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APPROXIMATING TOPOLOGICAL SURFACES IN 4-MANIFOLDS

BY

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ABSTRACT. Let M^2 be a compact, connected 2-manifold with $\partial M^2 \neq \varnothing$ and let h: $M^2 \to W^4$ be a topological embedding of M^2 into a 4-manifold. The main theorem of this paper asserts that if W^4 is a piecewise linear 4-manifold, then h can be arbitrarily closely approximated by locally flat PL embeddings. It is also shown that if the 4-dimensional annulus conjecture is correct and if W is a topological 4-manifold, then h can be arbitrarily closely approximated by locally flat embeddings. These results generalize the author's previous theorems about approximating disks in 4-space.

1. Introduction. Let M^2 denote a compact 2-manifold and W^4 a piecewise linear (PL) 4-manifold. The following question is studied in this paper: *If* h: $M^2 \to W^4$ *is a topological embedding, under what conditions can* h *be approximated by PL embeddings*?

The first answer to this question was given in **[10]** and **[11]** where it was proved that if M^2 is a disk, then h can be arbitrarily closely approximated by PL embeddings. Thus there are no local problems involved. However, Y. Matsumoto **[6**, §5] has recently used a construction of Giffen to show that if M^2 is any closed, orientable surface of positive genus, then there exist a topological embedding h: $M^2 \to \mathbf{R}^4$ and a positive number ε such that there is no PL embedding within ε of h. Hence the answer to the question above is not always positive, as is the answer to the analogous question for topological embeddings of PL manifolds in codimensions $\geqslant 3$ **[8]**. The main result of this paper extends the positive answer of **[10]** and **[11]** to any surface with nonempty boundary.

MAIN THEOREM. *Suppose* M^2 *is a compact, connected surface with* $\partial M \neq \varnothing$ *and that* h: $M^2 \to W^4$ *is a topological embedding of* M^2 *into a PL 4-manifold* W^4. *Then* h *can be arbitrarily closely approximated by PL embeddings.*

The PL approximation can easily be constructed to be locally flat and thus we have the following corollary (see **[5]** for example).

COROLLARY 1.1. *Suppose* M^2 *is a compact, connected surface with* $\partial M^2 \neq \varnothing$ *and that* h: $M^2 \to W^4$ *is a topological embedding into a differentiable 4-manifold* W^4. *Then* h *can be arbitrarily closely approximated by smooth embeddings.*

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In §4 of this paper, the problem of when a topological embedding of M^2 into a topological 4-manifold can be approximated by locally flat embeddings is investigated. Among the results of that section are the following two corollaries.

COROLLARY 1.2. *Suppose M^2 is a compact, connected surface with $\partial M^2 \neq \varnothing$ and W^4 is a stable topological 4-manifold. Then every topological embedding of M into W can be approximated arbitrarily closely by locally flat embeddings.*

COROLLARY 1.3. *Suppose the 4-dimensional annulus conjecture is correct. Then every topological embedding of a compact, connected surface with nonempty boundary into a topological 4-manifold can be approximated by locally flat embeddings.*

REMARK. Let $T = S^1 \times S^1$ and let $h: T \to \mathbf{R}^4$ be the Giffen-Matsumoto embedding mentioned previously which cannot be PL-approximated. Remove a small disk D from T to form $T' = \overline{T - D}$. Then $h|T'$ can be PL-approximated by the Main Theorem; say $g: T' \to \mathbf{R}^4$ is such an approximation. The simple closed curve $g(\partial T')$ does not bound a small PL disk in $\mathbf{R}^4 - g(\text{int } T')$ by Matsumoto's theorem. However, it is easy to see that $g(\partial T')$ is null-homotopic in $\mathbf{R}^4 - g(\text{int } T')$. Thus the approximation results described earlier are related to the failure of the 3-dimensional Dehn Lemma in dimension 4 (see [7]).

There are many unresolved problems related to these results. For example, it is not known whether the Main Theorem is true in the case in which M^2 is a 2-sphere or a nonorientable closed surface. It also seems reasonable to ask whether a topological embedding of a 2-complex into a PL 4-manifold can be PL-approximated if the 2-complex collapses to a 1-dimensional spine.

If M^2 is a compact, connected surface with $\partial M^2 \neq \varnothing$, then M^2 has a handle decomposition with one 0-handle and no 2-handles. The Main Theorem is proved by approximating h restricted to the 0-handle first and then extending that approximation to the 1-handles. Thus the approximation is constructed by approximating a disk at a time and there is therefore a large amount of overlap between the present paper and **[11]**. It is not possible, however, to extend to the 1-handles by a direct application of **[11]** and so every attempt has been made to make this paper as self-contained as possible and to incorporate various simplifications which have been made in the proofs in **[11]**.

The author wishes to thank Y. Matsumoto for many helpful conversations regarding the results in this paper.

2. Preliminaries. Throughout this paper, M^2 will denote a compact 2-manifold (= surface) while ∂M and int M respectively denote the manifold boundary and the manifold interior of M. The 4-manifold W^4 has a metric denoted by d. If $X \subset W$ and $\varepsilon > 0$, $N_\varepsilon(X) = \{x \in W | d(x, X) < \varepsilon\}$. A map $h: M \to W$ is an *embedding* if h is a homeomorphism onto $h(M)$. We say that $g: M \to W$ *ε-approximates h* if $d(g(x), h(x)) < \varepsilon$ for every $x \in M$. We will tacitly assume that $h(M) \subset$ int W since h can always be approximated by an embedding with that property. If $A \subset W$, then $\bar{A}$ denotes the closure of A in W. As usual, $\mathbf{R}^n$ denotes Euclidean n-space, S^n the standard n-sphere, B^n the unit n-ball, and $I = [0, 1]$.

