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SOME RESTRICTIONS ON FINITE GROUPS ACTING FREELY ON $(S^n)^k$

BY

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ABSTRACT. Restrictions other than rank conditions on elementary abelian subgroups are found for finite groups acting freely on $(S^n)^k$, with trivial action on homology. It is shown that elements x of order p, p an odd prime, with x in the normalizer of an elementary abelian 2-subgroup E of G, must act trivially on E unless $p|(n+1)$. It is also shown that if $p = 3$ or 7, x must act trivially, independent of n. This produces a large family of groups which do not act freely on $(S^n)^k$ for any values of n and k. For certain primes p, using the mod two Steenrod algebra, one may show that x acts trivially unless $2^{\mu(p)}|(n+1)$, where $\mu(p)$ is an integer depending on p.

Introduction. In this paper, we will describe some restrictions of the structure of finite groups G which act freely on spaces having the homotopy type $\prod_{j=1}^{k} S^n$. For the case of $k = 1$, it is well known that all abelian subgroups of G are cyclic, and this condition is sufficient to allow the construction of a finite CW-complex on which G acts freely. In **[2]**, it was proven that for k arbitrary, the elementary abelian 2-subgroups of G must have rank $\leqslant k$. However, for $k \geqslant 1$, more subtle restrictions than rank conditions are necessary. The following is known.

THEOREM **[5]**. *A_4 does not act freely on $S^n \times S^n$, with trivial action on* mod 2 *homology.*[2]

In this paper we apply the results of **[2]** to generalize this result considerably; thus, the main theorem, Theorem 4.4.

THEOREM. *Let G be a semidirect product $\mathbf{Z}/3 \times_T (\mathbf{Z}/2 \times \mathbf{Z}/2)$ or $\mathbf{Z}/7 \times_T (\mathbf{Z}/2 \times \mathbf{Z}/2 \times \mathbf{Z}/2)$, with nontrivial action. Then G does not act freely on a finite CW-complex X, $X \simeq \prod_{j=1}^{k} S^n$, with trivial action on* mod 2 *homology, for any values of k and n.*

For fixed values of n, we obtain stronger restrictions on G, in Theorem 3.5.

THEOREM. *Let G act freely on X, $X \simeq \prod_{j=1}^{k} S^n$, where X is finite, and acts trivially on $H_*(X; \mathbf{Z}/2)$. Let $x \in N_G(E)$ be an odd order element of order p, p prime, where N_G denotes normalizer and E is an elementary abelian subgroup of G. Then, unless $p|(n+1)$, x is in the centralizer of E. Moreover, there is a function $\mu: \mathbf{Z} \to \mathbf{Z}$, such that x is in the centralizer unless $2^{\mu(d)}|(n+1)$. (μ is nontrival, i.e. $\mu(31) = 3$.)*

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[2]R. Oliver informs me that he can also prove this theorem for $(S^n)^k$.

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EXAMPLE. Suppose $x \in N_G(E)$ has order 127, and acts nontrivially on E. Then $16 \cdot 127 | (n+1)$.

These conditions superficially resemble the Milnor condition (see [4]), for $k = 1$. However, the Milnor condition is a restriction on groups acting freely on a *manifold* M, $M \simeq S^n$. Our conditions depend only on the 2-local homotopy type of the space X. Secondly, the Milnor condition depends on k; the group D_6 will act freely on $S^n \times S^n$, but not on S^n. Our conditions are independent of k.

1. Preliminaries. Let Γ be a finite group of odd order, and let $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbf{Z}/2)$ be an n-dimensional linear representation of Γ over the field $k = \mathbf{Z}/2$. We define $\Sigma(\Gamma, \rho)$ by

$$\sum(\Gamma, \rho) = \Gamma \underset{\rho}{\times} (\mathbf{Z}/2)^n,$$

the semidirect product of Γ with $(\mathbf{Z}/2)^n$. Recall that the $\mathbf{Z}/2$-cohomology of $(\mathbf{Z}/2)^n$ is given by

$$H^*((\mathbf{Z}/2)^n; \mathbf{Z}/2) \simeq (\mathbf{Z}/2)[x_1, \ldots, x_n] \overset{(\mathrm{defn.})}{=} A_*,$$

the polynomial ring on n 1-dimensional generators $x_1, \ldots, x_n$. Note that if Γ is a group of automorphisms of $(\mathbf{Z}/2)^n$, then Γ acts on A_*; in fact, if M is the matrix describing the action of $\gamma \in \Gamma$ on $(\mathbf{Z}/2)^n$, then γ acts on $H^1((\mathbf{Z}/2)^n; \mathbf{Z}/2) = k \cdot x_1 + \cdots + k \cdot x_n$ by the matrix M^t, and the action extends to the rest of A_* by multiplicativity.

