NON-QUASI-WELL BEHAVED CLOSED • DERIVATIONS

BY

FREDERICK M. GOODMAN

ABSTRACT. Examples are given of a non-quasi-well behaved closed • derivation in $C([0, 1] \times [0, 1])$ extending the partial derivative, and of a compact subset $\Omega$ of the plane such that $C(\Omega)$ has no nonzero quasi-well behaved • derivations but $C(\Omega)$ does admit nonzero closed • derivations.

1. Introduction. A regularity condition which arises in the study of unbounded derivations in $C^*$ algebras is quasi-well behavedness. (A definition is given below.) Sakai asked in [S2] whether every closed • derivation in a $C^*$ algebra must be quasi-well behaved (qwb). Batty gave a counterexample: a compact subset $\mathbb{D}^2$ of the plane such that the partial derivative $\partial/\partial x$ defines a non-qwb closed • derivation in $C(\mathbb{D}^2)$ [B2, Example 5].

Most of this paper is devoted to two further examples. In §3, we present an example of a non-qwb closed • derivation in $C([0, 1] \times [0, 1])$ which is an extension of the partial derivative $\partial/\partial x$. This is interesting for two reasons. It shows that an extension of the qwb closed • derivative $\partial/\partial x$ need not be qwb. And it provides an example of a non-qwb closed • derivation in $C_0(M)$, where $M$ is a manifold. (The boundary of the unit square plays no role.) The second example, in §4, is of a compact subset $\mathbb{D}^2$ of the plane such that $C(\mathbb{D}^2)$ has no nonzero qwb • derivations, but does admit nontrivial closed • derivations.

§2 contains a brief discussion of qwb and non-qwb closed • derivations in $C[0, 1]$. The remainder of this introduction contains definitions and preliminary results.

We will be concerned exclusively with commutative $C^*$ algebras. Let $\Omega$ be compact Hausdorff. A linear map $\delta$ in $C(\Omega)$ is called a • derivation if its domain $\mathcal{D}(\delta)$ is a dense conjugate closed subalgebra of $C(\Omega)$, and $\delta$ satisfies $\delta(fg) = f\delta(g) + \delta(f)g$ and $\delta(f) = \delta(f)$ for all $f, g \in \mathcal{D}(\delta)$. If $\delta$ is a closed map, then $\mathcal{D}(\delta)$, with the graph norm $\| \cdot \|_\delta = \| \cdot \|_\infty + \| \delta(\cdot) \|_\infty$, is a Silov regular Banach algebra with structure space $\Omega$. The Silov algebra $\mathcal{D}(\delta)$ has a $C^*$ functional calculus. If $f, g \in \mathcal{D}(\delta)$ agree in a neighborhood of $\omega \in \Omega$, then $\delta(f)(\omega) = \delta(g)(\omega)$ [S2], [G2], [B3].

We let $\mathcal{D}(\delta)_{\omega,a}$ denote the set of real valued functions in $\mathcal{D}(\delta)$.

DEFINITION 1.1. Let $\delta$ be a • derivation in $C(\Omega)$ (not necessarily closed).

(i) $f \in \mathcal{D}(\delta)_{\omega,a}$ is said to be well behaved if $\exists \omega \in \Omega$ such that $\| f \|_\infty = |f(\omega)|$ and $\delta(f)(\omega) = 0$.

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(ii) \( f \in \mathcal{D}(\delta)_{s.a.} \) is said to be strongly well behaved if \( \forall \omega \in \Omega, \|f\|_{\infty} = |f(\omega)| \) implies \( \delta(f)(\omega) = 0 \).

(iii) A point \( \omega \in \Omega \) is said to be well behaved if \( \forall f \in \mathcal{D}(\delta)_{s.a.}, \|f\|_{\infty} = |f(\omega)| \) implies \( \delta(f)(\omega) = 0 \).

