THE $\aleph_2$-SOUSLIN HYPOTHESIS

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ABSTRACT. We prove the consistency with CH that there are no $\aleph_2$-Souslin trees.

The $\aleph_2$-Souslin hypothesis, $SH_{\aleph_2}$, is the statement that there are no $\aleph_2$-Souslin trees. In Mitchell’s model [5] from a weakly compact the stronger statement holds (Mitchell and Silver) that there are no $\aleph_2$-Aronszajn trees, a property which implies that $2^{\aleph_0} > \aleph_1$.

Theorem. Con(ZFC + there is a weakly compact cardinal) implies

\[ \text{Con}(ZFC + 2^{\aleph_1} = \aleph_1 + SH_{\aleph_2}). \]

In the forcing extension, $2^{\aleph_1}$ is greater than $\aleph_2$, and can be arbitrarily large. Analogues of this theorem hold with $\aleph_2$ replaced by the successor of an arbitrary regular cardinal. Strengthenings and problems are given at the end of the paper.

Let $\mathfrak{M}_k$ be a ground model in which $\kappa$ is a weakly compact cardinal. The extension which models $SH_{\aleph_2}$ and $CH$ is obtained by iteratively forcing $\geq \kappa^+$ times with certain $\kappa$cc, countably closed partial orders, taking countable supports in the iteration. For $\alpha > 1$, $(\mathfrak{a}_\alpha, \mathfrak{a})$ is the ordering giving the first $\alpha$ steps in the iteration. $\mathfrak{a}_\alpha$ is a set of functions with domain $\alpha$.

Let $L_{\aleph_\kappa}$ be the Levy collapse by countable conditions of each $\beta \in [\aleph_\alpha, \kappa)$ to $\aleph_1$ (so $\kappa$ is the new $\aleph_2$). Then $\mathfrak{P}_1$ (isomorphic to $L_{\aleph_\kappa}$) is $\{f: \text{dom} f = 1, f(0) \in L_{\aleph_\kappa}\}$, ordered by $f < g$ iff $f(0) < g(0)$. To define $\mathfrak{P}_{\beta + 1}$, choose a term $A_\beta$ in the forcing language of $\mathfrak{P}_\beta$ for a countably closed partial ordering (to be described later) and let $\mathfrak{P}_{\beta + 1} = \{f: \text{dom} f = \beta + 1, f \upharpoonright \beta \in \mathfrak{P}_\beta, f \upharpoonright \beta \upharpoonright \supseteq \mathfrak{a}_\beta f(\beta) \in A_\beta\}$, ordered by $f < g$ iff $f \upharpoonright \beta < g \upharpoonright \beta$ and $g \upharpoonright \beta \upharpoonright \supseteq \mathfrak{a}_\beta f(\beta) < g(\beta)$. For $\alpha$ a limit ordinal, $\mathfrak{P}_\alpha = \{f: \text{dom} f = \alpha, f \upharpoonright \beta \in \mathfrak{P}_\beta \text{ for all } \beta < \alpha, \text{ and } f(\beta) \text{ is (the term for) } \mathfrak{a}_\beta\}$, ordered by $f < g$ iff for all $\beta < \alpha, f \upharpoonright \beta < g \upharpoonright \beta$.

Each $\mathfrak{P}_\alpha$ is countably closed. We are done as in Solovay-Tennenbaum [7] if the $A_\beta$'s can be chosen so that each $\mathfrak{P}_\alpha$ has the $\kappa$cc, and therefore that every $\aleph_2$ (= $\kappa$)-Souslin tree which crops up gets killed by some $A_\beta$.