PROPOSITION 1. *Let Γ be an odd order group, $\rho\colon \Gamma \to \mathrm{GL}_n(k)$ a representation as above. Then $H^*(\Sigma(\Gamma, \rho); k) \cong A_*^\Gamma$, the ring of invariants under the action of Γ on A_*.*

PROOF. Consider the Hochschild-Serre spectral sequence for the group extension

$$(\mathbf{Z}/2)^n \to \sum(\Gamma, \rho) \to \Gamma.$$

$E_2^{p,q} = 0$ for $p \neq 0$ and $E_2^{0,q} = A_q^\Gamma$; hence the result. □

Recall that since $H^*(G; k)$ is the cohomology ring of the space BG for any group G, it is an algebra over the mod 2 Steenrod algebra $\mathcal{Q}(2)$. In particular, A_* is such an algebra, and the action of $\mathcal{Q}(2)$ on A_* is determined by the Cartan formula and the requirements $\mathrm{Sq}^1 x_j = x_j^2$, $\mathrm{Sq}^i x_j = 0$ for $i > 1$.

We recall the inductive definition of the Milnor primitives Q_i by

$$Q_1 = \mathrm{Sq}^1, \qquad Q_{i+1} = \left[Q_i, \mathrm{Sq}^{2^i}\right].$$

Conventionally, as in [1], we define a derivation Q_0 on A_* by $Q_0 x_i = x_i$, and the requirement that Q_0 be a derivation. We remark that under left multiplication of elements of A_*, the derivations of A_* to itself become a graded A_*-module.

PROPOSITION 2 (SEE [1]). (a) *Q_i is a derivation of A_*.*

(b) *There exist polynomials $\varphi_{lj} \in A_*$, $j = 0, \ldots, n-1$, $l > n$, such that $Q_l = \sum_{j=1}^{n-1} \varphi_{lj} Q_j$.*

(c) *$Q_0(z) = 0$ for $z \in A_{2k}$, $Q_0(z) = z$ for $z \in A_{2k+1}$.*

PROPOSITION 3 [1]. *Let $\theta \in A_*$ satisfy $Q_i \theta = 0$ $\forall i$. Then θ is a square.*

COROLLARY 4. *If $\theta \in A_*$ satisfies $Q_j\theta = 0$ for $j = 0, \ldots, n-1$, then θ is a square.*

Finally, we recall some results from [2]. Let G denote a finite group acting freely on a finite CW-complex X, where $X \simeq \prod_{j=1}^{k} S^n$ so that the action of G on $H^*(X; k)$ is trivial. Let $y_j \in H^n(X)$ denote the dual to the fundamental class of the jth sphere. Consider the Serre spectral sequence for the fibration

$$X \to EG \underset{G}{\times} X \to BG$$

where EG denotes a contractible space on which G acts freely. Define $f_j \in H^{n+1}(BG; k) = H^{n+1}(G; k)$ to be $d_{n+1}(y_j)$ in the above spectral sequence.

PROPOSITION 5 [2]. *The ideal in $H^*(G; k)$ generated by the f_j's is $\mathcal{Q}(2)$-invariant.*

Let C_* denote a graded k-algebra, and let $\{\theta_1, \ldots, \theta_k\} \in C_*$ be a collection of homogeneous elements in C_*.

DEFINITION 6. *We say $\{\theta_1, \ldots, \theta_k\}$ is a homogeneous system of parameters (h.s.o.p.) for C_* if C_* is a finitely generated module over the subalgebra generated by the θ_j's.*

REMARK. This terminology is nonstandard in that the usual definition would require the set $\{\theta_1, \ldots, \theta_k\}$ to be algebraically independent.

THEOREM 7 [2]. *If $G = (\mathbf{Z}/2)^n$, and X and the f_j's are as above, then the f_j's form an h.s.o.p. for $A_* = H^*((\mathbf{Z}/2)^n; k)$.*

COROLLARY 8. *If $G = \Sigma(\Gamma, \rho)$, and X and the f_j's are as above, then the f_j's form an h.s.o.p. for $A_*^\Gamma = H^*(\Sigma(\Gamma, \rho);\ k)$.*

PROOF. A_* is finitely generated as an A_*^Γ-module. □

2. Invariant theory for odd order cyclic groups. We wish to study groups of the type $\Sigma(\mathbf{Z}/n\mathbf{Z}; \rho)$, where ρ is a faithful, irreducible representation of $\mathbf{Z}/n\mathbf{Z}$ over k, n odd. The following four propositions are standard results in the representation theory of finite groups. We omit the proofs and refer the reader to [8]. We describe the faithful, irreducible representations of $\mathbf{Z}/n\mathbf{Z}$ over k. Let $k(\zeta_n)$ denote the field obtained from k by adjoining the nth roots of unity, and let $d_n = [k(\zeta_n): k]$. Let $T_n \subseteq k(\zeta_n)$ consist of the nth roots, and let $G = G(k(\zeta_n)|k) \simeq \mathbf{Z}/d_n\mathbf{Z}$ denote the Galois group of $k(\zeta_n)$ over k. Note that G acts on T_n, and consequently on $\mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}; T_n)$.

