemma 2.A.4, \( \pi_{p+q}M \) has a \( \pi \)-basis for \( 2 < p + q < 3n - 6. \) Therefore when \( 3 < p < n - 1, \) the condition \( 0 < q < 2n - 5 \) guarantees that \( \pi_{p+q}M \) has a \( \pi \)-basis. By Lemma 2.A.3, \( H^p(M; \pi_{p+q}M) = H^{p-1}(M; \pi_{p+q}M), \) so Lemma 1.C.1 applies. □

**Corollary 3.A.4.** For \( 1 < \alpha < n - 4, \) \( \pi_{\alpha}EM \equiv A_{n+q}, \) hence is finite.

**Proof.** For these dimensions, \( A_{q+2} = 0 = A_{q+1} \) by Lemma 2.A.4. Therefore \( H^1(M; \pi_{q+2}M) = 0 = H^1(M; \pi_{q+1}M), \) so \( \pi_{q}(EM) \equiv H^n(M; \pi_{n+q}M) \equiv A_{n+q}, \) using Lemma 2.A.3.

**3.B. Calculation of \( D_q.\)** To compute \( D_q, \) we first define a homomorphism \( k: \)

\[ H^1(M; \pi_{q+2}M) \rightarrow \pi_{q+1}(M^{[M^2,M^3]}(M)) \] such that \( d_1 \circ k = \text{identity}, \) where \( d_1 \) is the homomorphism of Lemma 1.B.1. Recall the cell structure for \( M \) described in §2.C. Given a generator \( x_i[S^i] \in H_{n-1}(M; \pi_{q+2}M) \approx H^1(M; \pi_{q+2}M), \) let \( f: ((M - \text{int}(X_i)) \cup \sigma_i) \times I^{q+1} \rightarrow M \) be a map such that

\[ f|(M - \text{int}(X_i)) \times I^{q+1} = \text{proj}_M, \text{ and } d_{q+2}(\text{proj}_M, f)[\sigma_i \times I^{q+1}] = x_i \in \pi_{q+2}(M, \sigma). \]

Define \( k(x_i[S^i]) = \langle f|_{M^{[M^2,M^3]} \times I^{q+1}} \rangle. \)

Since \( D_q \) is induced by \( \partial: \pi_{q+1}(M^{[M^2,M^3]}(M)) \rightarrow \pi_q(M^{[M^2,M^3]}(M)), \) we have \( D_q = d_n \circ \partial \circ k, \) where \( d_n: \pi_q(M^{[M^2,M^3]}(M)) \rightarrow H^n(M; \pi_{n+q}M) \) is defined in Lemma 1.C.1. The calculation of \( \partial\langle f|_{M^{[M^2,M^3]} \times I^{q+1}} \rangle \) is exactly analogous to the calculation of \( D_q \) in the example of §1.D. The generator \( z \) there of \( \pi_{n-1}(X) \) corresponds to the element \( z^i_\alpha \) of \( \pi_{n-1}(M, \sigma) \) (defined in §2.A). The group \( \pi_{q+2}X \) is replaced by \( \pi_{q+2}(M, \sigma) \) and \( \pi_q(X^{[\sigma] \cup [\sigma]}) \) is replaced by \( \pi_q(M^{[M^2,M^3]}(M)). \) Therefore \( \partial k(x_i[S^i]) \) is representable by a map \( \tilde{f} \) which equals \( \text{proj}_M \) on \( (M - \text{small ball in } X_i) \times I^q \) and such that \( d_{n+q}(\text{proj}_M, \tilde{f})([M \times I^q, M \times \partial I^q]) = [z^i_\alpha, x_i]. \) Hence \( D_q(x_i[S^i]) = d_k(\tilde{f}) = \{[z^i_\alpha, x_i]\} \) where the curly brackets indicate an equivalence class in \( H^n(M; \pi_{n+q}M) \approx \pi_{n+q}(M)/\pi_1(M) \approx A_{n+q}. \) We have shown

