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NON-QUASI-WELL BEHAVED CLOSED * DERIVATIONS

BY

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ABSTRACT. Examples are given of a non-quasi-well behaved closed * derivation in $C([0, 1] \times [0, 1])$ extending the partial derivative, and of a compact subset Ω of the plane such that $C(\Omega)$ has no nonzero quasi-well behaved * derivations but $C(\Omega)$ does admit nonzero closed * derivations.

1. Introduction. A regularity condition which arises in the study of unbounded derivations in C^* algebras is *quasi-well behavedness*. (A definition is given below.) Sakai asked in [**S2**] whether every closed * derivation in a C^* algebra must be quasi-well behaved (qwb). Batty gave a counterexample: a compact subset Ω of the plane such that the partial derivative $\partial/\partial x$ defines a non-qwb closed * derivation in $C(\Omega)$ [**B2**, Example 5].

Most of this paper is devoted to two further examples. In §3, we present an example of a non-qwb closed * derivation in $C([0, 1] \times [0, 1])$ which is an extension of the partial derivative $\partial/\partial x$. This is interesting for two reasons. It shows that an extension of the qwb closed * derivative $\partial/\partial x$ need not be qwb. And it provides an example of a non-qwb closed * derivation in $C_0(M)$, where M is a manifold. (The boundary of the unit square plays no role.) The second example, in §4, is of a compact subset Ω of the plane such that $C(\Omega)$ has *no* nonzero qwb * derivations, but does admit nontrivial closed * derivations.

§2 contains a brief discussion of qwb and non-qwb closed * derivations in $C[0, 1]$. The remainder of this introduction contains definitions and preliminary results.

We will be concerned exclusively with commutative C^* algebras. Let Ω be compact Hausdorff. A linear map δ in $C(\Omega)$ is called a * derivation if its domain $\mathfrak{D}(\delta)$ is a dense conjugate closed subalgebra of $C(\Omega)$, and δ satisfies $\delta(fg) = f\delta(g) + \delta(f)g$ and $\delta(\bar{f}) = \overline{\delta(f)}$ for all $f, g \in \mathfrak{D}(\delta)$. If δ is a closed map, then $\mathfrak{D}(\delta)$, with the graph norm $\|\cdot\|_\delta = \|\cdot\|_\infty + \|\delta(\cdot)\|_\infty$, is a Silov regular Banach algebra with structure space Ω. The Silov algebra $\mathfrak{D}(\delta)$ has a C^1 functional calculus. If $f, g \in \mathfrak{D}(\delta)$ agree in a neighborhood of $\omega \in \Omega$, then $\delta(f)(\omega) = \delta(g)(\omega)$ [**S2**], [**G2**], [**B3**].

We let $\mathfrak{D}(\delta)_{\text{s.a.}}$ denote the set of real valued functions in $\mathfrak{D}(\delta)$.

DEFINITION 1.1. Let δ be a * derivative in $C(\Omega)$ (not necessarily closed).

(i) $f \in \mathfrak{D}(\delta)_{\text{s.a.}}$ is said to be *well behaved* if $\exists \omega \in \Omega$ such that $\|f\|_\infty = |f(\omega)|$ and $\delta(f)(\omega) = 0$.

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(ii) $f \in \mathfrak{D}(\delta)_{\text{s.a.}}$ is said to be *strongly well behaved* if $\forall \omega \in \Omega$, $\|f\|_\infty = |f(\omega)|$ implies $\delta(f)(\omega) = 0$.

(iii) A point $\omega \in \Omega$ is said to be *well behaved* if $\forall f \in \mathfrak{D}(\delta)_{\text{s.a.}}$, $\|f\|_\infty = |f(\omega)|$ implies $\delta(f)(\omega) = 0$.

Notation. Denote the set of well behaved functions in $\mathfrak{D}(\delta)_{\text{s.a.}}$ by $WF(\delta)$ and the set of well behaved points in Ω by $WP(\delta)$. By int $WF(\delta)$, we mean the interior of $WF(\delta)$ in $\mathfrak{D}(\delta)_{\text{s.a.}}$ with respect to the sup-norm.

The following result is due to C. Batty [**B1**, Proposition 7], [**B2**, Propositions 2, 3 and Theorem 4].