If $T$ is a tree then $(T)_\kappa$ is the $\lambda$th level of $T$, $(T)_\kappa = \bigcup_{\mu < \kappa} T_\mu$. Regarding the previous problem, it is a theorem of Mitchell that if $CH$ and $\diamondsuit(\alpha < \omega_2: cf(\alpha) = \aleph_1)$ hold, then there are countably closed $\aleph_2$-Souslin trees $T_n, n < \omega$, such that for...
each \( m < \omega, \otimes_{n<m} T_n \) has the \( \aleph_2cc \), but \( \otimes_{n<\omega} T_n \) does not have the \( \aleph_2cc \). We give for interest his proof modulo the usual Jensen methods. At stage \( \mu < \omega_2 \) construct each \( (T_n)_\mu \) normally above \( (T_n)_< \). If \( \mu = \nu + 1 \) let each \( x \in (T_n)_\nu \) have at least two successors in \( (T_n)_\mu \). If \( cf(\mu) = \omega \) let all branches in \( (T_n)_< \) go through. If \( cf(\mu) = \omega_1 \) make sure that the antichain given by the \( \hat{\diamond} \)-sequence for \( \otimes_{n<m} T_n \) is taken care of, and choose \( \langle \mu^0_n: n < \omega \rangle \in \otimes_{n<\omega} (T_n)_\mu \) so that if \( \mu' < \mu, cf(\mu') = \omega_1 \), then \( \langle \mu^0_n: n < \omega \rangle \not\in \langle \mu^0_n: n < \omega \rangle \). We also carry along the following induction hypothesis: if \( \nu < \mu, \langle x_\xi: n < \omega \rangle \in \otimes_{n<\omega} (T_n)_\nu, m < \omega, \langle y_\xi: n < m \rangle \in \otimes_{n<m} (T_n)_\mu, x_n < y_n (n < m) \) and \( \langle x_n: n < \omega \rangle \not\in \langle c_{\alpha^n}: n < \omega \rangle \), for all \( \lambda < \nu \) with \( cf(\lambda) = \omega_1 \), then there are \( y_n \in (T_n)_\mu (m < n < \omega) \) with \( x_n < y_n \), such that \( \langle y_n: n < \omega \rangle \not\in \langle c_{\lambda^n}: n < \omega \rangle \), for all \( \lambda < \mu \) with \( cf(\lambda) = \omega_1 \).

If \( \delta \) is inaccessible, then forcing with \( L_{\text{infin}} \) (whence \( 2^\omega = \aleph_1, 2^{\aleph_1} = \aleph_2 = \delta \), and \( \langle \alpha < \omega_2, cf(\alpha) = \omega_1 \rangle \) hold) followed by forcing with the \( \otimes_{n<\omega} T_n \) constructed previously, gives a countably closed length \( \omega \) iteration of countably closed, \( \aleph_2cc \) partial orderings which does not have \( \deltacc \).

The previous theorem does not rule out that an iteration of \( \aleph_2 \)-Souslin trees can give \( CH \) and \( SH \); in this paper, though, the \( \aleph_2 \)-Souslin trees are killed by a different method. Let \( T \) be an \( \aleph_2 \)-Souslin tree (we may assume without loss of generality that \( T \) is normal and \( \text{Card}(T_1) = \aleph_1 \)). The antichain partial order \( A_T \) is defined to be \( \langle (x \subseteq T: x \text{ a countable antichain, root } T \notin x), \subseteq \rangle \). Now \( A_T \) need not have the \( \aleph_2cc \), as shown by the following result of the first author: \( \text{Con}(ZFC) \) implies \( \text{Con}(ZFC + \text{"there is an } \aleph_2 \text{-Souslin tree } T \text{ and a sequence } \langle d_{\alpha^n}: n < \omega \rangle \text{ from } (T)_\alpha, \text{ for each } \alpha < \omega_2, \text{ such that if } \alpha < \beta, \text{ there is an } m < \omega \text{ with } d_{\alpha^n} < d_{\beta^n}, \text{ for all } n > m".\rangle \). Namely, start with a model of \( CH \). Determine in advance that, say, \( (T)_\alpha = [\omega_1 \alpha, \omega_1 (\alpha + 1)] \) and that \( d_{\alpha^n} = \omega_1 \alpha + n \). Conditions are countable subtrees \( S \) of \( T \) such that if \( S \cap (T)_\alpha \neq \emptyset \) then \( \{d_{\alpha^n}: n < \omega \} \subseteq S \), which meet the requirements on the \( d_{\beta^n} \)'s.

Devlin [2] has shown that such a tree exists in \( L \).

We show now that if each \( A_\alpha \) is an \( A_T \), \( T \) an \( \aleph_2 \)-Souslin tree, then each \( P_\alpha \) has the \( \kappacc \), which will prove the theorem (we actually just use that Card \( T \) < the cardinal designated as the new \( 2^\omega \) and \( T \) has no \( \omega_2 \)-paths; see remarks at the end). This theorem was originally proved by the first author when \( \kappa \) is measurable; that the assumption can be weakened to weak compactness of \( \kappa \) is due to the second author.

We consider now only the case \( \alpha < \kappa^+ \) (which will suffice, assuming \( 2^\omega = \kappa^+ \) in \( \mathbb{M} \), for \( CH + SH \) and \( 2^{\omega_1} = \aleph_3 \); \( \alpha \) arbitrary will be dealt with at the end.

Fix \( \alpha \) for the rest of the proof. We assume by induction that

(1) For each \( \beta < \alpha, P_\beta \) has the \( \kappacc \).

(One more induction hypothesis is listed later.)