**Notation.** Denote the set of well behaved functions in \( \mathcal{D}(\delta)_{s.a.} \) by \( WF(\delta) \) and the set of well behaved points in \( \Omega \) by \( WP(\delta) \). By \( int WF(\delta) \), we mean the interior of \( WF(\delta) \) in \( \mathcal{D}(\delta)_{s.a.} \) with respect to the sup-norm.

The following result is due to C. Batty [B1, Proposition 7], [B2, Propositions 2, 3 and Theorem 4].

**Theorem 1.2.** Let \( \delta \) be a \( \ast \) derivation in \( C(\Omega) \).

1. Every element of \( int WF(\delta) \) is strongly well behaved.

2. The following conditions are equivalent:
   - (a) \( WP(\delta) = \Omega \),
   - (b) \( WF(\delta) = \mathcal{D}(\delta)_{s.a.} \).

3. The following conditions are equivalent:
   - (a) \( int WP(\delta) \) is dense in \( \Omega \),
   - (b) \( int WF(\delta) \) is dense in \( \mathcal{D}(\delta)_{s.a.} \) in the sup-norm.

**Definition 1.3.** A \( \ast \) derivation is called well behaved if it satisfies the conditions of 1.2(2). It is called quasi-well behaved if it satisfies the conditions of 1.2(3).

To give these definitions a context, we mention that a closed \( \ast \) derivation \( \delta \) is the infinitesimal generator of a strongly continuous one parameter group of \( \ast \) automorphisms (a \( C^* \) dynamics) if and only if

(i) \( \delta \) is well behaved, and

(ii) \( (\delta \pm 1)\mathcal{D}(\delta) = C(\Omega) \).

A qwb \( \ast \) derivation is always closable, and the closure is again qwb [S2], [B1].

**Lemma 1.4.** Let \( \delta \) be a closed \( \ast \) derivation in \( C(\Omega) \), and let \( \omega \in WP(\delta) \). If \( f \in \mathcal{D}(\delta)_{s.a.} \) has a local extremum at \( \omega \), then \( \delta(f)(\omega) = 0 \).

**Proof.** By replacing \( f \) by \(-f + c1\) if necessary, we can assume that \( f \) has a local maximum at \( \omega \) and \( f(\omega) > 0 \). Let \( U \) be an open neighborhood of \( \omega \) such that for all \( \omega' \in U, f(\omega') > f(\omega) > 0 \). There is an \( e \in \mathcal{D}(\delta) \) such that \( e = 1 \) near \( \omega \), \( 0 < e < 1 \), and \( \text{support}(e) \subseteq U \) (because \( \mathcal{D}(\delta) \) is a conjugate closed Silov algebra). Then \( ef \in \mathcal{D}(\delta)_{s.a.} \) and \( \|ef\|_{\infty} = (ef)(\omega) \). Since \( \omega \in WP(\delta) \), \( \delta(ef)(\omega) = 0 \). But \( f = ef \) near \( \omega \); so \( \delta(f)(\omega) = 0 \) also. \( \square \)

**Definition 1.5.** Let \( \delta \) be a closed \( \ast \) derivation in \( C(\Omega) \). A closed subset \( E \subseteq \Omega \) is called a restriction set for \( \delta \) if \( \delta(f)|_E = 0 \) whenever \( f|_E = 0 \). If \( E \) is a restriction set, then the formula \( \delta_E(f)(\omega) = \delta(f)|_E \) defines a \( \ast \) derivation in \( C(E) \) with domain \( \{f|_E : f \in \mathcal{D}(\delta)\} \).

If \( \delta \) is a closed \( \ast \) derivation in \( C(\Omega) \) and \( U \subseteq \Omega \) is open, then \( \bar{U} \) is a restriction set for \( \delta \) and \( \delta_{\bar{U}} \) is closable [B3].