**Proposition 3.B.1.** \( D_q(\Sigma^i_\alpha x_i[S^i]) = \Sigma^i_\alpha \{[z^i_\alpha, x_i]\} \in A_{n+q}. \)

We will now determine \( \ker D_{n-3} \) and \( \coker D_{n-3} \) using the results and notation of §2.A. For \( q = n - 3, \) we have \( [z^i_\alpha, z^j_\beta] = z^i_\alpha z^j_\beta, \) so
given on generators by

\[\gamma \mapsto (0, \ldots, (z^i), \ldots, 0) \in \{z^i\}_{\gamma} \] (where \(\gamma\) indicates that \(z^i\) appears in the \(i\)th slot) in all cases except both \(\gamma = e\) and \(i = j\). We will describe the inverse image of each of these summands in order to determine the kernel and cokernel of \(D_{n-3}\). Let \(B'_{n-3} = \text{ker}(D_{n-3}|_{\pi_{n-1}(S^i)}) = \{z^i\} = \pi_{n-1}(S^i) \rightarrow \pi_{2n-3}(S^i)\},\) which is 0 if \(n\) is odd and has index < 2 if \(n\) is even. Let \(C'_{n-3} = \text{coker}(D_{n-3}|_{\pi_{n-1}(S^i)})\), which is well known to be finite. Observe that

\[\bigoplus_{\gamma \in \Gamma} \pi_{2n-3}(S^i_{e,\gamma})\] is in the image of \(D_{n-3}\) since \(D_{n-3}(0, \ldots, (z^i), \ldots, 0) = \{z^i\}_{\gamma}\), and the inverse image of \(\{z^i\}_{\gamma}\) consists of \((0, \ldots, (z^i), \ldots, 0)\) and \(0, \ldots, (-1)^{n-1}(z^i_{e,\gamma-1}), \ldots, 0\). Therefore the kernel contains \(\bigoplus_{\gamma \in \Gamma} \{z^i\}_{\gamma}\), where \(\{z\}_{\gamma} \approx Z\) is generated by \((0, \ldots, (z^i_{e,\gamma-1}), \ldots, 0)\). Explicitly, we have

\[D_{n-3}(0, \ldots, (z^i_{e,\gamma-1}, \ldots, 0) = \{z^i\}_{\gamma} = (-1)^{n-1}z^i_{e,\gamma-1}\) (0, \ldots, (z^i_{e,\gamma-1}, \ldots, 0).

Collecting this information, we state

**Lemma 3.B.2.** \[\text{coker}(D_{n-3}) \cong \bigoplus_{i=1}^{r-1} C'_{n-3}\] and

\[\text{ker}(D_{n-3}) \cong \bigoplus_{i=1}^{r-1} B'_{n-3} \cong \bigoplus_{\gamma \in \Gamma} \{z^i\}_{\gamma} \cong \bigoplus_{1 \leq i < j < r-1} Z\].

**Corollary 3.B.3.** For \(n > 4\), \(\pi_{n-3}EM\) is finite.

**Proof.** In this case, the exact sequence of Theorem 3.A.1 yields \(\pi_{n-3}EM \cong \text{coker}(D_{n-3}) \cong \bigoplus_{i=1}^{r-1} C'_{n-3}\). □

**Corollary 3.B.4.** \(\pi_{n-3}EM\) is infinitely-generated as an abelian group.

**Proof.** In Theorem 3.A.1 we have a surjection \(\pi_{n-2}(EM) \rightarrow \text{ker}(D_{n-3})\) and \(\text{ker}(D_{n-3})\) is infinitely-generated by Lemma 3.B.2. □

**Corollary 3.B.5.** Let \(M^3\) be a connected sum of aspherical 3-manifolds. Then \(\pi_1(EM^3)\) is infinitely-generated.
PROOF. Take $n = 3$ in Corollary 3.B.4.

4. Homeomorphisms of nonirreducible 3-manifolds. Let $HX$ denote the path component of $id_X$ in the group $\text{Homeo}(X)$; then $HX \subset EX$. Throughout this section, we will assume $M = M_1 \# \cdots \# M_r$ is a connected sum of $r > 3$ aspherical 3-manifolds.

From Theorem 3.A.1 we have a homomorphism

$$D_0: H^1(M; \pi_2 M) \to H^3(M; \pi_3 M)$$

and a surjective homomorphism $j: \pi_1(EM) \to \text{kernel}(D_0)$. Let $i: \pi_1(HM, id_M) \to \pi_1(EM, id_M)$ denote the homomorphism induced by inclusion. The remainder of this section will be devoted to the proof of

**Theorem 4.1.** The image of $j \circ i$ contains an infinitely-generated direct summand of $\text{kernel}(D_0)$. Hence $\pi_1(HM)$ is not finitely-generated.