THEOREM 1.2. *Let δ be a $*$ derivation in $C(\Omega)$.*

(1) *Every element of* int $WF(\delta)$ *is strongly well behaved.*

(2) *The following conditions are equivalent*:

(a) $WP(\delta) = \Omega$,

(b) $WF(\delta) = \mathfrak{D}(\delta)_{s.a.}$

(3) *The following conditions are equivalent*:

(a) int $WP(\delta)$ *is dense in* Ω,

(b) int $WF(\delta)$ *is dense in* $\mathfrak{D}(\delta)_{s.a.}$ *in the sup-norm.*

DEFINITION 1.3. A $*$ derivation is called *well behaved* if it satisfies the conditions of 1.2(2). It is called *quasi-well behaved* if it satisfies the conditions of 1.2(3).

To give these definitions a context, we mention that a closed $*$ derivation δ is the infinitesimal generator of a strongly continuous one parameter group of $*$ automorphisms (a C^* dynamics) if and only if

(i) δ is well behaved, and

(ii) $(\delta \pm \mathbf{1})\mathfrak{D}(\delta) = C(\Omega)$.

A qwb $*$ derivation is always closable, and the closure is again qwb [**S2**], [**B1**].

LEMMA 1.4. *Let δ be a closed $*$ derivation in $C(\Omega)$, and let $\omega \in WP(\delta)$. If $f \in \mathfrak{D}(\delta)_{s.a.}$ has a local extremum at ω, then $\delta(f)(\omega) = 0$.*

PROOF. By replacing f by $-f + c\mathbf{1}$ if necessary, we can assume that f has a local maximum at ω and $f(\omega) > 0$. Let U be an open neighborhood of ω such that for all $\omega' \in U$, $f(\omega) \geqslant f(\omega') > 0$. There is an $e \in \mathfrak{D}(\delta)$ such that $e = 1$ near ω, $0 \leqslant e \leqslant 1$, and support$(e) \subseteq U$ (because $\mathfrak{D}(\delta)$ is a conjugate closed Silov algebra). Then $ef \in \mathfrak{D}(\delta)_{\text{s.a.}}$ and $\|ef\|_\infty = (ef)(\omega)$. Since $\omega \in WP(\delta)$, $\delta(ef)(\omega) = 0$. But $f = ef$ near ω; so $\delta(f)(\omega) = 0$ also. □

DEFINITION 1.5. Let δ be a closed $*$ derivation in $C(\Omega)$. A closed subset $E \subseteq \Omega$ is called a *restriction set* for δ if $\delta(f)|_E = 0$ whenever $f|_E = 0$. If E is a restriction set, then the formula $\delta_E(f|_E) = \delta(f)|_E$ defines a $*$ derivation in $C(E)$ with domain $\{f|_E : f \in \mathfrak{D}(\delta)\}$.

If δ is a closed $*$ derivation in $C(\Omega)$ and $U \subseteq \Omega$ is open, then $\bar{U}$ is a restriction set for δ and $\delta_{\bar{U}}$ is closable [**B3**].

LEMMA 1.6. *Let δ be a closed $*$ derivation in $C(\Omega)$, and let U be an open subset of Ω. Then $WP(\delta) \cap U \subseteq WP(\delta_{\bar{U}})$. Consequently if δ is qwb, then $\delta_{\bar{U}}$ is also qwb.*

PROOF. Let $\omega \in WP(\delta) \cap U$ and suppose $f \in \mathfrak{D}(\delta_{\bar{U}})_{\text{s.a.}}$ satisfies $\|f\|_{\bar{U}} = |f(\omega)|$. Let e be an element of $\mathfrak{D}(\delta)$ such that $e = 1$ near ω, $0 \leqslant e \leqslant 1$, and support$(e) \subseteq U$. Then $fe \in \mathfrak{D}(\delta)$ and $\|fe\|_\infty = |(fe)(\omega)| = |f(\omega)|$. Therefore $\delta_{\bar{U}}(f)(\omega) = \delta(fe)(\omega) = 0$. □

LEMMA 1.7. *Suppose δ is a closable $*$ derivation in $C(\Omega)$. Then* int $WP(\delta) =$ int $WP(\bar{\delta})$.