PROPOSITION 1. *Any injection $\varphi \in \mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, T_n)$ defines an irreducible, faithful representation of $\mathbf{Z}/n\mathbf{Z}$, with $k(\zeta_n) \cong k^{d_n}$ as representation space. Conversely, any irreducible, faithful representation of $\mathbf{Z}/n\mathbf{Z}$ is obtained in this way, and two injections $\varphi, \psi \in \mathrm{Hom}(\mathbf{Z}/n\mathbf{Z}, T_n)$ define isomorphic representations iff $\psi = \varphi^g$, for some $g \in G$.*

It is also useful to consider the faithful, irreducible $\bar{k}$-representations of $\mathbf{Z}/n\mathbf{Z}$, where $\bar{k}$ denotes the algebraic closure of k.

PROPOSITION 2. *All irreducible representations of $\mathbf{Z}/n\mathbf{Z}$ over $\bar{k}$ are one-dimensional. Each $\varphi \in \operatorname{Hom}(\mathbf{Z}/n\mathbf{Z}, T_n)$ defines an irreducible representation of dimension* 1, *and all these representations are distinct. $G(\bar{k}|k)$ acts on the irreducible representations.*

Given a k-representation (V, ρ), one may form a $\bar{k}$-representation $(V \otimes_k \bar{k}, \rho \otimes \mathrm{id})$, which we write $(\bar{V}, \bar{\rho})$. We describe $(\bar{V}, \bar{\rho})$, where $V = k(\zeta_n)$, and ρ is an injection $\rho\colon \mathbf{Z}/n\mathbf{Z} \to T_n$. Recall that the Frobenius map $\gamma(x) = x^2$ topologically generates $G(\bar{k}|k)$, and hence γ generates $G(\bar{k}|k)$, where K is any finite extension of k. Moreover, for $x \in T_n$, $\gamma^{d_n}(x) = x$, since $d_n = [k(\zeta_n)\colon k]$. Let $i\colon k(\zeta_n) \to \bar{k}$ be any embedding.

PROPOSITION 3. *$i \circ \rho$ defines an irreducible $\bar{k}$ representation of $\mathbf{Z}/n\mathbf{Z}$, say σ. Then $(\bar{V}, \bar{\rho}) \simeq \coprod_{j=0}^{d_n-1}(\bar{k}, \sigma^{\gamma^j})$. This is independent of i, since any two choices of i are conjugate under $G(\bar{k}|k)$.*

Thus, an irreducible k-representation amounts to a choice of primitive nth roots of unity ζ; the representation $(\bar{V}, \bar{\rho})$ is then defined by

$$\bar{V} = \bar{k}^{d_n}, \bar{\rho}(T)(x_1, \ldots, x_{d_n}) = \left(\zeta x_1, \zeta^2 x_2, \ldots, \zeta^{2^{(d_n-1)}} x_{d_n}\right).$$

If we let (V, ρ) be a finite-dimensional k-representation of a group G we may form the symmetric algebra $k[V]$, and extend the action of G to $k[V]$. $k[V]^G$ will denote the ring of invariants. If $G = \mathbf{Z}/n\mathbf{Z}$, and ρ is a faithful, irreducible representation on V,

$$H^*\left(\textstyle\sum(\mathbf{Z}/n\mathbf{Z}; \rho); k\right) \cong k[V]^{\mathbf{Z}/n\mathbf{Z}}.$$

The structure of $k[V]^{\mathbf{Z}/n\mathbf{Z}}$ may be complicated; for our purpose, however, it will suffice to know in what dimensions elements exist. For this purpose, we may tensor up to $\bar{k}$, for

PROPOSITION 4. $k[V]^G \otimes_k \bar{k} \cong k[V]^G$.

Let (W, σ) denote any representation of $\mathbf{Z}/n\mathbf{Z}$ over $\bar{k}$. Thus $W \cong \bigoplus_{j=1}^{s} \bar{k}$, and the representation is determined on a generator T of $\mathbf{Z}/n\mathbf{Z}$ by

$$\rho(T) = \Delta(\zeta_1, \ldots, \zeta_s),$$

where $\Delta(\zeta_1, \ldots, \zeta_s)$ denotes the diagonal matrix with entries $\zeta_1, \ldots, \zeta_s$, nth roots of unity. Choosing a particular primitive nth root ζ, we define integers modulo n λ_i by $\zeta_i = \zeta^{\lambda_i}$.