4.A. Isotopies of $S^2 \times I$. Let $X = S^2 \times I$. We regard $S^2$ as the unit sphere in $R^3$. Since $SO(3)$ preserves $S^2$ we have $SO(3) \subset HS^2$ (it is actually a deformation retract $[K]$). Let $SO(2) \subset SO(3)$ be the subgroup of rotations that leave the points $(0, 0, 1)$ and $(0, 0, -1)$ fixed. Let $\tau: (I, 0, 1) \to (SO(2), id, id)$ be the path such that $\tau(t)$ is rotation through an angle of $2\pi t$; then $\tau$ represents a generator of $\pi_1(SO(2)) \cong Z$ (and hence represents a generator of $\pi_1(SO(3)) \cong Z/2Z$). We will now use the results and notation of §1.D. We define a level-preserving homeomorphism $f: X \to X$ by $f(x, s) = (\tau(s)(x), s)$. Assuming that the 1-cell $a$ equals $(0, 0, 1) \times I$, we see that $f$ represents an element of $\pi_0(X^{[x,a \times x]}) \cong \pi_3(X) \cong Z$. It is known that the difference class $d_3(f, id_X)$ is a generator of this group (see $[HI, p. 85]$). In the exact sequence of §1.D,

$$\pi_2 X \to \pi_3 X \to \pi_0 (X^{[x,a \times x]}) \to 0$$

the homomorphism $[z, -]$ is well known to have image $2Z \subset Z$; hence $\pi_0(X^{[x,a \times x]})$ is isomorphic to $Z/2Z$ and $f$ represents a generator of this quotient group. Now if $\theta: (I \times I, \partial I \times I) \to (SO(3), id)$ is a nullhomotopy with $\theta(s, 0) = \tau^2(s)$, $\theta(s, 1) = id$, then $F: X \times I \to X$ defined by $F_t(x, s) = F((x, s), t) = (\theta(s, t(x), s)$ is an isotopy from $j^2$ to $id_X$. Under the identification $\pi_1(X^{[x,a \times x]}) \cong \pi_3(X)$, the restriction $F|_{\sigma \cup ax}$ represents a generator since $\langle F|_{\sigma} \rangle = \langle F_0 \rangle = \langle f^2 \rangle$. We choose the generator $z$ of $\pi_3(X)$ such that $d_3(F|_{\sigma})[\sigma \times I] = z$. Note that $F^{-1}$ is an isotopy from $(f^{-1})^2$ to $id_X$, with $d_3((F^{-1})|_{\sigma})[(\sigma \times I)] = -z$.

4.B. The 3-manifold $Z$. Let $X_1$ and $X_2$ be two copies of $S^2 \times I$, and let $B = D^2 \times I$. Let $Z$ denote the 3-manifold-with-boundary obtained by identifying $D_1 = D^2 \times \{0\}$ with a disc in $S^2 \times \{1\} \subset \partial X_1$ and $D_2 = D^2 \times \{1\}$ with a disc in $S^2 \times \{1\} \subset \partial X_2$, by orientation-reversing homeomorphisms. We assume these discs do not contain the 0-cells $\sigma_1 \cap S^2 \times \{1\} \subset \partial X_1$ and $\sigma_2 \cap S^2 \times \{1\} \subset \partial X_2$. We will use $S_1$ to denote $S^2 \times \{0\} \subset X_1$ and $S_2$ for $S^2 \times \{0\} \subset \partial X_2$. Let $S_3$ denote the remaining boundary component of $Z$, so that the oriented boundary of $Z$ is $\partial Z = S_1 \cup S_2 \cup (-S_3)$. We choose a nice collar neighborhood $X_3$ of $S_3$ so that $\sigma_1 \cap X_3 = \sigma_1$ has the form $(\sigma_1 \cap S_3) \times I = (0, 0, 1) \times I$ and $\sigma_2 \cap X_3 = \sigma_2$.
has the form \((\sigma_2 \cap S_3) \times I = (0, 0, -1) \times I\). We will define two isotopies of \(Z\). The isotopy \(G\) is defined by

\[
G_t(z) = G(z, t) = \begin{cases} 
F_t(x, s) & \text{if } z = (x, s) \in X_3, \\
z & \text{if } z \notin X_3,
\end{cases}
\]

and the isotopy \(H\) is defined by

\[
H_t(z) = H(z, t) = \begin{cases} 
F_t(x, s) & \text{if } z = (x, s) \in X_1, \\
F_t^{-1}(x, s) & \text{if } z = (x, s) \in X_2, \\
z & \text{if } (z, t) \in B \times I.
\end{cases}
\]

We make the important observation that \(G_0\) is isotopic to \(H_0\) by an isotopy which is fixed on \(\sigma_1 \cup \sigma_2 \cup \partial Z\). This observation appears in the thesis of Hendriks [HI, p. 103].