For \( \beta < \alpha \), let \( T_\beta \) be the \( \beta \)th \( \aleph_2 \)-Souslin tree, so \( P_{\beta + 1} = P_\beta \bigotimes A_\beta \), where \( A_\beta = A_{T_\beta} \). Assume without loss of generality that for each \( \lambda < \kappa, \)

\[
(T_\beta)_\lambda \subseteq \{\omega_1 \lambda, \omega_1 (\lambda + 1)\}.
\]

An \( f \in P_\beta, \beta < \alpha, \) is said to be determined if there is in \( \mathbb{M} \) a sequence \( \langle z_\gamma: \gamma \in \text{dom } f - \{0\} \rangle \) of countable sets of ordinals such that for all \( \gamma \in \text{dom } f - \{0\}, \)
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If $f_n : n < \omega$ is a sequence of determined members of $\mathcal{P}_\beta$, with $f_n \subseteq f_{n+1}$, then the coordinatewise union $f_\omega$ of the $f_n$'s is seen to be a determined member of $\mathcal{P}_\beta$ extending each $f_n$. From this it may be seen, by induction on $\beta < \alpha$, that the set of determined members of $\mathcal{P}_\beta$ is cofinal in $\mathcal{P}_\beta$. Redefine each $\mathcal{P}_\beta$ then to consist just of the determined conditions. Clearly Card $\mathcal{P}_\beta < \kappa$, for all $\beta < \alpha$.

For $f, g \in \mathcal{P}_\beta, f \sim g$ means that $f$ and $g$ are compatible.

Fix for the rest of the proof a one-one enumeration $\alpha = \{ \alpha_\mu : \mu \in S \}$, for some $S \subseteq \kappa$ (this induces a similar enumeration of each $\beta < \alpha$, the induction hypothesis (2) for $\beta$ below, is with respect to this induced enumeration). For notational simplicity we now assume that $S$ is some $\kappa' < \kappa$.

If $\lambda < \kappa, \beta < \alpha, f \in \mathcal{P}_\beta$, define $f|\lambda$ to be the function $h$ with domain $\beta$ such that $h(\gamma) = \emptyset$ unless $\gamma \in \{ \alpha_\mu : \mu < \lambda \} \cap \beta$, in which case,

$$\gamma = 0 \Rightarrow h(\gamma) = f(\gamma) \uparrow (\omega_1 \times \lambda), \quad \gamma > 0 \Rightarrow h(\gamma) = f(\gamma) \cap \lambda.$$

The function $f|\lambda$ need not be a condition, but for $g \in \mathcal{P}_\beta$, we will still write $f|\lambda \subseteq g$ to mean that $f|\lambda$ is coordinatewise a subset of $g$. Let $\mathcal{P}_\beta|\lambda = \{ f \in \mathcal{P}_\beta : f|\lambda = f \}$.

Suppose $0 < \beta < \alpha, \lambda < \kappa$. Define

$$\#^\beta_\lambda(f, g, h) \iff f, g \in \mathcal{P}_\beta, f|\lambda = g|\lambda = h,$$

$$\ast^\beta_\lambda(f, h) \iff f \in \mathcal{P}_\beta, h \in \mathcal{P}_\beta|\lambda \text{ and for every } h' \supseteq h \text{ with } h' \in \mathcal{P}_\beta|\lambda, h' \sim f,$$

$$\ast^\beta_\lambda(f, g, h) \iff \ast^\beta_\lambda(f, h) \text{ and } \ast^\beta_\lambda(g, h).$$

For $P \subseteq Q, Q$ a partial ordering, $P \subseteq_{\text{reg}} Q$ means that $P$ is a regular subordering of $Q$, that is, any two members of $P$ compatible in $Q$ are compatible in $P$, and every maximal antichain of $P$ is a maximal antichain of $Q$. If $\mathcal{P}_\beta|\lambda \subseteq_{\text{reg}} \mathcal{P}_\beta$, then $\ast^\beta_\lambda(f, h)$ states that $h \uparrow_{\mathcal{P}_\beta|\lambda} [f] \neq 0$.

Recall that the sets of the form $\{ \lambda < \kappa : (R_\lambda, \in, A \cap R_\lambda) \vDash \Phi \}$, where $A \subseteq \kappa, \Phi$ is $\pi_1^1$, and $(R_\kappa, \in, A) \vDash \Phi$, belong to a normal uniform filter $\mathcal{F}_\kappa$, the weakly compact filter on $\kappa$ (see [9], [10]). The second thing we assume by induction is

(2) for all $\beta < \alpha$, for $\mathcal{F}_\kappa$-almost all $\lambda < \kappa$, for all $f, g, h$, $\#^\beta_\lambda(f, g, h)$ implies that for some $h' \supseteq h, \ast^\beta_\lambda(f, g, h').$

If $\beta < \alpha, \lambda < \kappa$, say that $(T_\beta)_{<\lambda}$ is determined by $\mathcal{P}_\beta|\lambda$ if for each $\theta, \tau$ in $(T_\beta)_{<\lambda}$ there is a $\mathcal{P}_\beta$-maximal antichain $R$ of conditions deciding the ordering between $\theta$ and $\tau$ in $T_\beta$, such that $R \subseteq \mathcal{P}_\beta|\lambda$.