**Lemma 1.6.** Let \( \delta \) be a closed \( \ast \) derivation in \( C(\Omega) \), and let \( U \) be an open subset of \( \Omega \). Then \( WP(\delta) \cap U \subseteq WP(\delta_{\bar{U}}) \). Consequently if \( \delta \) is qwb, then \( \delta_{\bar{U}} \) is also qwb.
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Proof. Let \( \omega \in WP(\delta) \cap U \) and suppose \( f \in \mathcal{G}(\delta U)_{sa} \) satisfies \( \|f\|_U = |f(\omega)| \). Let \( e \) be an element of \( \mathcal{G}(\delta) \) such that \( e = 1 \) near \( \omega \), \( 0 < e < 1 \), and \( \text{support}(e) \subseteq U \). Then \( fe \in \mathcal{G}(\delta) \) and \( \|fe\|_\infty = |(fe)(\omega)| = |f(\omega)| \). Therefore \( \delta_f(f)(\omega) = \delta(fe)(\omega) = 0 \). \( \square \)

Lemma 1.7. Suppose \( \delta \) is a closable \( \ast \) derivation in \( C(\Omega) \). Then \( \text{int} \ WP(\delta) = \text{int} \ WP(\tilde{\delta}) \).

Proof. Suppose that \( f \in \mathcal{G}(\tilde{\delta})_{sa} \) attains its maximum value at a point \( \omega_0 \in \text{int} \ WP(\delta) \). We have to show that \( \tilde{\delta}(f)(\omega_0) = 0 \). Assume without loss of generality that \( f \leq 0 \) and \( f(\omega_0) = 0 \). Let \( U \) be an open neighborhood of \( \omega_0 \) in \( \text{int} \ WP(\delta) \), and let \( e \in \mathcal{G}(\tilde{\delta}) \) satisfy \( e = 1 \) near \( \omega_0 \), \( 0 < e < 1 \), and \( \text{support}(e) \subseteq U \). For each \( n \in \mathbb{N} \), choose \( f_n \in \mathcal{G}(\delta) \) satisfying:

1. \( \|f_n - (f + e/n)\|_\infty < 1/3n \), and
2. \( \|\tilde{\delta}(f_n) - \tilde{\delta}(f + e/n)\|_\infty < 1/n \).

Since \( (f + e/n)(\omega_0) = 1/n, f_n(\omega_0) > 2/3n \). For \( \omega \not\in U \),

\[ f_n(\omega) < f(\omega) + 1/3n < 1/3n. \]

Therefore \( f_n \) achieves its maximum value at a point \( \omega_n \in U \), and \( \tilde{\delta}(f_n)(\omega_n) = 0 \), since \( U \subseteq WP(\delta) \). It follows from (2) that

\[ |\tilde{\delta}(f)(\omega_n)| < n^{-1} + n^{-1} \|\tilde{\delta}(e)\|_\infty. \]

If \( \omega \) is an accumulation point of \( \langle \omega_n \rangle \), then \( \omega \in \text{cl}(U) \) and \( \tilde{\delta}(f)(\omega) = 0 \). Since \( U \) was an arbitrary neighborhood of \( \omega_0 \) in \( \text{int} \ WP(\delta) \), this shows that \( \tilde{\delta}(f)(\omega_0) = 0 \); thus \( \text{int} WP(\delta) \subseteq \text{int} WP(\tilde{\delta}) \). The opposite inclusion is evident. \( \square \)

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2. Closed \( \ast \) derivations in \( C([0, 1]) \). Let \( I \) denote the interval \([0, 1]\). The following conditions are equivalent for a closed \( \ast \) derivation in \( C(I) \) [S2]:

1. \( \mathcal{G}(\delta) \) contains a homeomorphism of \( I \) onto \( I \).
2. There is a \( \ast \) automorphism \( \alpha \) of \( C(I) \) such that \( \alpha C^1(I) \subseteq \mathcal{G}(\delta) \).