PROPOSITION 5. *$\bar{k}[V]^{\mathbf{Z}/n\mathbf{Z}} \subseteq k[V]$ may be identified with the subalgebra of $k[x_1, \ldots, x_s]$ consisting of all monomials $x_1^{e_1} \cdots x_s^{e_s}$ such that*

$$\sum_{j=1}^{s} e_j \lambda_j = 0 \pmod{n}.$$

PROOF. Let x_j represent a basis vector for the jth summand of (V, ρ), so $Tx_j = \zeta_j x_j = \zeta^{\lambda_j} x_j$.

It is evident that the action of T respects the monomial basis of $\bar{k}[V]$. Given a monomial $x_1^{e_1} \cdots x_s^{e_s}$,

$$T(x_1^{e_1} \cdots x_s^{e_s}) = \left(\prod_{j=1}^{s} \zeta_j^{e_j} \right) x_1^{e_1} \cdots x_s^{e_s} = \zeta^{\Sigma \lambda_j e_j} x_1^{e_1} \cdots x_s^{e_s},$$

so $x_1^{e_1} \cdots x_s^{e_s} \in \bar{k}[V]^{\mathbf{Z}/n\mathbf{Z}} \Leftrightarrow \Sigma_{i=1}^{d_n} \lambda_j e_j \equiv 0 \pmod n$. □

COROLLARY 6. *Let (V, ρ) be any faithful, irreducible $\mathbf{Z}/n\mathbf{Z}$-representation over k. Then $k[\bar{V}]^{\mathbf{Z}/n\mathbf{Z}}$ may be identified with the subalgebra of $k[x_1, \ldots, x_{d_n}]$ consisting of all monomials $x_1^{e_1} \cdots x_{d_n}^{e_{d_n}}$ so that $\Sigma_{i=1}^{d_n} 2^i e_i = 0 \pmod n$.*

DEFINITION 7. *Define the dyadic expansion number of n, $\nu(n)$, by $\nu(n) = \min_S\{\Sigma_{i=1}^{d_n} \alpha_i\}$ where $S = \{(\alpha_1, \ldots, \alpha_{d_n}) | \Sigma_{i=1}^{d_n} \alpha_i 2^i \equiv 0 \pmod n\}$.*

COROLLARY 8. *Let (V, ρ) be as above. Then the first dimension for which $k[V]^{\mathbf{Z}/n\mathbf{Z}}$ is nontrivial is $\nu(n)$.*

As remarked before, $k[V]^{\mathbf{Z}/n\mathbf{Z}}$ is endowed with an $\mathcal{A}(2)$-action. Extending the action k-linearly, we obtain an action of $\mathcal{A}(2)$ on $\bar{k}[\bar{V}]^{\mathbf{Z}/n\mathbf{Z}}$, which satisfies the Cartan formula and the axiom $\mathrm{Sq}^i(x) = 0$ if $\dim x < i$, but no longer satisfies the condition $\mathrm{Sq}^n(x) = x^2$ for $\dim x = n$. We wish to describe the $\mathcal{A}(2)$-action on $k[V]^{\mathbf{Z}/n\mathbf{Z}}$, when this is given in a basis which splits $\bar{V}$ as a direct sum of 1-dimensional representations, as in Corollary 6.

LEMMA 9. *In terms of a basis which splits $\bar{V}$ into a sum of one-dimensional representations, say $x_1, \ldots, x_{d_n}$, the action of $\mathcal{A}(2)$ on $\bar{k}[\bar{V}]$ is characterized by the Cartan formula, $\bar{k}$-linearity, and the condition $\mathrm{Sq}^1 x_j = x_{j-1}^2$. (Here x_0 is identified with x_{d_n}.)*

PROOF. Writing $(\bar{V}, \bar{\rho}) \cong \bigoplus_{j=1}^{d_n} (\bar{V}_j, \bar{\rho}_j)$, where $\bar{V}_i \cong \bar{k}$, and $\rho_i(T)(x) = \zeta^{2^{i-1}} x$, V may be identified with the invariants $[\bigoplus_{j=1}^{d_n} \bar{V}_j]^{G(\bar{k}|k)}$ where $G(\bar{k}|k)$ acts on $\bigoplus_{j=1}^{d_n} V_j$ by

$$(x_1, \ldots, x_{d_n})^\gamma = (x_{d_n}^\gamma, x_1^\gamma, \ldots, x_{d_n-1}^\gamma).$$