**Lemma 1.** There is a closed unbounded set of $\lambda < \kappa$ such that for all $\mu < \lambda, (T_\alpha)_{<\lambda}$ is determined by $\mathcal{P}_\alpha|\lambda$.

**Proof.** This is a consequence of the strong inaccessibility of $\kappa$ and the assumption that each $\mathcal{P}_\beta, \beta < \alpha$, has kcc.
Lemma 2. For $\mathfrak{P}_{\text{wc}}$-almost all $\lambda < \kappa$,
(a) $\lambda$ is strongly inaccessible.
(b) For all $\mu < \lambda$, $\mathfrak{P}_{\alpha_{\mu}}|\lambda$ has the $\lambda$-cc.
(c) For all $\mu < \lambda$, $\mathfrak{P}_{\alpha_{\mu}}|\lambda \subseteq \mathfrak{P}_{\alpha_{\mu}}$.
(d) For all $\mu < \lambda$, $\mathfrak{P}_{\alpha_{\mu}}|\lambda = \mathfrak{N}_2$.
(e) For all $\mu < \lambda$, $\mathfrak{P}_{\alpha_{\mu}}|\lambda (T_{\alpha_{\mu}} < \lambda)$ is an $\mathfrak{N}_2$-Souslin tree.

Proof. By $\pi_1$ reflection and the normality of $\mathfrak{P}_{\text{wc}}$.

Lemma 3. Let $\beta < \alpha$, $\lambda < \kappa$, $\mathfrak{P}_\beta|\lambda \subseteq \mathfrak{P}_\beta$.
(a) If $f \in \mathfrak{P}_\beta$, $j \in \mathfrak{P}_\beta|\lambda$, and $f \sim j$, then there is an $h > j$ with $\mathfrak{A}_\lambda(f, h)$.
(b) If $\mathfrak{A}_\lambda(f, g, h)$ and $D, E$ are cofinal subsets of $\mathfrak{P}_\beta$, then there exists $\langle f', g', h' \rangle \succ \langle f, g, h \rangle$ with $\mathfrak{A}_\lambda(f', g', h')$, $f' \in D$, $g' \in E$, $h < f', g'$.

Proof. These are standard facts about forcing.

The following is T. Carlson's version of the lemma we originally used here.

Lemma 4. Suppose $\lambda$ satisfies Lemma 1 and Lemma 2(c), $\mu < \lambda$, and $\mathfrak{A}_\lambda(f, g, h)$. Then $f|\lambda < h$.

Proof. Otherwise there is a $\nu < \lambda$ with $\alpha_{\nu} < \alpha_{\mu}$, and a $\theta \in f(\alpha_{\nu}) \cap \lambda$ such that $\theta \notin h(\alpha_{\nu})$. We have that $h \upharpoonright \alpha_{\nu} \nsubseteq \mathfrak{P}_{\alpha_{\nu}}|\theta$ is $T_{\alpha_{\nu}}$-incomparable with each member of $h(\alpha_{\nu})$; otherwise $\mathfrak{A}_\lambda(f, h)$ would be contradicted. Pick an $h' \in \mathfrak{P}_{\alpha_{\nu}}|\lambda$, $h' > h \upharpoonright \alpha_{\nu}$, and a $\theta' < \lambda$ such that $h' \parallel \mathfrak{P}_{\alpha_{\nu}}|\theta < h(\alpha_{\nu})$. Let $h'$ be $h' \succ \langle h(\alpha_{\nu}) \cup \{\theta'\} \rangle \succ h \upharpoonright [\alpha_{\nu} + 1, \alpha_{\mu}]$. Then $h \in \mathfrak{P}_{\alpha_{\mu}}|\lambda$, $h < h'$, and $h \sim f$, a contradiction.

Lemma 5. Suppose $\lambda$ satisfies Lemma 1 and Lemma 2(c), $\mu < \lambda$, and $\mathfrak{A}_\lambda(f, g, h)$. Then there is an $\langle f', g', h' \rangle > \langle f, g, h \rangle$ with $\mathfrak{A}_\lambda(f', g', h')$.

Proof. Choose $\langle f, g, h \rangle = (f_0, g_0, h_0) < \cdots < (f_n, g_n, h_n) < \cdots$ so that $\mathfrak{A}_\lambda(f_n, g_n, h_n)$, $h_n < f_{n+1}$, $h_n < g_{n+1}$. This is done by repeated applications of Lemma 3(a), (b). Then Lemma 4 implies that the coordinatewise union $\langle f', g', h' \rangle$ of the $(f_n, g_n, h_n)$'s is as desired.

