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RANDOM ERGODIC SEQUENCES ON LCA GROUPS

BY

JAKOB I. REICH

ABSTRACT. Let $\{X(t, \omega)\}_{t\in\mathbf{R}^+}$ be a stochastic process on a locally compact abelian group G, which has independent stationary increments. We show that under mild restrictions on G and $\{X(t, \omega)\}$ the random families of probability measures

$$\mu_T(\cdot, \omega) = B_T^{-1}\int_0^T f(t)x_{(\cdot)}(X(t, \omega))\,dt \quad \text{for } T > 0,$$

where $f(t)$ is a function from $\mathbf{R}^+$ to $\mathbf{R}^+$ of polynomial growth and $B_T = \int_0^T f(t)\,dt$, converge weakly to Haar measure of the Bohr compactification of G. As a consequence we obtain mean and individual ergodic theorems and asymptotic occupancy times for these processes.

0. Summary. Let G be an LCA group of the form $\mathbf{R}^n \times Z^m \times \mathcal{K}$ where $\mathcal{K}$ is a closed subgroup of $\mathfrak{U}^\infty$, the countable product of the unit circle. Let $\{X(t, \omega)\}_{t\in\mathbf{R}^+}$ be a stochastic process on a probability space $(\Omega, \mathfrak{F}, P)$ with independent, stationary increments and state space G.

For $\gamma \in \hat{G}$ let $\phi_t(\gamma) = E(\langle X(t, \omega), \gamma\rangle)$ be the characteristic function of the $X(t)$'s. Call a function f: $\mathbf{R}^+ \to \mathbf{R}^+$ a weight function if it has polynomial growth, i.e., if there exist positive constants $\underline{C}$, $\overline{C}$ and a nonnegative p such that $\underline{C}t^p < f(t) < \overline{C}t^p$. In this paper we show that for every weight function f there exists a set $\Omega_f \subset \Omega$ with $P(\Omega_f) = 1$ such that for $\omega \in \Omega_f$,

$$\lim_{T\to\infty} B_T^{-1}\int_0^T f(t)\langle X(t, \omega), \gamma\rangle\,dt = 0 \tag{1}$$

for all $\gamma \in \hat{G} - \{0\}$, where $B_T = \int_0^T f(t)\,dt$.

If for a given weight function f we define the random families of probability measures on G as

$$\mu_T(dx, \omega) = B_T^{-1}\int_0^T f(t)\chi_{(dx)}(X(t, \omega))\,dt, \tag{2}$$

then (1) says that for $\omega \in \Omega_f$ the Fourier transforms $\hat{\mu}_T(\gamma, \omega)$ satisfy

$$\lim_{T\to\infty} |\hat{\mu}_T(\gamma, \omega)| = 0 \quad \text{for } \gamma \in \hat{G} - \{0\}. \tag{3}$$

As a consequence we obtain mean ergodic theorems for unitary representations of G and weighted occupancy times for $\{X(t, \omega)\}$.

1. Preliminaries. Let G be an LCA-group of the form $\mathbf{R}^n \times Z^m \times \mathcal{K}$ with dual $\hat{G} = \mathbf{R}^n \times \mathfrak{U}^m \times \hat{\mathcal{K}}$. Since $\mathcal{K}$ is a closed subgroup of $\mathfrak{U}^\infty$, $\hat{\mathcal{K}}$ is countable. Let $\overline{G}$ be the Bohr compactification of G and m Haar measure on $\overline{G}$. For details see [4].

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We say that a family $\{\mu_T\}$ of probability measures on G is ergodic if

$$\lim_{T\to\infty} \hat{\mu}_T(\gamma) = 0 \quad \text{for } \gamma \in \hat{G} - \{0\}.$$

If we consider μ_T as measures on $\overline{G}$ this is equivalent to saying that weak $\lim_{T\to\infty} \mu_T = m$.

As shown in **[2]** ergodic families of measures provide mean ergodic theorems for unitary representations of G on a Hilbert space.