The following proposition is basic to the proof of the Main Theorem. The proposition is a radial engulfing theorem and could be proved by standard radial engulfing techniques (e.g. the techniques of [1]), but we prefer to include here a very simple proof which is adequate for the present situation.

PROPOSITION 2.1. *Let* W^4 *be a PL* 4*-manifold,* K^1 *a finite* 1*-complex, and* $f\colon K \times I \to \operatorname{int} W^4$ *a map such that* $f|K \times (\partial I)$ *is a PL embedding. Then for every* $\varepsilon > 0$ *there exists a PL isotopy* h_t *of* W^4 *such that*

(1) $h_0 = \mathrm{id}$,

(2) $h_1(f(x, 0)) = f(x, 1)$ *for all* $x \in K$*, and*

(3) *for each* $x \in W^4$ *either* $h_t(x) = x$ *for all* $t \in I$ *or there exist* $x_1, x_2 \in K$ *such that* $h_t(x) \in N_\varepsilon(f(\{x_1, x_2\} \times I))$ *for all* $t \in I$.

Furthermore, if there exist a closed set $C \subset W$ *and a neighborhood* U *of* C *such that* $f(\{x\} \times I) = f(x, 0)$ *whenever* $f(\{x\} \times I) \cap U \neq \varnothing$*, then* $h_t|C = \mathrm{id}$ *for all* $t \in I$.

COROLLARY 2.2. *Suppose that* $D\colon I \times I \to W^4$ *is a topological embedding into a PL* 4*-manifold and that* $0 < a \leqslant 1$*. For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* K *and* L *are disjoint, finite* 1*-polyhedra in* $N_\delta(D(I \times I))$*, then there exists a PL isotopy* h_t *of* W^4 *such that*

(i) $h_0 = \mathrm{id}$,

(ii) $h_t = \mathrm{id}$ *on* $N_\delta(D(I \times [a, 1])) \cup L$ *and outside of* $N_\varepsilon(D(I \times [0, a]))$,

(iii) $h_1(K) \subset N_\varepsilon(D(I \times [a, 1]))$*, and*

(iv) $h_t(N_\delta(D(\{x\} \times I))) \subset N_\varepsilon(D(\{x\} \times I))$ *for every* $x \in I$.

There exists a map $f\colon K \times I \to N_\varepsilon(D(I \times I))$ *such that* h_t *can be chosen to have support in an arbitrarily small neighborhood of* $f(K \times I)$.

PROOF OF PROPOSITION 2.1. First suppose that $C = \varnothing$. Shift f into general position so that the singular set of f consists of a finite set of points, each of which is in the interior of a disk of the form $\sigma \times I$ where σ is a 1-simplex of K. By adjusting f slightly, it can be arranged that no two points of the singular set lie on the same vertical segment $\{x\} \times I$. The points of the singular set come in pairs $((x_i, t_i), (y_i, s_i))$ such that $f(x_i, t_i) = f(y_i, s_i)$. By pushing $f(\{x_i\} \times I)$ over the end of $f(\{y_i\} \times I)$ for each i (as illustrated in Figure 1) we can construct a PL embedding $f'\colon K \times I \to W$.

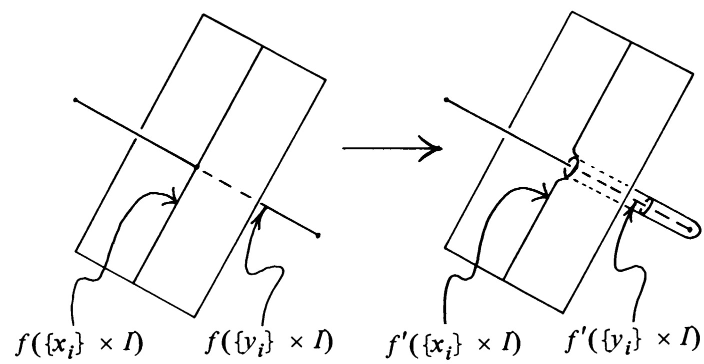

FIGURE 1

Working only in a small neighborhood of $f(\{y_i\} \times I)$, one constructs f' to have the following property: For each $x \in K$ there exists $\bar{x} \in K$ such that $f'(\{x\} \times I) \subset N_\varepsilon(f(\{x\} \times I) \cup f(\{\bar{x}\} \times I))$. Note that $f'|K \times \partial I = f|K \times \partial I$. Since f' is an embedding, the isotopy $h_t(x) = f'(x, t)$ of $f(K \times \{0\})$ can be extended to an ambient isotopy having the desired properties.

In case $C \neq \varnothing$, begin by subdividing K so that there exists a subcomplex L of K such that $L \times I$ is a neighborhood of $f^{-1}(C)$ and $f(L \times I) \subset U$. Put $f|(K - L) \times I$ into general position in the complement of C, keeping $f(L \times I)$ fixed. Then proceed as before to modify $f|(K - L) \times I$ and produce a map f': $K \times I \to \operatorname{int} W$ such that $f'|L \times I = f|L \times I$ and $f'|(K - L) \times I$ is an embedding. The rest of the proof is the same as in the preceding case. □

Proof of Corollary 2.2. We must construct the homotopy needed to apply Proposition 2.1.