PROOF. Suppose that $f \in \mathfrak{D}(\bar{\delta})_{\text{s.a.}}$ attains its maximum value at a point $\omega_0 \in$ int $WP(\delta)$. We have to show that $\bar{\delta}(f)(\omega_0) = 0$. Assume without loss of generality that $f \leqslant 0$ and $f(\omega_0) = 0$. Let U be an open neighborhood of ω_0 in int $WP(\delta)$, and let $e \in \mathfrak{D}(\bar{\delta})$ satisfy $e = 1$ near ω_0, $0 \leqslant e \leqslant 1$, and support$(e) \subseteq U$. For each $n \in \mathbf{N}$, choose $f_n \in \mathfrak{D}(\delta)$ satisfying:

(1) $\|f_n - (f + e/n)\|_\infty < 1/3n$, and

(2) $\|\delta(f_n) - \bar{\delta}(f + e/n)\|_\infty < 1/n$.

Since $(f + e/n)(\omega_0) = 1/n, f_n(\omega_0) > 2/3n$. For $\omega \notin U$,

$$f_n(\omega) < f(\omega) + 1/3n \leqslant 1/3n.$$

Therefore f_n achieves its maximum value at a point $\omega_n \in U$, and $\delta(f_n)(\omega_n) = 0$, since $U \subseteq WP(\delta)$. It follows from (2) that

$$|\bar{\delta}(f)(\omega_n)| < n^{-1} + n^{-1}\|\bar{\delta}(e)\|_\infty.$$

If $\bar{\omega}$ is an accumulation point of $\langle \omega_n \rangle$, then $\bar{\omega} \in \bar{U}$ and $\bar{\delta}(f)(\bar{\omega}) = 0$. Since U was an arbitrary neighborhood of ω_0 in int $WP(\delta)$, this shows that $\bar{\delta}(f)(\omega_0) = 0$; thus int $WP(\delta) \subseteq$ int $WP(\bar{\delta})$. The opposite inclusion is evident. □

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2. Closed $*$ derivations in $C([0, 1])$. Let I denote the interval $[0, 1]$. The following conditions are equivalent for a closed $*$ derivation in $C(I)$ [**S2**]:

(1) $\mathfrak{D}(\delta)$ contains a homeomorphism of I onto I.

(2) There is a $*$ automorphism α of $C(I)$ such that $\alpha C^1(I) \subseteq \mathfrak{D}(\delta)$.

Derivations meeting these conditions were investigated in [**G2**] and [**B3**]. Batty showed in [**B3**] that they are precisely the qwb closed $*$ derivations in $C(I)$. It follows from this that a closed $*$ derivation in $C(I)$ extending a qwb $*$ derivation is necessarily qwb. Using methods of [**G2**, §3] one can derive similar results for closed $*$ derivations in $C_0(\mathbf{R})$ and $C(\mathbf{T})$. ($\mathbf{T}$ denotes the circle.) *It is an open question whether there are any non-qwb closed $*$ derivations in these algebras.*

LEMMA 2.1. *Suppose $C(I)$ has a non-qwb closed $*$ derivation. Then $C(I)$ has a closed $*$ derivation D satisfying*:

(i) int $WP(D) = \varnothing$,

(ii) *if $f \in \mathfrak{D}(D)_{s.a.}$, then f is not one-to-one on any subinterval of I.*

PROOF. If δ is a closed non-qwb $*$ derivation in $C(I)$, let J be a closed interval such that $J \cap$ int $WP(\delta) = \varnothing$. Let D denote the closure of δ_J. It is easily seen that

$$\text{int } WP(D) \cap \text{int}(J) \subseteq \text{int } WP(\delta) \cap \text{int}(J) = \varnothing.$$

If $f \in \mathfrak{D}(D)_{\text{s.a.}}$ is one-to-one on a closed interval $K \subseteq J$, then by the remarks above, $\overline{D}_K$ is qwb. But int $WP(\overline{D}_K) \cap \operatorname{int}(K) \subseteq$ int $WP(D) = \varnothing$. This is a contradiction. □

This lemma shows that if there are any non-qwb closed * derivations in $C(I)$ at all, then there are some which are quite strange. A closed * derivation D in $C(I)$ such that int $WP(D) = \varnothing$ would surely have nothing at all to do with differentiation.