Derivations meeting these conditions were investigated in [G2] and [B3]. Batty showed in [B3] that they are precisely the qwb closed \( \ast \) derivations in \( C(I) \). It follows from this that a closed \( \ast \) derivation in \( C(I) \) extending a qwb \( \ast \) derivation is necessarily qwb. Using methods of [G2, §3] one can derive similar results for closed \( \ast \) derivations in \( C_0(\mathbb{R}) \) and \( C(T) \). (\( T \) denotes the circle.) It is an open question whether there are any non-qwb closed \( \ast \) derivations in these algebras.

Lemma 2.1. Suppose \( C(I) \) has a non-qwb closed \( \ast \) derivation. Then \( C(I) \) has a closed \( \ast \) derivation \( D \) satisfying:

1. \( \text{int} WP(D) = \varnothing \),
2. if \( f \in \mathcal{G}(D)_{sa} \), then \( f \) is not one-to-one on any subinterval of \( I \).

Proof. If \( \delta \) is a closed non-qwb \( \ast \) derivation in \( C(I) \), let \( J \) be a closed interval such that \( J \cap \text{int} WP(\delta) = \varnothing \). Let \( D \) denote the closure of \( \delta_J \). It is easily seen that

\[ \text{int} WP(D) \cap \text{int}(J) \subseteq \text{int} WP(\delta) \cap \text{int}(J) = \varnothing. \]
If $f \in \mathfrak{D}(D)_{s.a.}$ is one-to-one on a closed interval $K \subseteq J$, then by the remarks above, $D_K$ is qwb. But $\text{int } WP(D_K) \cap \text{int}(K) \subseteq \text{int } WP(D) = \emptyset$. This is a contradiction.

This lemma shows that if there are any non-qwb closed * derivations in $C(I)$ at all, then there are some which are quite strange. A closed * derivation $D$ in $C(I)$ such that $\text{int } WP(D) = \emptyset$ would surely have nothing at all to do with differentiation.

The following lemma will be used in §4.

**Lemma 2.2.** Let $\delta$ be a well behaved * derivation in $C(I)$. Then $\delta(f)(0) = \delta(f)(1) = 0$ for all $f \in \mathfrak{D}(\delta)$.

**Proof.** Since $\delta$ is closable and its closure is also well behaved [S2, Theorem 2.9], we can assume that $\delta$ is closed. It also suffices to prove the statement for $f$ real valued. If $f$ is one-to-one in some neighborhood of 0, then $f$ has a local extremum at 0, and $\delta(f)(0) = 0$ (1.4). If $f$ is never one-to-one in a neighborhood of 0, then in each neighborhood $f$ has a local extremum, and therefore in each neighborhood there is a point $p$ such that $\delta(f)(p) = 0$. By continuity, $\delta(f)(0) = 0$ in this case also. Similarly, $\delta(f)(1) = 0$. □

3. A non-quasi-well behaved closed * derivation in $C(I \times I)$. While any non-qwb closed * derivation in $C(I)$ must be fairly bizarre, there are rather tame examples of non-qwb closed * derivations in $C(I \times I)$. In fact there exist closed * derivations extending the partial derivative $\partial / \partial x$ in $C(I \times I)$ such that the interior of the set of well behaved points is empty. To give such an example, we require the following lemma, due to Batty [B3, Theorem 4.4].

**Lemma 3.1.** Let $\delta$ be a closed * derivation in $C(\Omega)$ and let $f \in \ker(\delta)_{s.a.}$. Let $E = f^{-1}(0)$ and let $\omega_0 \in \text{int } WP(\delta) \cap E$. If $h \in \mathfrak{D}(\delta)_{s.a.}$ and $h(\omega_0) = \sup\{h(\omega) : \omega \in E\}$, then $\delta(h)(\omega_0) = 0$.