Given $\varepsilon > 0$, use the uniform continuity of D and D^{-1} to find a number $\gamma > 0$ such that if $N_\gamma(D(\{x_1\} \times I)) \cap N_{2\gamma}(D(\{x_2\} \times I)) \neq \varnothing$ and $N_{2\gamma}(D(\{x_2\} \times I)) \cap N_{2\gamma}(D(\{x_3\} \times I)) \neq \varnothing$ for some $x_1, x_2, x_3 \in I$, then $N_{2\gamma}(D(\{x_2\} \times I)) \cup N_{2\gamma}(D(\{x_3\} \times I)) \subset N_\varepsilon(D(\{x_1\} \times I))$. Choose a number $b < a$ such that $D(I \times [b, 1]) \subset N_\varepsilon(D(I \times [a, 1]))$. Since $D(I \times I)$ is an ANR, there exist a neighborhood N of $D(I \times I)$ and a homotopy Γ_t: $N \to N_\varepsilon(D(I \times I))$ such that $\Gamma_0 = \mathrm{id}$, $\Gamma_1(N) = D(I \times I)$ and Γ_t moves no point more than γ. Let $\tilde{\Gamma}_t$ be the homotopy of N which is defined to be Γ_{2t} for $0 \leqslant t \leqslant 1/2$ and is the natural fiber-preserving homotopy of $D(I \times I)$ onto $D(I \times [b, 1])$ for $1/2 \leqslant t \leqslant 1$. Finally, choose δ, $0 < \delta \leqslant \gamma$, such that

$$\tilde{\Gamma}_t(N_{2\delta}(D(I \times [0, b]))) \cap N_\delta(D(I \times [a, 1])) = \varnothing, \tag{2.1}$$

and

$$\tilde{\Gamma}_t(N_\delta(D(I \times [b, a]))) \subset N_\varepsilon(D(I \times [a, 1])) \tag{2.2}$$

for all $t \in I$.

Now suppose that K and L are as in the statement of the corollary. Let u: $N_\delta(D(I \times I)) \to [0, 1]$ be a Urysohn function such that

$$u(N_\delta(D(I \times [0, b]))) = 1, \tag{2.3}$$

and

$$u(N_\delta(D(I \times I)) - N_{2\delta}(D(I \times [0, b]))) = 0. \tag{2.4}$$

Define f: $K \times I \to N_\varepsilon(D(I \times [0, a]))$ by $f(x, t) = \tilde{\Gamma}_{t \cdot u(x)}(x)$.

By general position we may assume that $f(K \times I) \cap L = \varnothing$. Also $f_0(x) = \tilde{\Gamma}_0(x) = \Gamma_0(x) = x$ for all $x \in K$. If $x \in K \cap N_\delta(D(I \times [0, b]))$, then $f_1(x) = \tilde{\Gamma}_1(x)$ by (2.3). If $x \in K \cap N_{2\delta}(D(I \times [0, b])) \cap N_\delta(D(I \times [b, a]))$, then $f_1(x) \in N_\varepsilon(D(I \times S[a, 1]))$ by (2.2). If $x \in K - N_{2\delta}(D(I \times [0, b]))$, then $f_1(x) = \tilde{\Gamma}_0(x) = x$ by (2.4). In every case, $f_1(x) \in N_\varepsilon(D(I \times [a, 1]))$. By (2.1) we have that if $f(\{x\} \times I) \cap N_\delta(D(I \times [a, 1]))$ for some $x \in K$, then $f(\{x\} \times I) = f(x, 0)$. Hence we can apply Proposition 2.1 with

$$C = L \cup N_\delta(D(I \times [a, 1])) \cup \{W - N_\varepsilon(D(I \times [0, a]))\}$$

and the ε of Proposition 2.1 equal to γ.

The isotopy h_t of W given by Proposition 2.1 obviously satisfies conclusions (i), (ii) and (iii) of Corollary 2.2. In addition, the choice of γ guarantees that h_t satisfies conclusion (iv) as well. $\square$

3. Proof of the Main Theorem. Let M^2 be a compact, connected surface with $\partial M^2 \neq \emptyset$ and let $h: M^2 \to W^4$ be a topological embedding of M^2 into a PL 4-manifold W^4. Choose a fixed handle decomposition of M^2, say

$$M^2 = H_0 \cup H_1 \cup \cdots \cup H_l$$

where each H_i is a disk, $H_i \cap H_j = \emptyset$ for $i, j \geqslant 1$ and $H_0 \cap H_i = \partial H_0 \cap \partial H_i \approx (\partial I) \times I$ for each $i \geqslant 1$. From **[10]** we know that $h|H_0$ can be ε-approximated by locally flat PL embeddings for every $\varepsilon > 0$. Thus it clearly suffices to prove the following lemma.

LEMMA 3.1. *Suppose an integer r, $1 \leqslant r \leqslant l$, and a neighborhood U of $H_r \cap H_0$ in H_0 are given. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $g: H_0 \to W$ is a locally flat PL embedding with $d(g, h|H_0) < \delta$ then there exists a locally flat PL embedding $\bar{g}: H_0 \cup H_r \to W$ such that $d(\bar{g}, h|H_0 \cup H_r) < \varepsilon$ and $\bar{g}|H_0 - U = g|H_0 - U$.*

REMARK. It is not necessary for the reader who is not familiar with **[11]** to appeal to **[11]** for a proof that Lemma 3.1 implies the Main Theorem. A proof of the result in **[11]** can be constructed from the techniques of this section.