The following lemma will be used in §4.

LEMMA 2.2. *Let δ be a well behaved * derivation in $C(I)$. Then $\delta(f)(0) = \delta(f)(1) = 0$ for all $f \in \mathfrak{D}(\delta)$.*

PROOF. Since δ is closable and its closure is also well behaved [**S2**, Theorem 2.9], we can assume that δ is closed. It also suffices to prove the statement for f real valued. If f is one-to-one in some neighborhood of 0, then f has a local extremum at 0, and $\delta(f)(0) = 0$ (1.4). If f is never one-to-one in a neighborhood of 0, then in each neighborhood f has a local extremum, and therefore in each neighborhood there is a point p such that $\delta(f)(p) = 0$. By continuity, $\delta(f)(0) = 0$ in this case also. Similarly, $\delta(f)(1) = 0$. □

3. A non-quasi-well behaved closed * derivation in $C(I \times I)$. While any non-qwb closed * derivation in $C(I)$ must be fairly bizarre, there are rather tame examples of non-qwb closed * derivations in $C(I \times I)$. In fact there exist closed * derivations extending the partial derivative $\partial/\partial x$ in $C(I \times I)$ such that the interior of the set of well behaved points is empty. To give such an example, we require the following lemma, due to Batty [**B3**, Theorem 4.4].

LEMMA 3.1. *Let δ be a closed * derivation in $C(\Omega)$ and let $f \in \ker(\delta)_{s.a.}$. Let $E = f^{-1}(0)$ and let $\omega_0 \in$ int $WP(\delta) \cap E$. If $h \in \mathfrak{D}(\delta)_{s.a.}$ and $h(\omega_0) = \sup\{h(\omega): \omega \in E\}$, then $\delta(h)(\omega_0) = 0$.*

Consequently, if int $WP(\delta) \cap E$ is dense in E, then E is a restriction set for δ and δ_E is qwb. If $E \subseteq$ int $WP(\delta)$, then δ_E is well behaved.

Let us write ∂ for $\partial/\partial x$. The natural domain for ∂ is $\{f$: ∂f exists and is continuous on $I \times I\}$, and with this domain, ∂ is a closed * derivation.

Let Y and Z be compact Hausdorff spaces. We say a continuous function Φ: $I \times Y \to Z$ is a *generalized Cantor function* (gcf) if each fiber $\Phi^{-1}(z)$ $(z \in Z)$ is a connected subset of $I \times \{y\}$ for some $y \in Y$ and Φ is not one-to-one on any open subset of $I \times Y$. It was shown in [**G2**] that for any gcf Φ: $I \times I \to Z$ there is a unique closed * derivation D extending ∂ such that $\mathfrak{D}(D) = \mathfrak{D}(\partial) + \Phi^0(C(Z))$ and $\ker(D) = \Phi^0(C(Z))$.

We will produce a gcf Φ: $I \times I \to Z$ such that the set

$$S = \{y \in I: x \mapsto \Phi(x, y) \text{ is injective on } I\}$$

is dense in I. Suppose for the moment that this has been done. Let D be the closed * derivation in $C(I \times I)$ extending ∂ and with $\ker(D) = \Phi^0(C(Z))$. Assume that int $WP(D)$ is not empty and let J and K be closed subintervals of I such that

$J \times K \subseteq \operatorname{int} WP(D)$. Then $D_{J\times K}$ is well defined and closable, and

$$\operatorname{int}(J \times K) \subseteq \operatorname{int} WP(D_{J\times K}) \qquad \text{(Lemma 1.6)}$$
$$\subseteq \operatorname{int} WP(\overline{D_{J\times K}}) \qquad \text{(Lemma 1.7).}$$

Let $y_0 \in S \cap \operatorname{int}(K)$. The set $J \times \{y_0\}$ is the zero set of the function $(x, y) \mapsto y - y_0$, which is an element of the kernel of $\overline{D_{J\times K}}$. By Lemma 3.1, $J \times \{y_0\}$ is a restriction set for $\overline{D_{J\times K}}$, and therefore for D. On the one hand, $D_{J\times\{y_0\}}$ extends $\partial_{J\times\{y_0\}}$, a nonzero derivation. On the other hand, $\ker(D)$ separates points of $J \times \{y_0\}$. Hence

$$\ker(D_{J\times\{y_0\}}) \supseteq \{f|_{J\times\{y_0\}}: f \in \ker(D)\} = C(J \times \{y_0\}).$$

That is, $D_{J\times\{y_0\}}$ is zero. This contradiction shows that in fact $\operatorname{int} WP(D) = \varnothing$.