We will now construct a nontrivial element of \(\pi_1(Z^{[Z, \partial Z]}\)). Let \(\sigma = \sigma_1 \cup \sigma_2 \subset Z\). From the fibration \(Z^{[Z, \sigma \cup \partial Z]} \rightarrow Z^{[Z, \partial Z]} \rightarrow Z^{[\sigma \cup \partial Z, \partial Z]}\) we obtain a commutative diagram

\[
\begin{CD}
\pi_1(Z^{[Z, \partial Z]}) @>>> \pi_1(Z^{[\sigma \cup \partial Z, \partial Z]}) @>>> \pi_0(Z^{[\sigma \cup \partial Z, \partial Z]}) @>>> \pi_0(Z^{[Z, \partial Z]}) \\
\downarrow j @> \partial_1 \downarrow >> @> \partial_3 \downarrow >> @> D_0 \downarrow >> \pi_1(Z, \partial Z; \pi_3(Z)) \rightarrow \pi_3(Z, \partial Z; \pi_3(Z))
\end{CD}
\]

Now \(Z \simeq S_1 \setminus \setminus S_2\), so using the results of Hilton [H2] we may write \(\pi_2(Z) \approx \pi_2(S_1) \oplus \pi_2(S_2)\) and \(\pi_3(Z) \approx \pi_3(S_1) \oplus \pi_3(S_2) \oplus \pi_3(S_{1,2})\) where \(S_{1,2}\) is a 3-sphere. Let \(z_1, z_2\) be generators of \(\pi_2(S_1)\) and \(\pi_2(S_2)\), respectively, such that the homotopy class represented by the oriented sphere \(S_3\) equals \(z_1 + z_2 \in \pi_2(Z)\). Then the Whitehead product \(z_1 \circ z_2 = [z_1, z_2]\) corresponds to a generator of \(\pi_3(S_{1,2}) \subset \pi_3(Z)\).

Let \(z_i\), \(1 \leq i \leq 2\), be the generators of \(\pi_3(S_i)\) so that \([z_i, z_i] = 2z_i\). We have

\[H^1(Z, \partial Z; \pi_3(Z)) \approx \pi_3(Z) \approx \pi_3(S_1) \oplus \pi_3(S_2) \oplus \pi_3(S_{1,2})\]

where the summands are generated by the cocycles \(c_{i,j}\), \(1 \leq i, j \leq 2\), such that \(d_j[a \times I] = z_j\) and \(d_i[a \times I] = 0\) if \(k \neq i\).

We also have \(H^3(Z, \partial Z; \pi_3(Z)) \approx \pi_3(Z) \approx Z \otimes Z \otimes Z\) generated by the cocycles \(c_{i,1}\) and \(c_{1,2}\), where \(c_{i,1}([Z, \partial Z]) = z_{1,1}\) and so on. Let \(c_{2,1} = c_{1,2}\). As in §3.B, it follows that the homomorphism \(D_0\) is given by \(D_0(c_{i,j}) = c_{i,j}\) if \(i \neq j\) while \(D_0(c_{i,i}) = 2c_{i,i}\). Therefore the kernel of \(D_0\) is generated by \(d_{1,2} - d_{2,1}\). From the discussion of \(F: X \times I \rightarrow X\), we have \(d_1(<H|_o>) = d_{1,1} - d_{2,2}\). To find \(d_1(<G|_o>)\), we see from the definition of \(G\) that \(d_1(<G|_o>)[(a_1 \times I]) = z_1 + z_2\); hence \(d_1(<G|_o>) = d_{1,1} + d_{1,2} + ad_{2,1} + \beta d_{2,2}\) for some \(a, \beta \in Z\). Since \(G_0\) is isotopic to \(H_0\) with \(\sigma\) held fixed, we have \(D_0(d_1(<G|_o>)) = D_0(d_1(<H|_o>)) = 2c_{1,1} - 2c_{2,2}\). From our formula for \(D_0\) we also have

\[
D_0(d_1(<G|_o>)) = D_0(d_{1,1} + d_{1,2} + ad_{2,1} + \beta d_{2,2})
\]

\[
= 2c_{1,1} + c_{1,2} + ac_{2,1} + 2\beta c_{2,2};
\]