Definition. Suppose $\lambda < \kappa$, $\mu < \lambda$, $f, g \in \mathfrak{P}_{\alpha_{\mu}}$, and suppose $\theta, \tau$ are nodes of $(T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}} (\theta = \tau$ allowed). Then $\langle f, g \rangle$ is said to $\lambda$-separate $\langle \theta, \tau \rangle$ if there is a $\gamma < \lambda$ and $\theta', \tau' \in (T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}}$ with $\theta' \neq \tau'$, such that

Proof. Choose $(f, g, h) = (f_0, g_0, h_0) < \cdots < (f_n, g_n, h_n) < \cdots$ so that $\mathfrak{A}_\lambda(f_n, g_n, h_n)$, $h_n < f_{n+1}$, $h_n < g_{n+1}$. This is done by repeated applications of Lemma 3(a), (b). Then Lemma 4 implies that the coordinatewise union $\langle f', g', h' \rangle$ of the $(f_n, g_n, h_n)$'s is as desired.

Lemma 6. Suppose $\lambda$ satisfies Lemmas 1 and 2, $\mu < \lambda$, $\mathfrak{A}_\lambda(f, g, h)$, $(\theta, \tau) \subseteq (T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}}$ with $\theta = \tau$ allowed. Then there is an $\langle f', g', h' \rangle > \langle f, g, h \rangle$ such that $\mathfrak{A}_\lambda(f', g', h')$ and $\langle f', g' \rangle \lambda$-separates $\langle \theta, \tau \rangle$.

Proof. Choose $(f, g, h) = (f_0, g_0, h_0) < \cdots < (f_n, g_n, h_n) < \cdots$ so that $\mathfrak{A}_\lambda(f_n, g_n, h_n)$, $h_n < f_{n+1}$, $h_n < g_{n+1}$. This is done by repeated applications of Lemma 3(a), (b). Then Lemma 4 implies that the coordinatewise union $\langle f', g', h' \rangle$ of the $(f_n, g_n, h_n)$'s is as desired.

Definition. Suppose $\lambda < \kappa$, $\mu < \lambda$, $f, g \in \mathfrak{P}_{\alpha_{\mu}}$, and suppose $\theta, \tau$ are nodes of $(T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}}$ with $\theta = \tau$ allowed. Then $\langle f, g \rangle$ is said to $\lambda$-separate $\langle \theta, \tau \rangle$ if there is a $\gamma < \lambda$ and $\theta', \tau' \in (T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}}$ with $\theta' \neq \tau'$, such that

Proof. Consider the result of taking a generic set $G_{\alpha_{\mu}}|\lambda$ over $\mathfrak{P}_{\alpha_{\mu}}|\lambda$ which contains $h$. In $\mathfrak{M}[G_{\alpha_{\mu}}|\lambda]$, $(T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}}$ is a $\lambda$-Souslin tree. In the further extension $\mathfrak{M}[G_{\alpha_{\mu}}|\lambda]$, $\theta$ determines a $\lambda$-path through $(T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}}$ Since this path is not in $\mathfrak{M}[G_{\alpha_{\mu}}|\lambda]$, there must be $h \in G_{\alpha_{\mu}}|\lambda$, $h > h$, $f_0, f_1 > f$, $\gamma < \lambda$, $\theta_0, \theta_1 \in (T_{\alpha_{\mu}})_{\mathfrak{P}_{\alpha_{\mu}}}$, $\theta_0 \neq \theta_1$, with
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$\exists\theta_0 < \tau_\theta$, $\exists\theta_1 < \tau_\theta$, such that $\psi_\theta^\mathbb{N}(f_0, \xi)$ and $\psi_\theta^\mathbb{N}(f_1, \xi)$. This gives the claim.

Now, by Lemma 3, choose $(g', h') > (g, h)$ and a $\tau' \in (T_\alpha)_\gamma$ so that $\psi_\theta^\mathbb{N}(g', h')$ and $g'' \uparrow \tau' < \tau_\theta$. Pick $i \in \{0, 1\}$ with $\tau' \neq \theta_i$. Let $f_i' = f_i$, $\tau_i' = \theta_i$. Then $(f', g', h')$ are as desired. This proves the lemma.