It will be useful in the following proofs to have a fixed parametrization of $H_r \cup U$. The parametrization is indicated in Figure 2. It is just $(H_r \cup U, H_r) = (I \times I \cup [0, 1/3] \times [-1, 0] \cup [2/3, 1] \times [-1, 0], I \times I)$. Of course there is no loss of generality in assuming that U is of that form.

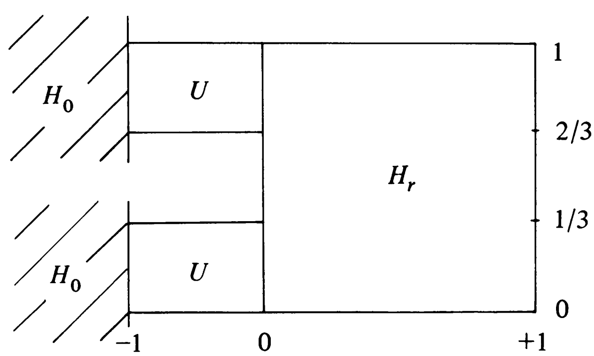

FIGURE 2

Notation. Suppose k and j are integers such that $0 \leqslant j \leqslant k$. We use the notation U_j^k to denote the subset

$$[0, 1/3] \times [-1 + j/(k+1), 0] \cup [2/3, 1] \times [-1 + j/(k+1), 0].$$

For each $x \in I$, the *fiber over* x is $F(x) = \{x\} \times [-1, 1]$ if $x \in [0, 1/3] \cup [2/3, 1]$ or $F(x) = \{x\} \times [0, 1]$ if $x \in (1/3, 2/3)$.

The proof of Lemma 3.1 will be based on the following lemma.

LEMMA 3.2. *Suppose* $0 = a_0 < a_1 < \cdots < a_k = a_{k+1} = 1$ *is a partition of* $[0, 1]$ *and* $0 \leqslant j \leqslant k$. *For every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that if* $g\colon H_0 \cup H_r \to W^4$ *is a locally flat PL embedding satisfying*

$$d\big(h|H_0 - U_j^k, g|H_0 - U_j^k\big) < \delta, \tag{3.1}$$

$$g\big(I \times [a_i, a_{i+1}]\big) \subset N_\delta\big(h\big(I \times [a_i, a_{i+1}]\big)\big) \quad \textit{for all } i > j, \tag{3.2}$$

$$g\big(U_j^k \cup I \times [0, a_i]\big) \subset N_\delta\big(h\big(U_j^k \cup I \times [0, a_i]\big)\big) \quad \textit{for every } i, \textit{ and} \tag{3.3}$$

$$g(F(x)) \subset N_\delta(h(F(x))) \quad \textit{for every } x \in I, \tag{3.4}$$

then g *can be replaced by a locally flat PL embedding* g' *such that* g' *satisfies* (3.1) – (3.4) *with* j *replaced by* $j - 1$ *and* δ *replaced by* ε. *Also* $g'|H_0 - U_{j-1}^k = g|H_0 - U_{j-1}^k$.

PROOF OF LEMMA 3.2. Let $\varepsilon > 0$ be given. It may be assumed that

$$N_\varepsilon\big(h\big(H_0 - U_{j-1}^k\big)\big) \cap N_\varepsilon\big(h\big(U_j^k \cup H_r\big)\big) = \varnothing.$$

Let

$$L = g\big(\{1/3\} \times [-1, 0] \cup [1/3, 2/3] \times \{0\} \cup \{2/3\} \times [-1, 0]\big).$$

By Corollary 2.2, there exists $\gamma_0 > 0$ such that if K is any finite 1-complex in $N_{\gamma_0}(h(U_j^k \cup I \times [0, a_{j-1}]))$, then there exists a PL isotopy h_t of W with support in $N_\varepsilon(h(U_j^k \cup I \times [0, a_{j-1}])) - N_{\gamma_0}(h(I \times [a_{j-1}, 1]))$ such that $h_0 = \mathrm{id}$, $h_t|L = \mathrm{id}$, $h_1(K) \subset N_\varepsilon(h(I \times [a_{j-1}, 1]))$ and $h_t(N_{\gamma_0}(h(F(x)))) \subset N_\varepsilon(h(F(x)))$ for all $x \in I$. Let V be a PL manifold neighborhood of $h(U_j^k \cup I \times [0, a_j])$ in $N_{\gamma_0}(h(U_j^k \cup I \times [0, a_j]))$ and choose $\gamma_1 > 0$ such that $N_{\gamma_1}(h(U_j^k \cup I \times [0, a_j])) \subset V$.

Let $b_{j-1} = a_{j-1}$ and let b_i, $j \leqslant i \leqslant k - 1$, be a number such that $b_i > a_i$ and $h(I \times [a_i, b_i]) \subset N_\varepsilon(h(I \times \{a_i\}))$. Find a number $\gamma_2 > 0$ so small that the sets $N_{\gamma_2}(h(I \times [b_{i-1}, a_i]))$, $j \leqslant i \leqslant k$, are pairwise disjoint and $N_{\gamma_2}(H(I \times [a_{i-1}, b_i])) \subset N_\varepsilon(h(I \times [a_{i-1}, a_i]))$ for each i. Set $\varepsilon' = \min\{\gamma_0/2, \gamma_1, \gamma_2\}$. For each i, $j \leqslant i \leqslant k$, apply Corollary 2.2 with $\varepsilon = \varepsilon'$, D defined by $D(s, t) = h(s, a_i(1 - t))$, $a = b_{i-1}$, and $L = \varnothing$ to produce a number $\delta_i > 0$. Define $\delta = \min\{\delta_i\}$.[2]

Suppose g is as in the statement of Lemma 3.2. Triangulate V with mesh less than δ. Let P be the union of all simplices of V in $N_\delta(h(I \times [a_{j-1}, a_j]))$ plus all the 1-simplices of V and let P_* be the dual skeleton of V. Then $\dim P_* = 2$ and V is equal to the join of P and P_*.