Consequently, if $\text{int } WP(\delta) \cap E$ is dense in $E$, then $E$ is a restriction set for $\delta$ and $\delta_E$ is qwb. If $E \subseteq \text{int } WP(\delta)$, then $\delta_E$ is well behaved.

Let us write $\partial$ for $\partial / \partial x$. The natural domain for $\partial$ is $\{f : \partial f$ exists and is continuous on $I \times I\}$, and with this domain, $\partial$ is a closed * derivation.

Let $Y$ and $Z$ be compact Hausdorff spaces. We say a continuous function $\Phi : I \times Y \to Z$ is a generalized Cantor function (gcf) if each fiber $\Phi^{-1}(z) (z \in Z)$ is a connected subset of $I \times \{y\}$ for some $y \in Y$ and $\Phi$ is not one-to-one on any open subset of $I \times Y$. It was shown in [G2] that for any gcf $\Phi : I \times I \to Z$ there is a unique closed * derivation $D$ extending $\partial$ such that $\mathfrak{D}(D) = \mathfrak{D}(\partial) + \Phi^0(C(Z))$ and $\ker(D) = \Phi^0(C(Z))$.

We will produce a gcf $\Phi : I \times I \to Z$ such that the set

$$S = \{y \in I : x \mapsto \Phi(x, y) \text{ is injective on } I\}$$

is dense in $I$. Suppose for the moment that this has been done. Let $D$ be the closed * derivation in $C(I \times I)$ extending $\partial$ and with $\ker(D) = \Phi^0(C(Z))$. Assume that $\text{int } WP(D)$ is not empty and let $J$ and $K$ be closed subintervals of $I$ such that
Let $J \times \{y_0\} \subseteq \text{int} WP(D)$. Then $D_{J \times K}$ is well defined and closable, and
\[
\text{int}(J \times K) \subseteq \text{int} WP(D_{J \times K}) \quad \text{(Lemma 1.6)}
\]
\[
\subseteq \text{int} WP(D_{J \times K}) \quad \text{(Lemma 1.7)}.
\]
Let $y_0 \in S \cap \text{int}(K)$. The set $J \times \{y_0\}$ is the zero set of the function $(x,y) \mapsto y - y_0$, which is an element of the kernel of $D_{J \times K}$. By Lemma 3.1, $J \times \{y_0\}$ is a restriction set for $D_{J \times K}$, and therefore for $D$. On the one hand, $D_{J \times \{y_0\}}$ extends $D_{J \times \{y_0\}}$, a nonzero derivation. On the other hand, $\ker(D)$ separates points of $J \times \{y_0\}$. Hence
\[
\ker(D_{J \times \{y_0\}}) \supseteq \{ f|_{J \times \{y_0\}} : f \in \ker(D) \} = C(J \times \{y_0\}).
\]
That is, $D_{J \times \{y_0\}}$ is zero. This contradiction shows that in fact $\text{int} WP(D) = \emptyset$.

We now turn to the construction of $\Phi$. Let $\langle f_i : I \to I \rangle_{i \in \mathbb{N}}$ be a sequence of nondecreasing gcf's which collectively separate points of $I$ [G2, 1.3.3]. Define
\[
g_n = \sum_{i=1}^{n} 2^{-i} f_i \quad (n \in \mathbb{N}), \quad \text{and} \quad g_\infty = \sum_{i=1}^{\infty} 2^{-i} f_i.
\]
Then each $g_n$ is a nondecreasing gcf, but $g_\infty$ is injective. Define $H : I \times I \to \mathbb{R}$ by the following rules.

(a) $H(x, \frac{1}{2} + n^{-1}) = g_n(x) (n \in \mathbb{N})$.

(b) $H(x, \frac{1}{2}) = g_\infty(x)$.

(c) For $\frac{1}{2} + (n + 1)^{-1} \leq y \leq \frac{1}{2} + n^{-1}$, $H(x, y)$ is to be affine in $y$ for each fixed $x$.