We now turn to the construction of Φ. Let $\langle f_i: I \to I\rangle_{i\in\mathbf{N}}$ be a sequence of nondecreasing gcf's which collectively separate points of I [**G2**, 1.3.3]. Define

$$g_n = \sum_{i=1}^{n} 2^{-i} f_i \qquad (n \in \mathbf{N}), \quad \text{and} \quad g_\infty = \sum_{i=1}^{\infty} 2^{-i} f_i.$$

Then each g_n is a nondecreasing gcf, but g_∞ is injective. Define $H: I \times I \to \mathbf{R}$ by the following rules.

(a) $H(x, \frac{1}{2} + n^{-1}) = g_n(x)$ $(n \in \mathbf{N})$.

(b) $H(x, \frac{1}{2}) = g_\infty(x)$.

(c) For $\frac{1}{2} + (n+1)^{-1} \leqslant y \leqslant \frac{1}{2} + n^{-1}$, $H(x, y)$ is to be affine in y for each fixed x.

(d) $H(x, \frac{1}{2} - t) = H(x, \frac{1}{2} + t)$ $(x \in I, 0 \leqslant t \leqslant \frac{1}{2})$.

Then H is continuous, $x \mapsto H(x, y)$ is a nondecreasing gcf for each $y \neq \frac{1}{2}$, and $x \mapsto H(x, \frac{1}{2}) = g_\infty(x)$ is injective. Note also that

$$H(x, 0) = H(x, 1) = g_1(x) = f_1(x).$$

Now let n, k be odd positive integers, with $1 \leqslant k \leqslant 2^n - 1$. Let

$$J_{n,k} = \left[k \cdot 2^{-n} - 2^{-(n+1)}, k \cdot 2^{-n} + 2^{-(n+1)}\right],$$

and let $T_{n,k}$ be the following affine transformation of $\mathbf{R}$ which maps $J_{n,k}$ onto $[0, 1]$:

$$T_{n,k}(y) = 2^n y + \tfrac{1}{2} - k.$$

Define

$$\phi_{n,k}(x, y) = \begin{cases} f_1(x) & (y \notin J_{n,k}), \\ H(x, T_{n,k}(y)) & (y \in J_{n,k}). \end{cases}$$

If $y \neq k \cdot 2^{-n}$, then $x \mapsto \phi_{n,k}(x, y)$ is a gcf, but

(1) the function $x \mapsto \phi_{n,k}(x, k \cdot 2^{-n})$ is injective.

Let A be the C^* algebra generated by $\{\phi_{n,k}\}$ and the 2nd coordinate function $(x, y) \mapsto y$, and let $\Phi: I \times I \to Z$ be a continuous function such that $A = \Phi^0(C(Z))$. We claim that Φ is a gcf. Since it is clear that each fiber of Φ is a connected subset of $I \times \{y\}$ for some y, to prove the claim it will suffice to show that

(2) for each *even* positive integer m and each odd j with $1 \leqslant j \leqslant 2^m - 1$, the function $x \mapsto \Phi(x, j \cdot 2^{-m})$ is a gcf.

Let m and j be given. The fibers of $\Phi(\cdot, j \cdot 2^{-m})$ are the same as those of the function

$$x \mapsto \sum_{\substack{n,k \text{ odd} \\ 1 \leqslant k \leqslant 2^n - 1}} 2^{-(n+k)} \phi_{n,k}(x, j \cdot 2^{-m}).$$

Suppose n, k are odd positive integers with $n > m$ and $1 \leqslant k \leqslant 2^n - 1$. Since $j \cdot 2^{-m} \neq k \cdot 2^{-n}$,

$$|j \cdot 2^{-m} - k \cdot 2^{-n}| = |j \cdot 2^{n-m} - k| \cdot 2^{-n} \geqslant 2^{-n}.$$