Our first objective is to produce a locally flat PL embedding $\hat{g}\colon H_0 \cup H_r \to W$ such that $\hat{g}$ satisfies (3.1)–(3.4) with δ replaced by ε' and such that $\hat{g}(I \times [a_{j-1}, a_j]) \cap P_* = \varnothing$. Put $g(I \times [a_{j-1}, a_j])$ into general position with respect to P_*. Then $g(I \times [a_{j-1}, a_j]) \cap P_*$ consists of a finite number of points. Let Σ_j denote the shadow of $g^{-1}(g(I \times [a_{j-1}, a_j]) \cap P_*)$ down to the a_{j-1} level of $I \times I$. Let $f_j\colon \Sigma_j \times I \to N_{\varepsilon'}(h(I \times [0, a_j]))$ be the map promised by the application of Corollary 2.2 associated with δ_j. Now consider $f_j(\Sigma_j \times I) \cap g(I \times [a_j, a_{j+1}])$. That set is another

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[2]Notice that the orientation of D is different from that of h and therefore the isotopies associated with the δ_i's and that associated with γ_0 will push in opposite directions.

finite collection of points. Let Σ_{j+1} denote the shadow of

$$g^{-1}(f_j(\Sigma_j \times I) \cap g(I \times [a_j, a_{j+1}]))$$

down to the a_j level of $I \times I$. Let f_{j+1}: $\Sigma_{j+1} \times I \to N_{\varepsilon'}(h(I \times [0, a_{j+1}]))$ be the map promised by the application of Corollary 2.2 associated with δ_{j+1}. Continue in this way and identify a Σ_i and an f_i for each $i = j, \dots, k$.

Now begin with $f_k(\Sigma_k \times I)$. By Corollary 2.2 there exists an isotopy which pushes $g(\Sigma_k)$ along $f_k(\Sigma_k \times I)$ into $N_{\varepsilon'}(h(I \times [0, b_{k-1}]))$. Define g_k to be g followed by that push. By reparametrizing the image of g_k, it can be arranged that $g_k(I \times [a_{k-1}, a_k]) \cap f_{k-1}(\Sigma_{k-1} \times I) = \varnothing$. This reparametrization is accomplished by shoving $I \times \{a_i\}$ out over Σ_k as illustrated in Figure 3. Since Σ_k was first pushed into $N_{\varepsilon'}(h(I \times [0, b_{k-1}]))$, that can be done without destroying property (3.3). Next use $f_{k-1}(\Sigma_{k-1} \times I)$ to push $g_k(\Sigma_{k-1})$ into $N_{\varepsilon'}(h(I \times [0, b_{k-2}]))$. Define g_{k-1} to be g_k following by that push. The first push has pushed $g_k(I \times [a_{k-1}, a_k])$ off $f_{k-1}(\Sigma_{k-1} \times I)$, so the second push can be done without destroying property (3.2). By reparametrizing again, it can be arranged that $g_{k-1}(I \times [a_{k-2}, a_{k-1}4.]) \cap f_{k-2}(\Sigma_{k-2} \times I) = \varnothing$. Continue this process back to the jth level. Now g_j has the property that $g_j(\Sigma_j) \subset N_{\varepsilon'}(h(I \times [0, a_{j-1}]))$, so a final reparametrization produces the desired $\hat{g}$.

By Corollary 2.2 and the choice of ε', there exists an isotopy h_t of W with support in $N_\varepsilon(h(U_j^k \cup I \times [0, a_{j-1}])) - N_{\gamma_0}(h(I \times [a_{j-1}, 1]))$ such that $h_0 = \text{id}$, $h_t|L = \text{id}$, $h_t(N_{\varepsilon'}(h(F(x)))) \subset N_\varepsilon(H(F(x)))$, and $h_1(P) \subset N_\varepsilon(h(I \times [a_{j-1}, 1]))$. Since $\hat{g}(I \times [a_{j-1}, a_j]) \cap P_* = \varnothing$, we can push $\hat{g}(I \times [a_{j-1}, a_j])$ across the join structure between P and P_* until $h_1(\hat{g}(I \times [a_{j-1}, a_j])) \subset N_\varepsilon(h(I \times [a_{j-1}, a_j]))$. Let $g' = h_1 \circ \hat{g}$. Note that g' satisfies (3.1), (3.2) and (3.4) with δ replaced by ε and j replaced by $j - 1$. Since $h_1 = \text{id}$ on $L \cup N_\varepsilon(h(H_0 - U_{j-1}^k))$, one last reparametrization produces g' satisfying (3.3) as well. □