Now, $\mathrm{Sq}^1 x = x^2$ for $x \in V = A_1$, so it suffices to check that the formula of the theorem satisfies this condition on V. But every element in $[\bigoplus_{j=1}^{d_n} \bar{V}_j]^{G(\bar{k}|k)}$ is of the form

$$(\alpha, \alpha^\gamma, \alpha^{\gamma^2}, \ldots, \alpha^{\gamma^{d_n-1}}), \quad \text{for } \alpha \in k(\zeta_n).$$

If $(\alpha_1, \ldots, \alpha_{d_n}) = (\alpha_{d_n}^\gamma, \alpha_1^\gamma, \ldots, \alpha_{d_n-1}^\gamma)$, we obtain $\alpha_1 = \alpha_{d_n}^\gamma = \alpha_{d_n-1}^{\gamma^2} = \cdots = \alpha_1^{\gamma^{d_n}}$. Thus, α_0 belongs to the fixed field of γ^{d_n}, which is $k(\zeta_n)$. Applying the formula of the theorem to elements of this type, we obtain

$$\mathrm{Sq}^1\left(\sum_{j=1}^{d_n} \alpha^{\gamma^j} x_{j+1} \right) = \sum_{j=1}^{d_n} \alpha^{\gamma^j} x_j^2 = \sum_{j=1}^{d_n} \alpha^{\gamma^{j+1}} x_{j+1}^2 = \left(\sum_{j=1}^{d_n} \alpha^{\gamma^j} x_{j+1} \right)^2,$$

where we interpret x_{d_n+1} as x_1. □

3. Systems of parameters in $k[V]^{\mathbf{Z}/n\mathbf{Z}}$. Recall from §1 the definition of an h.s.o.p. $\{\theta_1, \ldots, \theta_k\}$ of a graded k-algebra. We say $\{\theta_1, \ldots, \theta_k\}$ is $\mathcal{A}(2)$-invariant if the ideal $(\theta_1, \ldots, \theta_k)$ is preserved by the $\mathcal{A}(2)$-action, provided the given graded k-algebra is an $\mathcal{A}(2)$-algebra.

LEMMA 1. *Let* $\{\theta_1, \ldots, \theta_k\}$ *be an h.s.o.p. for a graded algebra* C_*, *and* j: $C_* \to C'_*$ *a surjective map of graded rings. Then* $\{j\theta_1, \ldots, j\theta_k\}$ *is an h.s.o.p. for* C'_*. *Similarly, if* C_* *and* C'_* *are* $\mathcal{A}(2)$-*algebras, and* j *commutes with the* $\mathcal{A}(2)$-*action, then* $\{j\theta_1, \ldots, j\theta_k\}$ *is an* $\mathcal{A}(2)$-*invariant h.s.o.p.*

PROOF. Immediate. □

PROPOSITION 2. *Suppose* $\{\theta_1, \ldots, \theta_k\}$ *is an h.s.o.p. for* $k[V]^{\mathbf{Z}/n\mathbf{Z}}$, *and suppose* $\dim(\theta_j) = d$ *for all* $j = 1, \ldots, k$. *Then* $n|d$.

PROOF. It is clear that $\{\theta_1, \ldots, \theta_k\}$ also forms an h.s.o.p. for $\bar{k}[\bar{V}]^{\mathbf{Z}/n\mathbf{Z}}$. Perform a basis change so that $\bar{k}[\bar{V}]$ is identified with the subalgebra of $\bar{k}[x_1, \ldots, x_{d_n}]$ defined in Corollary 2.6. Consider the ideal $I \subseteq \bar{k}[\bar{V}]$ generated by all monomials $x_i x_j$; $i \neq j$. Then $\bar{k}[\bar{V}]/I$ has the elements x_i^s as basis in dimension s. The image of $\bar{k}[\bar{V}]^{\mathbf{Z}/n\mathbf{Z}}$ consists of the subalgebra generated by $x_1^n, \ldots, x_{d_n}^n$. In particular, $k[V]^{\mathbf{Z}/n\mathbf{Z}}/_{I \cap k[V]^{\mathbf{Z}/n\mathbf{Z}}}$ is zero in all dimensions except multiples of n, and hence admits h.s.o.p. $\{\eta_1, \ldots, \eta_k\}$ with $\dim(\eta_1) = \cdots = \dim(\eta_k) = d$ only for $n|d$. □

Consider $A_* = \bar{k}[x_1, \ldots, x_{d_n}]$, with $\mathbf{Z}/n\mathbf{Z}$ acting by $Tx_i = \zeta^{2^{i-1}}x_i$, ζ a primitive nth root of unity, and $\mathcal{A}(2)$ acting as in Lemma 2.9.

LEMMA 3. *The ideal* J *generated by the elements* $\{x_i + x_j\}_{1 \leq i,j \leq d_n}$ *is* $\mathcal{A}(2)$-*invariant. Hence* A_*/J *is an algebra over* $\mathcal{A}(2)$. *Moreover,* $A_*/J \cong \bar{k}[x]$, *with* $\mathrm{Sq}^1 x = x^2$.