We claim that the induction hypotheses (1) and (2) automatically pass up to $\alpha$ if $\text{cf}(\alpha) > \omega$. Namely, (1) holds at $\alpha$ by a $\Delta$-system argument. For (2), suppose that for an $\mathcal{F}_w$-positive set $W$ of $\lambda$'s there is a counterexample $\langle f_\alpha, g_\alpha, h_\alpha \rangle$. Let $N_\lambda = (\text{support } f_\lambda \cup \text{support } g_\lambda)$. If $\text{cf}(\alpha) = \kappa$ then for some $\beta < \alpha$ and $\mathcal{F}_w$-positive $V \subseteq W$, $\lambda \in V$ implies $N_\lambda \subseteq \beta$, and we are done. If $\text{cf}(\alpha) = \kappa$, pick a closed unbounded set $C \subseteq k$ such that $\langle \sup\{\alpha_v : \nu < \lambda\} : \lambda \in C \rangle$ is increasing, continuous and cofinal in $\alpha$ and an $\mathcal{F}_w$-positive $V \subseteq W \cap C$ such that for some $\beta < \alpha$ and all $\lambda \in V$, $N_\lambda \cap \sup\{\alpha_v : \nu < \lambda\} \subseteq \beta$, then apply (2) at $\beta$.

Thus, we may assume for the rest of the proof that $\alpha$ is a successor ordinal or $\text{cf}(\alpha) = \omega$. Fix $\langle \mu_\alpha : n < \omega \rangle$ such that if $\alpha = \beta + 1$ then each $\mu_n$ is the $\mu$ with $\alpha_\mu = \beta$, and if $\text{cf}(\alpha) = \omega$ then $\langle \alpha_\mu : n < \omega \rangle$ is an increasing sequence converging to $\alpha$.

**Lemma 7.** For $\mathcal{F}_w$-almost all $\lambda$, the following holds: if $f, g, h \in \mathcal{F}_\alpha$, $h \in \mathcal{F}_\alpha$, $\#_\lambda(f, g, h)$ then there exists $\langle f', g', h' \rangle >\langle f, g, h \rangle$ such that $\#_\lambda(f', g', h')$ and such that for each $\mu < \lambda$ with $\alpha_\mu \neq 0$, and each $\theta \in f'(\alpha_\mu) - \lambda$, $\langle f' \uparrow \alpha_\mu, g' \uparrow \alpha_\mu \rangle \lambda$-separates $\langle \theta, \tau \rangle$.

**Proof.** We prove the lemma for $\lambda$, assuming that $\lambda$ satisfies Lemmas 1 and 2, $\lambda > \mu_n$ ($n < \omega$) and for each $n < \omega$, $\lambda$ is in the $\mathcal{F}_w$ set given by induction hypothesis (2) for $\alpha_\mu$. Construct $\langle f_n, g_n, h_n \rangle$, $n < \omega$, so that

(a) $f_n, g_n, h_n \in \mathcal{F}_\alpha$, $\#_\lambda(f_n, g_n, h_n)$,
(b) $\langle f \uparrow \alpha_\mu, g \uparrow \alpha_\mu, h \uparrow \alpha_\mu \rangle \leq \langle f_n, g_n, h_n \rangle$,
(c) $\langle f_n, g_n, h_n \rangle \leq \langle f_{n+1}, g_{n+1}, h_{n+1} \rangle$,
(d) if, at stage $n > 1$, $\langle \theta_n, \tau_n \rangle$ is the $\mu^\text{th}$ pair (in the appropriate bookkeeping list for exhausting them) with $\theta_n \in f_n(\alpha_\mu) - \lambda$, $\tau_n \in g_n(\alpha_\mu) - \lambda$, $\nu_\mu < \lambda$, $\alpha_\nu \leq \alpha_\mu$, then

$$\langle f_n \uparrow \alpha_\mu, g_n \uparrow \alpha_\mu \rangle \lambda$$

-separates $\langle \theta, \tau \rangle$.

Let $f_0 = f \uparrow \alpha_\mu$, $g_0 = g \uparrow \alpha_\mu$, $h_0 = h \uparrow \alpha_\mu$. Suppose $n > 1$ and $f_{n-1}$, $g_{n-1}$, $h_{n-1}$ have been constructed. Let

$$f_n' = f_{n-1} \uparrow [\alpha_{n-1}, \alpha_n), \quad g_n' = g_{n-1} \uparrow [\alpha_{n-1}, \alpha_n), \quad h_n' = h_{n-1} \uparrow [\alpha_{n-1}, \alpha_n).$$

Then $\#_\lambda(f_n', g_n', h_n')$. By induction hypothesis (2), there is an $h_n > h_n'$ such that $\psi_\theta^\mathbb{N}(f_n', g_n', h_n')$. By Lemma 6, there is $\langle f_n'', g_n'', h_n'' \rangle > \langle f_n', g_n', h_n' \rangle$ such that $\psi_\theta^\mathbb{N}(f_n'', g_n'', h_n'')$ and

$$\langle f_n'' \uparrow \alpha_\mu, g_n'' \uparrow \alpha_\mu \rangle \lambda$$

-separates $\langle \theta, \tau \rangle$.