hence \(\alpha = \beta = -1\) and \(d_1(<G|_o>)(\sigma_2 \times I) = -z_1 - z_2\). The identity map of \(Z\) is isotopic to \(G_0 \circ H_0^{-1}\) by an isotopy which is fixed on \(\sigma \cup \partial Z\). Define an isotopy \(J: Z \times I \rightarrow Z\) so that for \(0 \leq t \leq \frac{1}{2}\), \(J\) is such an isotopy, while \(J(z, t) = (G_{2t-1} \circ H_{2t-1})^{-1}(z, t)\) for \(\frac{1}{2} < t < 1\). Then \(J\) is a loop in \(\text{Homeo}(Z)\) and regarding \(J\)
as a loop in $Z^{[Z,\mathbb{Z}]}$ we have $d_1\langle J|_\sigma \rangle = d_1\langle G|_\sigma \rangle - d_1\langle H|_\sigma \rangle = d_{1,2} - d_{2,1}$. That is, $j\langle J \rangle$ is a generator of kernel$(D_0)$.

4.C. Isotopies of $M$. To construct isotopies of $M$, we will use the cell-complex structure of $M$ defined in §2.C. Since $\bigvee_{i=1}^{r-1} X_i \subset M$ is simply-connected and contains the basepoint $\ast$ of $M$, a path in $M$ with endpoints in $\bigvee_{i=1}^{r-1} X_i$ represents a well-defined element of $\pi_1(M, \ast)$. We may represent any element $\langle \alpha \rangle \in \pi_1(M_3) \ast \cdots \ast \pi_1(M_1)$ by a nicely-imbedded arc $\alpha$ in $M$ that runs from $X_1$ to $X_2$ intersecting them only in its boundary. We can imbed $Z$ in $M$ so that

1. $X_1 \subset Z$ is mapped homeomorphically to $X_1 \subset M$, carrying the basepoint $\sigma_1 \cap S_1$ to $\ast$.
2. $D_2 \times I \subset Z$ is mapped to a tubular neighborhood of $\alpha$ that intersects $X_1 \cup X_2$ in $D_2 \times \partial I$.
3. $X_2 \subset Z$ is mapped homeomorphically into $X_2 - \ast$. (This will reverse the local orientation, when $\alpha$ is orientation-reversing.)

Such an imbedding induces an injection $\pi_2(Z, \sigma_1 \cap S_3) \to \pi_2(M, \ast)$ given on generators by $z_1 \to z_1^2$ and $z_2 \to z_2^2$.

The isotopy $J_\alpha$ of $M$ is of course defined to be $J$ on $Z \subset M$ and the identity outside $Z$. Now $d_1\langle J_\alpha|_{(M \setminus Z)} \rangle = d_\alpha$ is the element of $H^1(M; \pi_2 M)$ such that $d_\alpha([\sigma_k \times I]) = 0$ if $3 < k < r - 1$, $d_\alpha([\sigma_k \times I]) = z_2^2$, and $d_\alpha([\sigma_k \times I]) = -z_2^{2,1}$. The last formula differs by the action of $\alpha^{-1}$ from the corresponding calculation for $\langle J|_\sigma \rangle \in \pi_1(Z^{[\alpha,\beta]}\setminus Z)$, since in $M$ we use a path in $X_2$ to base the homotopy class that is the value of the difference cohomology class $d_1\langle J_\alpha|_{\text{proj}_M} \rangle$ on $[\sigma_2 \times I]$, rather than a path in $Z$ that follows along $\alpha$ back to $X_1$. Under the isomorphism $H^1(M; \pi_2 M) \cong \bigoplus_{i=1}^{r-1} \pi_2(M)$, $d_1\langle J_\alpha|_{(M \setminus Z)} \rangle$ corresponds to $(z_2^2, -z_2^{1,1}, 0, \ldots, 0)$. Regarding the $J_\alpha$ as loops in $EM$, we have $j\langle J_\alpha \rangle = (z_2^2, -z_2^{1,1}, 0, \ldots, 0)$; hence the elements $j\langle J_\alpha \rangle$, $\alpha \in \pi_3(M) \ast \cdots \ast \pi_1(M)$, generate an infinitely-generated summand of kernel$(D_0)$, using Lemma 3.B.2. This concludes the proof of Theorem 4.1. \[\Box\]

Question. Can the generators of kernel$(D_0)$ of the form $(z_1^2 - z_1^{1,1}, 0, \ldots, 0)$, or of the form $(z_2^2, -z_2^{1,1}, 0, \ldots, 0)$ where $\gamma$ involves elements of $\pi_1M_1 \ast \pi_1M_2$, be realized as the images $j\langle J_\gamma \rangle$ of loops $J_\gamma$ in $HM$?

BIBLIOGRAPHY


1I learned from Frank Quinn that he has encountered and used the isotopy $J$ in another context.

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