(d) $H(x, \frac{1}{2} - t) = H(x, \frac{1}{2} + t) (x \in I, 0 < t < \frac{1}{2})$.

Then $H$ is continuous, $x \mapsto H(x, y)$ is a nondecreasing gcf for each $y \neq \frac{1}{2}$, and $x \mapsto H(x, \frac{1}{2}) = g_\infty(x)$ is injective. Note also that
\[
H(x, 0) = H(x, 1) = g_1(x) = f_1(x).
\]

Now let $n, k$ be odd positive integers, with $1 < k < 2^n - 1$. Let
\[
J_{n,k} = \left[ k \cdot 2^{-n} - 2^{-(n+1)}, k \cdot 2^{-n} + 2^{-(n+1)} \right],
\]
and let $T_{n,k}$ be the following affine transformation of $\mathbb{R}$ which maps $J_{n,k}$ onto $[0, 1]$:
\[
T_{n,k}(y) = 2y + \frac{1}{2} - k.
\]
Define
\[
\phi_{n,k}(x, y) = \begin{cases} f_1(x) & (y \notin J_{n,k}), \\
H(x, T_{n,k}(y)) & (y \in J_{n,k}).
\end{cases}
\]
If $y \neq k \cdot 2^{-n}$, then $x \mapsto \phi_{n,k}(x, y)$ is a gcf, but

(1) the function $x \mapsto \phi_{n,k}(x, k \cdot 2^{-n})$ is injective.

Let $A$ be the $C^*$ algebra generated by $\{\phi_{n,k}\}$ and the 2nd coordinate function $(x, y) \mapsto y$, and let $\Phi : I \times I \to Z$ be a continuous function such that $A = \Phi(C(Z))$. We claim that $\Phi$ is a gcf. Since it is clear that each fiber of $\Phi$ is a connected subset of $I \times \{y\}$ for some $y$, to prove the claim it will suffice to show that

(2) for each even positive integer $m$ and each odd $j$ with $1 < j < 2^m - 1$, the function $x \mapsto \Phi(x, j \cdot 2^{-m})$ is a gcf.
Let $m$ and $j$ be given. The fibers of $\Phi(\cdot, j \cdot 2^{-m})$ are the same as those of the function
\[ x \mapsto \sum_{\substack{n,k \text{ odd} \\ 1 \leq n \leq m \\ 1 \leq k < 2^n - 1}} 2^{-(n+k)}\phi_{n,k}(x, j \cdot 2^{-m}). \]

Suppose $n, k$ are odd positive integers with $n > m$ and $1 \leq k < 2^n - 1$. Since $j \cdot 2^{-m} \neq k \cdot 2^{-n}$,
\[ |j \cdot 2^{-m} - k \cdot 2^{-n}| = |j \cdot 2^{n-m} - k| \cdot 2^{-n} > 2^{-n}. \]
Therefore $j \cdot 2^{-m} \not\in J_{n,k}$ and $\phi_{n,k}(x, j \cdot 2^{-m}) = f_1(x)$. It follows that the fibers of $\Phi(\cdot, j \cdot 2^{-m})$ are the same as those of the generalized Cantor function
\[ x \mapsto f_1(x) + \sum_{\substack{n,k \text{ odd} \\ 1 \leq n \leq m \\ 1 \leq k < 2^n - 1}} \phi_{n,k}(x, j \cdot 2^{-m}). \]
This proves (2) and shows that $\Phi$ is a gcf.

From (1) it follows that $x \mapsto \Phi(x, k \cdot 2^{-n})$ is injective for each odd $n$ and $k$ with $1 \leq k < 2^n - 1$. Thus $\Phi$ has all the desired properties.

4. An example. An example is given here of a closed subset $\Omega$ of $I \times I$ such that $C(\Omega)$ has no nonzero closed quasi-well behaved $\ast$ derivation but does admit nontrivial closed $\ast$ derivations.