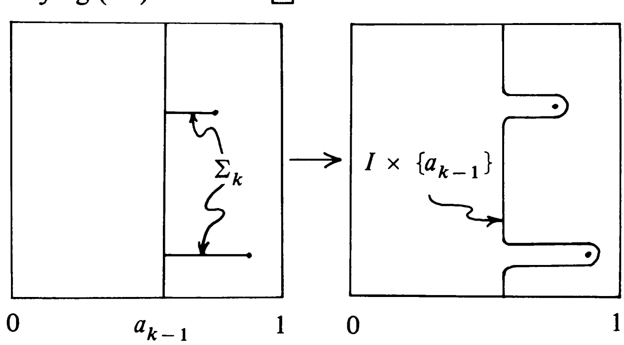

FIGURE 3

PROOF OF LEMMA 3.1. Let $\varepsilon > 0$ be given. It is possible to choose a number $\varepsilon_0 > 0$ and a partition $0 = a_0 < a_1 < \cdots < a_k = a_k = 1$ of $[0, 1]$ such that if g: $H_0 \cup H_r \to W$ is any map satisfying conditions (3.1)–(3.4) of Lemma 3.2 with $\delta = \varepsilon_0$ and $j = 0$, then $d(g, h|H_0 \cup H_r) < \varepsilon$. First choose U to be a very small neighborhood of $H_0 \cup H_r$ in H_0 so that (3.3) for the case $i = j = 0$ plus (3.4) imply that $d(g|U, h|U) < \varepsilon$. Then choose $\varepsilon_0 > 0$ and the partition so that (3.2) and (3.4) imply that $d(g|H_r, h|H_r) < \varepsilon$. As long as $\varepsilon_0 \leqslant \varepsilon$, (3.1) implies that $d(g|H_0 - U, h|H_0 - U) < \varepsilon$.

Inductively apply Lemma 3.2 to find positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ such that if g satisfies (3.1)–(3.4) with $\delta = \varepsilon_p$ and $j = p$, then there exists a g' satisfying (3.1)–(3.4) with $\delta = \varepsilon_{p-1}$ and $j = p - 1$. Assume ε_k is so small that $N_{\varepsilon_k}(h(H_r)) \cap N_{\varepsilon_k}(h(H_0 - U_k^k)) = \varnothing$. Use Corollary 2.2 (with $D = h|H_r$ and $a = 1$) to choose a number δ corresponding to $\varepsilon = \varepsilon_k$.

Now suppose $g: H_0 \to W$ is a locally flat PL embedding with $d(g, h|H_0) < \delta$. Since $I \times \{0\}$ is 1 dimensional, it is possible to extend g to a PL embedding of $H_0 \cup I \times \{0\}$. Then g can be extended to $\hat{g}: H_0 \cup H_r \to W$ such that $\hat{g}$ satisfies (3.3) and (3.4). (Just make each $\hat{g}(\{x\} \times I)$ very short.) Let $L = g(\{1/3\} \times [-1, 0] \cup [1/3, 2/3] \times \{0\} \cup \{2/3\} \times [-1, 0])$ and let $K = \hat{g}(I \times \{1\})$. By Corollary 2.2 and the choice of δ there exists an isotopy h_t of W such that $h_1(K) \subset N_{\varepsilon_k}(h(I \times \{1\}))$. Let $\tilde{g} = h_1 \circ \hat{g}$. Note that $\tilde{g}$ satisfies (3.1), (3.2) and (3.4) with $j = k$ and $\delta = \varepsilon_k$. A simple reparametrization of $g(H_r \cup U_k^k)$ makes $\tilde{g}$ satisfy (3.3) as well (since $h_t|L \cup \hat{g}(H_0 - U_k^k) = \mathrm{id}$). Now the choices of $\varepsilon_k, \varepsilon_{k-1}, \ldots, \varepsilon_0$ guarantee that there exists a $\bar{g}$ as in the conclusion of Lemma 3.1. $\square$

4. Topological 4-manifolds. In this section we investigate conditions under which the Main Theorem can be applied to a surface topologically embedded in a topological 4-manifold. The technique is to try to find a neighborhood of the embedded surface which is triangulable. By a 1-complex in a topological 4-manifold W we mean a subset of W which is homeomorphic with a compact 1-dimensional polyhedron.

THEOREM 4.1. *Suppose W^4 is a topological 4-manifold such that every 1-complex in* int W^4 *has a neighborhood which is triangulable as a PL manifold. Then every topological embedding of a compact, connected surface with nonempty boundary into W can be approximated by locally flat embeddings.*

A *pseudoisotopy* of W is a homotopy $h_t: W \to W$, $0 \leqslant t \leqslant 1$, such that $h_0 = \mathrm{id}$ and h_t is a homeomorphism for every $t < 1$.

LEMMA 4.2. *Let W^4 be a topological 4-manifold, M^2 a compact, connected surface with $\partial M^2 \neq \varnothing$ and let $g: M^2 \to \operatorname{int} W^4$ be a topological embedding. Then for every neighborhood U of $g(M^2)$ there exists a pseudoisotopy h_t of W with support in U such that $h_1(g(M^2))$ is a 1-complex and $h_1|W - g(M^2)$ is a homeomorphism onto $W - h_1(g(M^2))$.*

PROOF. It is enough to show that every 1-complex in $g(M^2)$ can be approximated by a tame 1-complex in $g(M^2)$ because then the pseudoisotopy h_t can be constructed exactly as in the proof of [**4**, Theorem 11].