Therefore $j \cdot 2^{-m} \notin J_{n,k}$ and $\phi_{n,k}(x, j \cdot 2^{-m}) = f_1(x)$. It follows that the fibers of $\Phi(\cdot, j \cdot 2^{-m})$ are the same as those of the generalized Cantor function

$$x \mapsto f_1(x) + \sum_{\substack{n,k \text{ odd} \\ 1 \leqslant n \leqslant m \\ 1 \leqslant k \leqslant 2^n - 1}} \phi_{n,k}(x, j \cdot 2^{-m}).$$

This proves (2) and shows that Φ is a gcf.

From (1) it follows that $x \mapsto \Phi(x, k \cdot 2^{-n})$ is injective for each odd n and k with $1 \leqslant k \leqslant 2^n - 1$. Thus Φ has all the desired properties.

4. An example. An example is given here of a closed subset Ω of $I \times I$ such that $C(\Omega)$ has no nonzero closed quasi-well behaved $*$ derivation but does admit nontrivial closed $*$ derivations.

We construct an Ω with the following properties:

(i) The projection of Ω on the second coordinate axis is totally disconnected.

(ii) Each nonempty (relatively) open subset of Ω contains a nonempty compact-open subset of Ω. (But Ω is not totally disconnected.)

(iii) Ω is the closure of a union of horizontal line segments.

Let $\beta = \langle \beta_i \rangle_{i \in \mathbf{N}}$ be any sequence with $\beta_i \in \{0, 2\}$ for all $i \in \mathbf{N}$. (Thus $\sum_{i=1}^{\infty} \beta_i 3^{-i}$ is an arbitrary element of the Cantor set Δ.) For each $n \in \mathbf{N}$ let

$$a_{n,\beta} = \sum_{i=1}^{n} \beta_i 3^{-i} \quad \text{and} \quad b_{n,\beta} = \sum_{i=1}^{n} \beta_i 3^{-i} + 3^{-(n+1)}.$$

For each such β and n, and for each odd k $(1 \leqslant k \leqslant 3^n - 2)$, let

$$G_{n,k,\beta} = \left]k \cdot 3^{-n}, (k+1) \cdot 3^{-n}\right[\times \left[a_{n,\beta}, b_{n,\beta}\right].$$

Define

$$\Omega = (I \times \Delta) \setminus \left(\bigcup_{n,k,\beta} G_{n,k,\beta} \right),$$

the union being over all allowed values of (n, k, β).

Some further notation will facilitate the discussion of Ω. For n and β as above and for odd k $(1 \leqslant k \leqslant 3^n)$ define:

$p_{n,k,\beta} = (k \cdot 3^{-n}, b_{n,\beta})$,
$E_{n,k,\beta} = [(k-1) \cdot 3^{-n}, k \cdot 3^{-n}] \times \{b_{n,\beta}\}$, and
$H_{n,k,\beta} = [(k-1) \cdot 3^{-n}, k \cdot 3^{-n}] \times [a_{n,\beta}, b_{n,\beta}]$.

Note that for all n, β,

$$\left]a_{n,\beta} - 3^{-(n+1)}, a_{n,\beta}\right[\subseteq \mathbf{R} \setminus \Delta, \quad \text{and} \quad \left]b_{n,\beta}, b_{n,\beta} + 3^{-(n+1)}\right[\subseteq \mathbf{R} \setminus \Delta.$$

Hence, if for each odd k $(1 \leqslant k \leqslant 3^n - 2)$ we let

$$K_{n,k,\beta} = \left]k \cdot 3^{-n}, (k+1) \cdot 3^{-n}\right[\times \left]a_{n,\beta} - 3^{-(n+1)}, b_{n,\beta} + 3^{-(n+1)}\right[,$$

then

$$\Omega = (I \times \Delta) \setminus \Big(\bigcup_{n,k,\beta} K_{n,k,\beta} \Big).$$

This shows that Ω is a closed set.