We construct an $\Omega$ with the following properties:
(i) The projection of $\Omega$ on the second coordinate axis is totally disconnected.
(ii) Each nonempty (relatively) open subset of $\Omega$ contains a nonempty compact-open subset of $\Omega$. (But $\Omega$ is not totally disconnected.)
(iii) $\Omega$ is the closure of a union of horizontal line segments.

Let $\beta = (\beta_i)_{i \in \mathbb{N}}$ be any sequence with $\beta_i \in \{0, 2\}$ for all $i \in \mathbb{N}$. (Thus $\sum_{i=1}^{\infty} \beta_i 3^{-i}$ is an arbitrary element of the Cantor set $\Delta$.) For each $n \in \mathbb{N}$ let
\[ a_{n,\beta} = \sum_{i=1}^{n} \beta_i 3^{-i} \quad \text{and} \quad b_{n,\beta} = \sum_{i=1}^{n} \beta_i 3^{-i} + 3^{-(n+1)}. \]
For each such $\beta$ and $n$, and for each odd $k$ ($1 \leq k < 3^n - 2$), let
\[ G_{n,k,\beta} = [(k-1) \cdot 3^{-n}, k \cdot 3^{-n}] \times [a_{n,\beta}, b_{n,\beta}]. \]
Define
\[ \Omega = (I \times \Delta) \setminus \left( \bigcup_{n,k,\beta} G_{n,k,\beta} \right), \]
the union being over all allowed values of $(n, k, \beta)$.

Some further notation will facilitate the discussion of $\Omega$. For $n$ and $\beta$ as above and for odd $k$ ($1 \leq k < 3^n$) define:
\[ p_{n,k,\beta} = (k \cdot 3^{-n}, b_{n,\beta}), \]
\[ E_{n,k,\beta} = [(k-1) \cdot 3^{-n}, k \cdot 3^{-n}] \times \{ b_{n,\beta} \}, \]
\[ H_{n,k,\beta} = [(k-1) \cdot 3^{-n}, k \cdot 3^{-n}] \times [a_{n,\beta}, b_{n,\beta}]. \]
Note that for all $n, \beta$,
\[ a_{n,\beta} - 3^{-(n+1)} \leq a_{n,\beta} \leq R \setminus \Delta, \quad \text{and} \quad b_{n,\beta}, b_{n,\beta} + 3^{-(n+1)} \leq R \setminus \Delta. \]
Hence, if for each odd \( k \) \((1 \leq k \leq 3^a - 2)\) we let
\[
K_{n,k,\beta} = \left] k \cdot 3^{-n}, (k + 1) \cdot 3^{-n} \right[ \times a_{n,\beta} - 3^{-(n+1)}, b_{n,\beta} + 3^{-(n+1)}] ,
\]
then
\[
\Omega = (I \times \Delta) \setminus \left( \bigcup _{n,k,\beta} K_{n,k,\beta} \right).
\]
This shows that \( \Omega \) is a closed set.

We next observe that for each \((n, k, \beta)\), the set \( H_{n,k,\beta} \cap \Omega \) is open and closed in \( \Omega \). It is clearly closed, and it is open because
\[
H_{n,k,\beta} \cap \Omega = \left( \left] (k - 2) \cdot 3^{-n}, (k + 1)3^{-n} \right[ \times a_{n,\beta} - 3^{-(n+1)}, b_{n,\beta} + 3^{-(n+1)} \right] \right) \cap \Omega.
\]
One can show that \( \Omega \) has the following property. The details can be found in [G1, pp. 76–81].

**Lemma.** Let \( p \in \Omega \). For each \( \varepsilon > 0 \) there is a triplet \((n, k, \beta)\) such that
1. \( E_{n,k,\beta} \subseteq \Omega \),
2. \( \text{diameter}(H_{n,k,\beta}) < \varepsilon \),
3. \( \text{distance}(p, p_{n,k,\beta}) < \varepsilon \).