Let K^1 be a finite 1-complex and let $f: K^1 \to g(M^2)$ be a topological embedding. Triangulate K^1 with such small mesh that the star of each vertex has a neighborhood in W^4 which is homeomorphic with $\mathbf{R}^4$. Let v be a vertex of K^1 and let βK^1 denote the first barycentric subdivision of K^1. Then by [**9**, Theorem 2], $f|\mathrm{st}(v, \beta K^1)$ can be approximated by a tame embedding $f_v: \mathrm{st}(v, \beta K^1) \to g(M)$ such that

$f_v|\partial(\mathrm{st}(v, \beta K^1)) = f|\partial(\mathrm{st}(v, \beta K^1))$. Choose an approximation f_v for each vertex v in such a way that $\hat{f}: K^1 \to g(M)$ defined by $\hat{f}|\mathrm{st}(v, \beta K^1) = f_v$ is an embedding. Then $\hat{f}$ is tame by [3]. □

PROOF OF THEOREM 4.1. Let $g: M^2 \to W^4$ be a topological embedding. Without loss of generality we may assume that $g(M^2) \subset \mathrm{int}\ W^4$. By Lemma 4.2, there exists a pseudoisotopy h_t such that $h_1(g(M^2))$ is a 1-complex. Let U be a neighborhood of $h_1(g(M^2))$ such that U is a PL manifold. There exists a $t \in [0, 1]$ such that $h_t(g(M^2)) \subset U$. Then $h_t^{-1}(U)$ is a PL manifold neighborhood of $g(M)$ in W. The Main Theorem implies that g can be approximated by locally flat embeddings in $h_t^{-1}(U)$ and hence in W. □

Let U be an open subset of a 4-manifold W. An *immersion* of U into $\mathbf{R}^4$ is a map $h: U \to \mathbf{R}^4$ such that for each $x \in U$ there exists a neighborhood U_x such that $h|U_x$ is an embedding.

LEMMA 4.3. *Let W^4 be a topological 4-manifold. Every contractible 1-complex in* int W *has a neighborhood which can be immersed in* $\mathbf{R}^4$.

COROLLARY 4.4. *Every contractible 1-complex in a topological 4-manifold has a neighborhood which is triangulable as a PL manifold.*

PROOF OF LEMMA 4.3. (This proof was shown to the author by J. Cannon.) Let X be a contractible 1-complex in int W^4. Then X can be written as $X = X_1 \cup X_2 \cup \cdots \cup X_n$ where each $X_i \approx I$, X_i has a neighborhood homeomorphic to B^4 and $X_i \cap (X_{i-1} \cup X_{i-2} \cup \cdots \cup X_1)$ is an endpoint of X_i. The proof is by induction on n. If $n = 1$, the result is obvious. Let V be a neighborhood of $X_1 \cup \cdots \cup X_{n-1}$ for which there is an immersion $h: V \to \mathbf{R}^4$. There exists a neighborhood B of X_n such that $B \approx B^4$. Let C be a small locally flat ball neighborhood of $X_n \cap (X_1 \cup \cdots \cup X_{n-1})$ such that $C \subset V$, $h|C$ is an embedding and $\overline{B - C} \approx S^3 \times I$. Then $h|C$ can be extended to an embedding $g: B \to \mathbf{R}^4$. Let U_1 and U_2 be small neighborhoods of $X_1 \cup \cdots \cup X_{n-1}$ and X_n respectively such that if $U = U_1 \cup U_2$, then $\bar{h}: U \to \mathbf{R}^4$ defined by $\bar{h}|U_1 = h|U_1$ and $\bar{h}|U_2 = g|U_2$ is a well-defined immersion. □

An attempt to use the technique of proof of Lemma 4.3 to prove that every 1-complex in a 4-manifold has a neighborhood which is triangulable leads to difficulties because the 4-dimensional annulus conjecture is not known to be true.

ANNULUS CONJECTURE. *If $g: B^4 \to$ int B^4 is a locally flat topological embedding, then $\overline{B^4 - g(B^4)} \approx S^3 \times I$.*

THEOREM 4.5. *The 4-dimensional Annulus Conjecture is correct if and only if every 1-complex in a topological 4-manifold has a neighborhood which is triangulable as a PL 4-manifold.*

PROOF OF COROLLARY 1.3. Corollary 1.3 follows immediately from Theorems 4.1 and 4.5. □

Before proving Theorem 4.5, we review the definitions of stable homeomorphism and stable manifold from [2]. A homeomorphism $h: \mathbf{R}^4 \to \mathbf{R}^4$ is *stable* if h can be written as $h = h_1 \circ \cdots \circ h_n$ where each h_i is the identity on some open set. If U

and V are open sets in $\mathbf{R}^4$ and $f\colon U \to V$ is a homeomorphism, then f is *stable at* $x \in U$ provided there exist a neighborhood U_x of x and a stable homeomorphism h of $\mathbf{R}^4$ such that $h|U_x = f|U_x$; f is *stable* if f is stable at x for every $x \in U$ [**2**, p. 27]. A 4-manifold W^4 is *stable* if W^4 can be covered by open sets $\{U_i\}_{i \in I}$ such that for each $i \in I$ there exists a homeomorphism h_i from an open subset $h_i^{-1}(U_i)$ of $\mathbf{R}^4$ onto U_i and $h_j^{-1} \circ h_i$ is a stable homeomorphism from $h_i^{-1}(U_i \cap U_j)$ to $h_j^{-1}(U_i \cap U_j)$ for all $i, j \in I$ [**2**, p. 32].

PROOF OF THEOREM 4.5. First assume that every 1-complex in a topological 4-manifold has a neighborhood which is triangulable. Let W^4 be a closed, orientable 4-manifold and let X be a locally flat simple closed curve in W^4. By hypothesis, there exists a neighborhood U of X in W^4 such that U is a PL manifold. But U is a stable manifold [**2**, Theorem II.10.4] and so X has a trivial tubular neighborhood in U [**2**, Theorem III.3.6]. But then W^4 is stable by the converse of [**2**, Theorem III.3.6]. If every closed, orientable 4-manifold is stable, then the 4-dimensional annulus conjecture must be true [**2**, II, §18].