A measurable subset I of G is called a p-set if there exists $p \in [0, 1]$ such that for every ergodic family (or sequence) $\{\mu_T\}$, $\lim_{T\to\infty} \mu_T(I) = p$. If $\overline{B}$ is a continuity set in $\overline{G}$, i.e., its boundary has measure zero, then, by the Paul Lévy continuity theorem, $B = \overline{B} \cap G$ is a p-set with $p = m(\overline{B})$.

Reich constructed in **[3]** large classes of p-sets; the simplest construction can be obtained as follows: let $\gamma \in \hat{G}$ be of infinite order and I an interval in $\mathfrak{U}$. Then $\{g \in \overline{G} | \langle g, \gamma \rangle \in I\}$ is a continuity set of measure $|I|$ and therefore $\{g \in G | \langle g, \gamma \rangle \in I\}$ is a p-set with $p = |I|$.

2. The main results. Let $X(t, \omega) = (X_1(t, \omega), \ldots, X_{n+m+1}(t, \omega))$, i.e., the jth coordinate X_j has state space $\mathbf{R}$, Z, $\mathcal{K}$ for $1 \leqslant j \leqslant n$, $n + 1 \leqslant j \leqslant n + m$, $j = n + m + 1$ respectively.

By a well-known argument, using stationarity and independence of the increments, we can show that

$$|\phi_t(\gamma)| = |\phi_1(\gamma)|^t. \tag{1}$$

THEOREM 1. *If $|\phi(\gamma)| < 1$ for $\gamma \in \hat{G} - \{0\}$ and $E|X_j(t, \omega)| = O(t)$ for $t \geqslant 0$ and $j = 1, 2, \ldots, n + m$, then for every weight function f of polynomial growth, there exists a set $\Omega_f \subset \Omega$ with $P(\Omega_f) = 1$ such that for $\omega \in \Omega_f$, $\lim_{T\to\infty} |\hat{\mu}_T(\gamma, \omega)| = 0$ for all $\gamma \in \hat{G} - \{0\}$.*

REMARK. Note that $|\phi_1(\gamma)| < 1$ for $\gamma \neq 0$ is merely a condition to ensure that $X(t, \omega)$ is not distributed on a proper closed subgroup of G.

3. Some lemmas. The first two lemmas are from **[3]**.

LEMMA 1. *Let l be a positive integer and $\delta_j = \pm 1$, $j = 1, 2, \ldots, 2l$, such that $\sum_{j=1}^{2l} \delta_j = 0$. Define $k_j = -\sum_{i=1}^{j} \delta_i$ for $j = 1, 2, \ldots, 2l - 1$. Then for indeterminates $x_1, \ldots, x_{2l}$,*

$$\sum_{j=1}^{2l} \delta_j x_j = \sum_{j=1}^{2l-1} k_j (x_{j+1} - x_j).$$

Furthermore, $|k_j| \leqslant l$ for all j and $k_{2j-1} \neq 0$ for $j = 1, \ldots, l$.

The proof is obvious.

LEMMA 2. *Let g be a continuous function from $\mathbf{R}^n \times \mathfrak{U}^m$ into the complex plane. Suppose K is a cube in $\mathbf{R}^n \times \mathfrak{U}^m$, i.e., $K = \prod_{j=1}^{n+m} I_j$ where the I_j's are intervals in $\mathbf{R}$,*

$\mathfrak{U}$, *respectively. Suppose* $\max_{j=1,\ldots,n+m} |\partial g(\alpha)/\partial\alpha_j| < C$ *for all* α; *then for any* α, $\beta \in K$,

$$|g(\alpha)| \leqslant |g(\beta)| + C \sum_{j=1}^{n+m} |I_j|.$$

PROOF. By induction on $n + m$, the case $n + m = 1$ follows from the mean value theorem applied to the real and imaginary part of f.

LEMMA 3. *Let* L *be a positive integer*, f *a weight function of polynomial growth*, $0 < r < 1$,

$$S = \left\{(t_1, \ldots, t_{2l}) \in [0, T]^{2l} | 0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{2l} \leqslant T\right\}$$

and dt^{2l} *Lebesgue measure on* $\mathbf{R}^{2l}$; *then*

$$B_T^{-2l} \int_S \prod_{j=1}^{2l} f(t_j) \prod_{j=1}^{l} r^{t_{2j}-t_{2j-1}}\, dt^{2l} \leqslant C|\ln(r)|^{-l} T^{-l},$$

where C *only depends on* f *and* l.