Now suppose that \( \delta \) is a closed \( * \) derivation in \( C(\Omega) \) and that \( p \in \text{int} WP(\delta) \). If \( U \) is an open neighborhood of \( p \) in \( \text{int} WP(\delta) \), then by the lemma there is a triplet \((n, k, \beta)\) such that \( E_{n,k,\beta} \subseteq \Omega \) and \( H_{n,k,\beta} \cap \Omega \subseteq U \). Let \( H = H_{n,k,\beta} \cap \Omega \). \( H \) is a restriction set for \( \delta \), and \( \delta_H \) is well behaved (1.6). Since \( \mathcal{S}(\delta) \) is a Silov algebra and \( H \) is open and closed, the characteristic function \( 1_H \) of \( H \) is an element of \( \mathcal{S}(\delta) \). It follows from this that \( \delta_H \) is also closed.

Let \( \pi \) denote the second coordinate projection on \( H \); \( \pi(\Omega) \) is totally disconnected and therefore \( C(\pi(\Omega)) \) is the uniform closure of the subalgebra generated by its projections. If \( e \in C(\pi(\Omega)) \) is a projection, then \( \pi^0(e) \) is a projection in \( C(H) \). Since \( \delta_H \) is a closed \( * \) derivation, \( \ker(\delta_H) \) contains the \( C^* \) algebra generated by these projections; that is
\[
\ker(\delta_H) \supseteq \pi^0(C(\pi(\Omega))) = \pi^0(C(I)).
\]
It follows that each set \( H' = (I \times \{ y \}) \cap H \) has the form \( f^{-1}(0) \) for some real valued \( f \in \ker(\delta_H) \). By 3.1, if \( H' \neq \emptyset \), then \( H' \) is a restriction set for \( \delta_H \), and the induced derivation \( (\delta_H)_{H'} = \delta_{H'} \) is well behaved. Taking \( y = b_{n,\beta} \), we have \( H' = E_{n,k,\beta} \).

Let \( f \in \mathcal{S}(\delta) \). Since \( \delta_{E_{n,k,\beta}} \) is well behaved, Lemma 2.2 implies
\[
\delta(f)(p_{n,k,\beta}) = \delta_{E_{n,k,\beta}}(f|_{E_{n,k,\beta}})(p_{n,k,\beta}) = 0.
\]
Thus
\[
p_{n,k,\beta} \in Z = \{ \omega \in \Omega : \delta(f)(\omega) = 0 \ \forall f \in \mathcal{S}(\delta) \}.
\]
This shows that \( Z \) intersects each neighborhood of the point \( p \) in \( \text{int} WP(\delta) \). Since \( Z \) is closed, \( p \in Z \); that is \( \text{int} WP(\delta) \subseteq Z \). It follows that if \( \delta \) is quasi-well behaved, then \( Z = \Omega \), and \( \delta = 0 \).
It is easy to produce a nontrivial closed $\ast$ derivation in $C(\Omega)$. By the lemma $\bigcup \{ E_{n,k,\beta} : E_{n,k,\beta} \subseteq \Omega \}$ is dense in $\Omega$. Define $\mathcal{A}$ to be the set of $f \in C(\Omega)$ such that $\partial f/\partial x$ exists on each $E_{n,k,\beta} \subseteq \Omega$ and $\partial f/\partial x$ extends to a continuous function on $\Omega$. Note that $\mathcal{A}$ contains $\{ f|_\Omega : f \in C^1(I \times I) \}$ and therefore $\mathcal{A}$ is dense in $C(\Omega)$. The partial derivative $\partial/\partial x$ defines a closed $\ast$ derivation in $C(\Omega)$ with domain $\mathcal{A}$. This $\ast$ derivation is of course not qwb. But it does satisfy a weaker condition defined by Batty in [B2]; it is pseudo-well behaved.

References