Next assume that the Annulus Conjecture is correct and let X^1 be a 1-complex in a topological 4-manifold W^4. We may as well assume that X^1 is connected and thus X^1 can be written as $X^1 = \hat{X} \cup \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_n$ where $\hat{X}$ is a contractible 1-complex, the σ_i's are pairwise disjoint, σ_i is homeomorphic to a 1-simplex, σ_i has a neighborhood homeomorphic to $\mathbf{R}^4$, and $\sigma_i \cap \hat{X} = \{a_i, b_i\} = \partial\sigma_i$. By Corollary 4.4, $\hat{X}$ has a neighborhood U which is a PL manifold.

Let A_1 and B_1 be disjoint 4-simplices in U which contain a_1 and b_1 in their respective interiors and let V be a neighborhood of σ_1 such that $V \approx \mathbf{R}^4$. By the Annulus Conjecture, the PL structure on $A_1 \cup B_1$ can be extended to V. By cutting down to smaller neighborhoods U' of $\hat{X}$ and V' of σ_1, we find a neighborhood $U' \cup V'$ of $\hat{X} \cup \sigma_1$ which is triangulable. Similarly there exist a triangulable neighborhood of $\hat{X} \cup \sigma_1 \cup \sigma_2$ and (by induction) a triangulable neighborhood of X. $\square$

THEOREM 4.6. *Let W^4 be a connected topological 4-manifold. Then W^4 is stable if and only if every 1-complex in W^4 has a neighborhood which can be immersed in S^4.*

PROOF OF COROLLARY 1.2. Corollary 1.2 follows from Theorems 4.6 and 4.1. $\square$

COROLLARY 4.7. *Every topological embedding of a compact, connected surface with nonempty boundary into a simply connected 4-manifold can be approximated by locally flat embeddings.*

PROOF OF COROLLARY 4.7. By [**2**, Theorem II, 10.3], every simply connected 4-manifold is stable. $\square$

PROOF OF THEOREM 4.6. First suppose that W^4 is a 4-manifold such that every 1-complex in W has a neighborhood which immerses in S^4. Let L be a loop in W. Then L can be homotoped to a locally flat embedded loop L_1 [**2**, Theorem III.3.1]. Fix a neighborhood U of L_1 and an immersion $h\colon U \to S^4$. Since L_1 is locally flat, we can move L_1 slightly so that h is an embedding on some neighborhood V of L_1. By [**2**, Theorem III.3.4], $h(L_1)$ has a trivial tubular neighborhood in W which implies that W is stable [**2**, Theorem III.3.3].

Now suppose that W is stable and let X be a 1-complex in W. As in the proof of Theorem 4.5, we may assume that X is connected and thus X can be written as $X = \hat{X} \cup \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_n$ where $\hat{X}$ is a contractible 1-complex, the σ_i's are pairwise disjoint, each σ_i is homeomorphic to a 1-simplex, σ_i has a neighborhood which is homeomorphic with $\mathbf{R}^4$ and $\sigma_i \cap \hat{X} = \{a_i, b_i\} = \partial\sigma_i$. By Lemma 4.3 there exist a neighborhood U of $\hat{X}$ and an immersion $h: U \to S^4$. We first show how to find a neighborhood of $\hat{X} \cup \sigma_1$ which immerses in S^4. Let V be a neighborhood of σ_1 such that $V \approx \mathbf{R}^4$. Choose an arc α in U from a_1 to b_1 such that h is an embedding on some neighborhood N of α. Let $f: (B^3 \times [0, 5], \{0\} \times [1, 4]) \to (N, \alpha)$ be a locally flat embedding such that $A = f(B^3 \times [0, 2]) \subset U$ and $B = f(B^3 \times [3, 5]) \subset U$. By [2, Theorem II.14.1], there exists a homeomorphism $g: W \to W$ such that g has support in a compact subset of V and $g \circ f(x, t) = f(x, t + 3)$ for all $(x, t) \in B^3 \times [0, 2]$. Fix a homeomorphism $k: S^3 \to \partial A$ and let S denote the one point compactification of V ($S \approx S^4$). By [2, Theorem I.3.5] and [2, Corollary, p. 8], there exists a homeomorphism $G: S^3 \times I \to \overline{S - A \cup B}$ such that $G(x, 0) = k(x)$ and $G(x, 1) = g \circ k(x)$ for every $x \in S^3$. Similarly there exists $G': S^3 \times I \to \overline{S^4 - h(A) \cup h(B)}$ such that $G'(x, 0) = h \circ k(x)$, $G'(x, 1) = h \circ g \circ k(x)$. Now $G' \circ G^{-1}|\overline{V - A \cup B}$ extends $h|A \cup B$ to V. Let U' be a neighborhood of $\hat{X}$ in U and let V' be a neighborhood of σ_1 in V such that $V' \cap U' \subset A \cup B$. Then we can define an immersion $\bar{h}: U' \cup V' \to S^4$ by $\bar{h}|U' = h|U'$ and $\bar{h}|V' = G' \circ G^{-1}|V'$.

The argument above can be applied to σ_2 to find a neighborhood of $\hat{X} \cup \sigma_1 \cup \sigma_2$ which immerses in S^4 and (by induction) a neighborhood of X which immerses in S^4. $\square$

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