PROOF. Clear. □

PROPOSITION 4. *Let* $\{\theta_1, \ldots, \theta_k\}$ *be an* $\mathcal{A}(2)$-*invariant h.s.o.p. for* $A_*^{\mathbf{Z}/n\mathbf{Z}}$, *with* $\dim(\theta_1) = \cdots = \dim(\theta_k) = S$. *Then* $2^{t+1}|S$, *where* t *is the largest integer satisfying* $2^t < \nu(n)$.

PROOF. According to Corollary 2.8, $A_*^{\mathbf{Z}/n\mathbf{Z}} = 0$ for $* < \nu(n)$. Consequently, $B_* = 0$ also for $* < \nu(n)$, where $B_* = A_*^{\mathbf{Z}/n\mathbf{Z}}/J \cap A_*^{\mathbf{Z}/n\mathbf{Z}}$. Let $\{\eta_1, \ldots, \eta_l\}$ be any $\mathcal{A}(2)$-invariant h.s.o.p. for B_*, $\dim(\eta_1) = \cdots = \dim(\eta_l) = S$. Since $\{\eta_1, \ldots, \eta_l\}$ is $\mathcal{A}(2)$-invariant,

$$\mathrm{Sq}^i(\eta_j) = \sum \lambda_{jk}^i \eta_k,$$

where $\lambda_{jk}^i \in B_i$. But for $i < \nu(n)$, $B_i = 0$, so $\mathrm{Sq}^i(\eta_j) = 0$ for $i < \nu(n)$. Using the decomposability of Sq^i for i not a power of 2, this is equivalent to the requirement $\mathrm{Sq}^{2^q}(\eta_j) = 0\ \forall q \leq t$. But, as is well known,

$$\mathrm{Sq}^{2^q}(x^m) = 0 \qquad \forall q \leq t \Rightarrow 2^{t+1}|m.$$

Since a basis for B_m is given by x^m, for $m > \nu(n)$, if the η_j's are to form an h.s.o.p., we must have $2^{t+1}|s$. Applying the second half of Lemma 1, we see that $\dim(\theta_1) = \cdots = \dim(\theta_k) = S$ is divisible by 2^{t+1}. □

We interpret this geometrically.

THEOREM 5. *Let $\Sigma(\mathbf{Z}/n\mathbf{Z}, \rho)$ act freely on $X \simeq \prod_{j=1}^k S^m$, X finite, and trivially on $H_*(X; k)$, where ρ is an irreducible, faithful representation of $\mathbf{Z}/n\mathbf{Z}$, n odd. Then*
(a) $n|(m+1)$,
(b) $2^{t+1}|(m+1)$, *where t satisfies $2^t < \nu(n) < 2^{t+1}$.*

PROOF. By Proposition 1.5 and Theorem 1.7, the f_j's ($f_j = d_{m+1}(z_j)$, z_j the dual to the fundamental class of the jth sphere) form an $\mathcal{Q}(2)$-invariant h.s.o.p. for $H^*(\Sigma(\mathbf{Z}/n\mathbf{Z}, \rho); k)$ with $\dim(f_j) = m + 1$. Proposition 2 and Proposition 4 now imply the result. □

EXAMPLE 6. Let $G = \Sigma(\mathbf{Z}/31, \rho)$, so $G = \mathbf{Z}/31 \times_\rho (\mathbf{Z}/2)^5$. G acts freely on $\prod_{j=1}^k S^m$, for some k, with trivial action on homology only if $8|(m+1)$, since $\nu(31) = 5$.

COROLLARY 7. *Let G be any finite group acting freely on $\prod_{j=1}^k S^{2^q-1}$, with trivial action on $H_*(\prod_{j=1}^k S^{2^q-1}; k)$. Let $E \subseteq G$ be any elementary abelian 2-subgroup, and $\mathcal{O} \subseteq N_G(E)$ an odd order subgroup. Then $\mathcal{O}$ acts trivially on E.*

PROOF. Apply Theorem 5(a), noting that on $\prod_{j=1}^k S^{2^q-1}$, none of the groups $\Sigma(\mathbf{Z}/n\mathbf{Z}; \rho)$ act freely, and if there were an $\mathcal{O} \subseteq N_G(E)$ which acted nontrivially on E, some $\Sigma(\mathbf{Z}/n\mathbf{Z}; \rho)$ would embed in G. □