Finally, by Lemma 5 we may choose $\langle f_n, g_n, h_n \rangle > \langle f_n'', g_n'', h_n'' \rangle$ so that $\#_\lambda(f_n, g_n, h_n)$. 

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Taking $f'$, $g'$, $h'$ to be the coordinatewise unions of the $f_n$'s, $g_n$'s, $h_n$'s gives the lemma.

We now verify the two induction hypotheses.

(1) $\mathfrak P_\alpha$ has the kcc.

**Proof.** Given $f_\lambda \in \mathfrak P_\alpha$, $\lambda < \kappa$. For each $\lambda$ which satisfies Lemmas 1, 2 and 7, with $\lambda > \mu_n$ ($n < \omega$), apply Lemma 7 to the triple $<f_\lambda, f_\lambda, f_\lambda \upharpoonright \lambda>$, obtaining a triple $<f_\lambda^*, f_\lambda^*, f_\lambda>$ (so $f_\lambda < f_\lambda^*, f_\lambda^*, f_\lambda = f_\lambda^|\lambda = f_\lambda^*|\lambda$).

Let

$$B_\lambda = (\text{support} f_\lambda^* \cup \text{support} f_\lambda^* ) \cap \{\alpha_\mu: \mu < \lambda\}.$$  

If $0 \neq \alpha_\mu \in B_\lambda$, write

$$f_\lambda^*(\alpha_\mu) - \lambda = \{\theta_\mu: n < r_\mu\}, \quad r_\mu < \omega,$$

$$f_\lambda^*(\alpha_\mu) - \lambda = \{\tau_\mu: n < s_\mu\}, \quad s_\mu < \omega.$$  

To each pair $<\theta_\mu: m < r_\mu\>, \quad n < r_\mu, m < s_\mu, <f_\lambda^*, f_\lambda^*>$ assigns a separating pair $<\theta_\mu: m < r_\mu\>, \quad n < r_\mu, m < s_\mu, <f_\lambda^*, f_\lambda^*>$.

Let $J_\lambda = (\text{dom} f_\lambda^*(0) \cup \text{dom} f_\lambda^*(0)) - (\omega_1 \times \lambda)$.

By the normality of $\mathfrak F_{\omega_1}$, there is an $\mathfrak F_{\omega_2}$-positive set $U$ such that on $U$, the sets $B_\lambda, r_\mu, s_\mu, \theta_\mu, \tau_\mu, J_\lambda$ are independent of $\lambda$, and such that if $\lambda, \lambda' \in U$, $\lambda < \lambda'$, then (support $f_\lambda^* \cup$ support $f_\lambda^*$) $\cap$ (support $f_\lambda^* \cup$ support $f_\lambda^*$) = $B_\lambda$, and $J_\lambda \cap J_{\lambda'} = \emptyset$.

By induction on $\gamma < \alpha$ it is seen that if $\lambda, \mu \in U$ and $\lambda < \mu$, then $f_\lambda^* \sim f_\mu^*$. Namely, there is no trouble with coordinates in the support of at most one of these functions; coordinates in both supports, being in $B_\lambda$, are taken care of by the construction. Since $f_\lambda < f_\lambda^*$ and $f_\mu < f_\mu^*$, we are done.

The following strengthening of kcc for $\mathfrak P_\alpha$ has thus been proved: if for an $\mathfrak F_{\omega_2}$-positive set $W$ of $\lambda$'s, $\#_{\omega_2}(f_\lambda, g_\lambda, h_\lambda)$, then there is an $\mathfrak F_{\omega_2}$-positive $U \subseteq W$ and $(f_\lambda, g_\lambda', h_\lambda')$, $\lambda \in U$, such that $<f_\lambda, g_\lambda, h_\lambda> < f_\lambda, g_\lambda, h_\lambda>$, $\#_{\omega_2}(f_\lambda, g_\lambda', h_\lambda')$, and so that if $\lambda, \mu \in W$, $\lambda < \mu$, then $f_\lambda \sim g_\mu$ in the strong sense that the coordinatewise union of $f_\lambda$ and $g_\mu$ is a condition extending both $f_\lambda$ and $g_\mu$.

Lastly, we prove the second induction hypothesis for $\alpha$.

(2) For $\mathfrak F_{\omega_2}$-almost all $\lambda < \kappa$, for all $f, g, h$, $\#_{\omega_2}(f, g, h)$ implies that for some $h' > h, \#_{\omega_2}(f, g, h')$.