We next observe that for each (n, k, β), the set $H_{n,k,\beta} \cap \Omega$ is open and closed in Ω. It is clearly closed, and it is open because

$$H_{n,k,\beta} \cap \Omega = \left(\left](k-2) \cdot 3^{-n}, (k+1)3^{-n}\right[\times \left]a_{n,\beta} - 3^{-(n+1)}, b_{n,\beta} + 3^{-(n+1)}\right[\right) \cap \Omega.$$

One can show that Ω has the following property. The details can be found in [**G1**, pp. 76–81].

LEMMA. *Let $p \in \Omega$. For each $\varepsilon > 0$ there is a triplet (n, k, β) such that*
(i) $E_{n,k,\beta} \subseteq \Omega$,
(ii) $\text{diameter}(H_{n,k,\beta}) < \varepsilon$,
(iii) $\text{distance}(p, p_{n,k,\beta}) < \varepsilon$.

Now suppose that δ is a closed $*$ derivation in $C(\Omega)$ and that $p \in \operatorname{int} WP(\delta)$. If U is an open neighborhood of p in $\operatorname{int} WP(\delta)$, then by the lemma there is a triplet (n, k, β) such that $E_{n,k,\beta} \subseteq \Omega$ and $H_{n,k,\beta} \cap \Omega \subseteq U$. Let $H = H_{n,k,\beta} \cap \Omega$. H is a restriction set for δ, and δ_H is well behaved (1.6). Since $\mathfrak{D}(\delta)$ is a Silov algebra and H is open and closed, the characteristic function $\mathbf{1}_H$ of H is an element of $\mathfrak{D}(\delta)$. It follows from this that δ_H is also closed.

Let π denote the second coordinate projection on H; $\pi(H)$ is totally disconnected and therefore $C(\pi(H))$ is the uniform closure of the subalgebra generated by its projections. If $e \in C(\pi(H))$ is a projection, then $\pi^0(e)$ is a projection in $C(H)$. Since δ_H is a closed $*$ derivation, $\ker(\delta_H)$ contains the C^* algebra generated by these projections; that is

$$\ker(\delta_H) \supseteq \pi^0(C(\pi(H))) = \pi^0(C(I)).$$

It follows that each set $H^y = (I \times \{y\}) \cap H$ has the form $f^{-1}(0)$ for some real valued $f \in \ker(\delta_H)$. By 3.1, if $H^y \neq \varnothing$, then H^y is a restriction set for δ_H, and the induced derivation $(\delta_H)_{H^y} = \delta_{H^y}$ is well behaved. Taking $y = b_{n,\beta}$, we have $H^y = E_{n,k,\beta}$.

Let $f \in \mathfrak{D}(\delta)$. Since $\delta_{E_{n,k,\beta}}$ is well behaved, Lemma 2.2 implies

$$\delta(f)(p_{n,k,\beta}) = \delta_{E_{n,k,\beta}}(f|_{E_{n,k,\beta}})(p_{n,k,\beta}) = 0.$$

Thus

$$p_{n,k,\beta} \in Z = \{\omega \in \Omega \colon \delta(f)(\omega) = 0 \ \forall f \in \mathfrak{D}(\delta)\}.$$

This shows that Z intersects each neighborhood of the point p in $\operatorname{int} WP(\delta)$. Since Z is closed, $p \in Z$; that is $\operatorname{int} WP(\delta) \subseteq Z$. It follows that if δ is quasi-well behaved, then $Z = \Omega$, and $\delta = 0$.

It is easy to produce a nontrivial closed $*$ derivation in $C(\Omega)$. By the lemma $\bigcup\{E_{n,k,\beta}: E_{n,k,\beta} \subseteq \Omega\}$ is dense in Ω. Define $\mathfrak{A}$ to be the set of $f \in C(\Omega)$ such that $\partial f/\partial x$ exists on each $E_{n,k,\beta} \subseteq \Omega$ and $\partial f/\partial x$ extends to a continuous function on Ω. Note that $\mathfrak{A}$ contains $\{f|_\Omega: f \in C^1(I \times I)\}$ and therefore $\mathfrak{A}$ is dense in $C(\Omega)$. The partial derivative $\partial/\partial x$ defines a closed $*$ derivation in $C(\Omega)$ with domain $\mathfrak{A}$. This $*$ derivation is of course not qwb. But it does satisfy a weaker condition defined by Batty in **[B2]**; it is *pseudo-well behaved.*

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