4. The groups A_4 and $\Sigma(\mathbf{Z}/7\mathbf{Z}; \rho)$. In this section, we will prove that the groups $A_4 = \Sigma(\mathbf{Z}/3\mathbf{Z}; \rho)$ and $\Sigma(\mathbf{Z}/7\mathbf{Z}; \rho)$ do not act freely on $\prod_{j=1}^k S^n$ for any values of k and n, where as usual ρ denotes an irreducible, faithful representation of the odd order cyclic group in question. Using the terminology of §2, we observe that $\nu(3) = 2$, and $\nu(7) = 3$. Consequently $H^1(A_4; k) = H^1(\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho); k) = H^2(\Sigma(\mathbf{Z}/7\mathbf{Z}; \rho); k) = 0$. Let $A_* = \bar{k}[x_1, x_2]$ for A_4, and $A_* = \bar{k}[x_1, x_2, x_3]$ for $\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho)$. Let $B_* = A_*^{\mathbf{Z}/3\mathbf{Z}}$ and $B_* = A_*^{\mathbf{Z}/7\mathbf{Z}}$ respectively.

PROPOSITION 1. *Let $\{\theta_1, \ldots, \theta_t\}$ be a collection of homogeneous elements in $H^*(G; k)$, where $G = A_4$ or $\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho)$, so that $\dim(\theta_1) = \cdots = \dim(\theta_t) = 2d$. Suppose the ideal $(\theta_1, \ldots, \theta_t)$ is $\mathcal{Q}(2)$-invariant. Then θ_i is a square in A_* for all i.*

PROOF. By the above remarks, since $H^*(A_4; k) = 0$ for $* = 1$, and $H^*(\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho); k) = 0$ for $* = 1, 2$, if $(\theta_1, \ldots, \theta_t)$ is $\mathcal{Q}(2)$-invariant, we must have $\mathrm{Sq}^1(\theta_i) = 0$ in the case of A_4, and $\mathrm{Sq}^1(\theta_i) = \mathrm{Sq}^2(\theta_i) = 0$ in the case of $\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho)$. Since $\dim(\theta_i) = 2d$, Q_0 vanishes on θ_i by Proposition 1.2(c). In the case of $\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho)$, $Q_2 = [\mathrm{Sq}^1, \mathrm{Sq}^2]$ vanishes on θ_i as well. $H^*(A_4; k)$ is a subalgebra of $H^*(\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}, k)$, consequently Q_j may be expressed as a linear combination of Q_0 and Q_1 for all j. Hence $Q_j\theta_i = 0\ \forall j$, which by Corollary 1.4 implies that θ_i is a square. Similarly, $[k(\zeta_7): k] = 3$, so on $H^*(\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho))$, Q_j may be expressed as a linear combination of Q_0, Q_1, and Q_2, implying $Q_j\theta_i = 0\ \forall_i$, so θ_i is a square in A_*. □

PROPOSITION 2. *Let $\theta \in B_*$ be a homogeneous element in B_*, with $\theta = \eta^2$ for some $\eta \in A_*$. Then $\eta \in B_*$.*

PROOF. The map $x \to x^2$ from A_S to A_{2S} is k-linear and an injection. Hence η is uniquely determined. If $\theta \in B_{2S}$, then θ is a sum $\sum a_{(e_1,\ldots,e_l)} x_1^{e_1} \cdots x_l^{e_l}$, where the l-tuples $(e_1, \ldots, e_l)$ range over all solutions to the equation $\sum 2^{i-1} e_i \equiv 0 \pmod n$, and $n = 3$ or 7, $l = 2$ or 3. θ is a square in A_* if and only if each e_j is even, say $e_j = 2f_j$. Thus, $\eta = \sum \sqrt{a_{(e_1,\ldots,e_l)}}\, x_1^{f_1} \cdots x_l^{f_l}$. Now, $\sum 2^{i-1} f_i = \frac{1}{2} \sum 2^{i-1} e_i \equiv 0$ (mod n), since n is odd. □

Let A_*^2 and B_*^2 denote the subalgebras of squares in A_* and B_*. The above proposition shows that $A_*^2 \cap B_* = B_*^2$.

PROPOSITION 3. *Let* $\theta_1, \ldots, \theta_s \in B_*^2$, $\dim(\theta_1) = \cdots = \dim(\theta_s)$, *and suppose that* $\xi = \sum \lambda_i \theta_i$, *where* $\xi \in B_*^2$ *is homogeneous, and the* λ_i*'s are homogeneous elements of* A_*. *Then there is an expression* $\xi = \sum \lambda_i' \theta_i$, *with* λ_i' *homogeneous,* $\lambda_i' \in B_*^2$.