**Proof.** Otherwise for an $\mathfrak F_{\omega_2}$-positive set $W$ of $\lambda$'s there exists a counterexample $<f_\lambda, g_\lambda, h_\lambda>$. We may assume that for each $\lambda \in W$, $\lambda > \mu_n$ ($n < \omega$) and $\lambda$ satisfies Lemmas 1, 2 and 7. Furthermore, since we have already proved that $\mathfrak P_\alpha$ has the kcc, we may assume that for each $\lambda \in W$, $\mathfrak P_\alpha|\lambda \subseteq \mathfrak P_\alpha$ and $\mathfrak P_\alpha|\lambda$ has the $\kappa c c$. If $f_\lambda$ or $g_\lambda$ equals $h_\lambda$ we are done, so assume, for each $\lambda \in W$, that $f_\lambda, g_\lambda \in \mathfrak P_\alpha|\lambda$.

Apply Lemma 7 to each $<f_\lambda, g_\lambda, h_\lambda>, \lambda \in W$, getting $<f_\lambda', g_\lambda', h_\lambda'>$. Now uniformize as in (a) to get an $\mathfrak F_{\omega_2}$-positive $V \subseteq W$ such that if $\lambda, \mu \in V$ and $\lambda < \mu$ then $f_\lambda \sim g_\mu$. Since $<f_\lambda, g_\lambda, h_\lambda>$ is a counterexample to (b), there is a maximal antichain $H_\lambda$ of $\{h \in \mathfrak P_\alpha|\lambda: h > h_\lambda\}$ such that for each $h \in H_\lambda$, $h \sim f_\lambda$ or $h \sim g_\lambda$. Then $H_\lambda$ is a maximal antichain of $\{h \in \mathfrak P_\alpha: h > h_\lambda\}$, and Card $H_\lambda < \lambda$. Pick an $\mathfrak F_{\omega_2}$-positive $U \subseteq V$ on which $H_\lambda = H$ is independent of $\lambda$ and such that for each $h \in H$, the questions, whether or not $h \sim f_\lambda, h \sim g_\lambda$, are independent of $\lambda$. Pick
THE \( \aleph_2 \)-SOUSLIN HYPOTHESIS

\( \lambda, \mu \in U, \lambda < \mu, \) and let \( j > f_\lambda, g_\mu \). Now \( j > h_\lambda \), and \( j \notin \mathcal{P}_\alpha \lambda \) (whence \( j \notin H \)). But for each \( h \in H \), either \( h \sim g_\lambda \) (whence \( h \sim g_\mu \)) or \( h \sim f_\lambda \). In either case, \( h \sim j \) since \( j > f_\lambda, g_\mu \), so \( H \) is not maximal, a contradiction.

This completes the proof of the theorem.

Denote by an \( \omega_2 \)-tree a tree \( T \) of any cardinality with no paths of length \( \omega_2 \). An \( \omega_2 \)-tree \( T \) is special if there is an \( f: T \to \omega_1 \) such that \( x < T y \) implies \( f(x) \neq f(y) \). By the previous methods, using countable specializing functions instead of countable antichains, the consistency of \( 2^{\omega_1} = \aleph_1, 2^{\omega_1} > \aleph_2, \) and every \( \omega_2 \)-tree of cardinality \( < 2^{\omega_1} \) is special” — the analogous theorem for the \( \aleph_1 \) case being Baumgartner-Malitz-Reinhardt [1]. We can also get this model to satisfy the “generalized Martin’s axioms” (which are consistent relative to just ZFC but which do not imply \( \text{SH}_{\omega_2} \)) that have been considered by the first author and by Baumgartner (see Tall [8]). Desirable, of course, would be the consistency of a generalized MA which is both simple and powerful.

The partial orderings appropriate for the prior methods can be iterated an arbitrary number of times, giving generalized MA models in which \( 2^{\omega_1} \) is arbitrarily large. The ordering \( \mathcal{R}_\alpha \) giving the first \( \alpha \) steps of the iteration need not be of cardinality \( < \kappa \), but, assuming each \( \mathcal{R}_\beta, \beta < \alpha, \) has \( \text{ccc} \), any sequence \( \langle p_\lambda: \lambda < \kappa \rangle \) from \( \mathcal{R}_\alpha \) is a subset of a sufficiently closed model of power \( \kappa \), in which the proof that two \( p_\lambda \)'s are compatible can be carried out.

Regarding the analog of these results where \( \aleph_2 \) is replaced by \( \gamma^+ \)—the relevant forcing is \( \gamma \)-directed closed, so by upward Easton forcing we may guarantee that, for example, \( \gamma \) remains supercompact if it was in the ground model.

For results involving consequences of \( \text{SH}_{\omega_1} \): with GCH, see Gregory [3], [4] (\( \text{Con} (\text{SH}_{\omega_1} \text{ and GCH}) \) is open); with just CH, see a forthcoming paper by Stanley and the second author.

References

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