PROOF. We first point out that $A_* = \bar{k}[x_1, \ldots, x_l]$ is free as an A_*^2-module, with basis consisting of all monomials $x_1^{d_1} \cdots x_l^{d_l}$, with $d_j = 0$ or 1. Given a monomial μ in the above basis, let π_μ denote the projection on the μth factor in the A_*^2 module A_*. π_μ is A_*^2-linear. If we have an expression $\xi = \sum \lambda_i \theta_i$, as in the theorem, we apply π_1 to the expression to obtain $\pi_1(\xi) = \sum \pi_1(\lambda_i)\theta_i$. Since ξ is a square, $\pi_1(\xi) = \xi$, and $\pi_1(\lambda_i) \in A_*^2$, so we may suppose that the λ_i's are squares.

Let $\{\chi_i\}$ be a complete set of irreducible characters of $\mathbf{Z}/n\mathbf{Z}$ over $\bar{k}$. Let p_i denote the projection on the χ_i-isotypical component (see [6]). The p_i's are B_*-linear, and moreover $p_i(A_*^2) \subseteq A_*^2$. If $\xi = \sum \lambda_i \theta_i$, with $\xi, \theta_1, \ldots, \theta_s \in B_*^2$, and $\lambda_i \in A_*^2$, by applying p_0, if χ_0 is the trivial character, we obtain $p_0(\xi) = \sum p_0(\lambda_i)\theta_i$, and $p_0(\xi) = \xi$, since $\xi \in B_*$, so $\xi = \sum p_0(\lambda_i)\theta_i$. Setting $\lambda_i' = p_0(\lambda_i)$, we obtain the theorem, since $\lambda_i' \in A_*^2 \cap B_* = B_*^2$, by Proposition 2. □

THEOREM 4. *There does not exist a free G-action on* $\prod_{i=1}^k S^n$, *with trivial action on homology, for* $G = A_4$ *or* $\Sigma(\mathbf{Z}/7\mathbf{Z}, \rho)$.

PROOF. According to Proposition 1.5 and Theorem 1.7, it suffices to prove that $H^*(G;k)$ admits no $\mathcal{Q}(2)$-invariant system of parameters $\{\theta_1, \ldots, \theta_s\}$ with $\dim(\theta_1) = \dim(\theta_2) = \cdots = \dim(\theta_s) = n + 1$. Proposition 1 and Proposition 2 guarantee that each θ_i is a square in B_*, say $\theta_i = \zeta_i$. We claim that the ideal $(\zeta_1, \ldots, \zeta_s)$ is $\mathcal{Q}(2)$-invariant. Let $\mathrm{Sq}^j(\theta_i) = \sum \lambda_{ik}^j \theta_k$. Since the θ's are squares, $\mathrm{Sq}^{2j}(\theta_i) = (\mathrm{Sq}^j(\zeta_i))^2$, and $\mathrm{Sq}^{2j+1}(\theta_i) = 0$. To obtain an expression for $\mathrm{Sq}^j(\zeta_i)$ in terms of $\zeta_1, \ldots, \zeta_s$, we write $(\mathrm{Sq}^j(\zeta_i))^2 = \sum \lambda_{ik}^{2j}\theta_k = \sum \lambda_{ik}^{2j}\zeta_k^2$. But Proposition 3 gives that λ_{ik}^{2j} may be chosen to be a square, say $\lambda_{ik}^{2j} = (\mu_{ik}^j)^2$, so $(\mathrm{Sq}^j(\zeta_i))^2 = \sum (\mu_{ik}^j)^2 \zeta_k^2$, or $\mathrm{Sq}^j(\zeta_i) = \sum \mu_{ik}^j \zeta_k$, which was to be shown. Moreover, $\{\zeta_1, \ldots, \zeta_s\}$ is an h.s.o.p. for $H^*(G;k)$, since $\{\zeta_1^2, \ldots, \zeta_s^2\} = \{\theta_1, \ldots, \theta_s\}$ is. Consequently, if $\dim(\theta_1) = \cdots = \dim(\theta_s) = 2^m l$, where l is odd, we have demonstrated the existence of an $\mathcal{Q}(2)$-invariant h.s.o.p. $\{\zeta_1, \ldots, \theta_s\}$, with $\dim(\theta_i) = 2^{m-1} l$. Inductively, we may produce an h.s.o.p. $\{\theta_1, \ldots, \theta_s\}$ with $\dim(\theta_i) = l$, where l is odd. We claim, however, that this is impossible. Let J be as in Lemma 3.3. Then $B_*/J \cap B_*$ is a graded $\mathcal{Q}(2)$-invariant subalgebra of $k[x]$, and $B_*/J \cap B_* = 0$ for $* = 1$. Consequently, $B_*/J \cap B_*$ admits no $\mathcal{Q}(2)$-invariant h.s.o.p. in odd dimensions, for Sq^1 is nonzero on x^{2i+1}. By Lemma 3.1, B_* admits no $\mathcal{Q}(2)$-invariant h.s.o.p. in odd dimensions, which gives a contradiction. □

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