PROOF. From $\underline{C}t^p \leqslant f(t) \leqslant \overline{C}t^p$ we obtain

$$\underline{C}T^{p+1}/(p+1) \leqslant B_T \leqslant \overline{C}T^{p+1}/(p+1). \tag{1}$$

Now by induction on l, let $l = 1$ and $p > 0$. Then

$$\begin{aligned} \int_0^T \int_{t_1}^T f(t_1)f(t_2)r^{t_2-t_1}\, dt_2\, dt_1 &\leqslant \overline{C}^2 \int_0^T t_1^p \int_{t_1}^T t_2^p r^{t_2-t_1}\, dt_2\, dt_1 \\ &= \overline{C}^2 \int_0^T t_1^p \left[\left. \frac{t_2^p r^{t_2-t_1}}{\ln(r)} \right|_{t_1}^T - \frac{p}{\ln(r)} \int_{t_1}^T t_2^{p-1} r^{t_2-t_1}\, dt_2 \right] dt_1 \qquad (2) \\ &\leqslant \overline{C}^2 \int_0^T t_1^p \frac{t_1^p + T^p}{|\ln(r)|}\, dt_1 \leqslant 2\overline{C}^2 T^{2p+1} |\ln(r)|^{-1}. \end{aligned}$$

Now divide both sides by the lower bound in (1) to obtain the inequality.

For the case $p = 0$ we can compute the iterated integral directly.

Now assume true for l, to prove the inequality for $l + 1$. Write $\int_S \ldots dt^{2l}$ as an iterated integral, split off the two innermost integrals which are handled as for $l = 1$, then apply the induction hypothesis.

LEMMA 4.

$$E\left(\sup_{\gamma \in \hat{G}} \max_{j=1,\ldots,n+m} \left| \frac{\partial}{\partial \gamma_j} \hat{\mu}_T(\gamma, \omega) \right| \right) = O(T).$$

PROOF. By hypothesis there is some positive C such that

$$\max_{j=1,\ldots,n+m} E|X_j(t, \omega)| \leqslant C \cdot t. \tag{1}$$

For $\gamma \in \hat{G}$, $\gamma = (\gamma_1, \ldots, \gamma_{n+m}, \gamma_{n+m+1})$, hence

$$\langle X(t, \omega), \gamma \rangle = \prod_{j=1}^{n+m+1} \langle X_j(t, \omega), \gamma_j \rangle$$

and, therefore,

$$\frac{\partial}{\partial \gamma_j}\langle X(t, \omega), \gamma\rangle = iX_j(t, \omega)\langle X(t, \omega), \gamma\rangle \quad \text{for } j = 1, \ldots, n + m.$$

From the last equation it follows that

$$\left|\frac{\partial}{\partial \gamma_j} \hat{\mu}_T(\gamma, \omega)\right| = \left|B_T^{-1}\int_0^T f(t)\frac{\partial}{\partial \gamma_j}\langle X(t, \omega), \gamma\rangle\, dt\right| \leqslant CB_T^{-1}\int_0^T f(t)|X(t, \omega)|\, dt.$$

Taking expectations on both sides, using (1) and the fact that f has polynomial growth finishes the proof.

LEMMA 5. *Let l be a positive integer and $\gamma \in \hat{G}$ such that $k\gamma \neq 0$ for $1 \leqslant |k| \leqslant l$. Then*

$$E|\hat{\mu}_T(\gamma, \omega)|^{2l} \leqslant C \cdot \left|\ln\left(\max_{1 \leqslant |k| \leqslant l} |\phi_1(k\gamma)|\right)\right|^{-l} T^{-l}$$

where C is independent of T and γ.

PROOF.

$$\begin{aligned}
|\hat{\mu}_T(\gamma, \omega)|^{2l} &= \prod_{j=1}^{l} B_T^{-1}\int_0^T f(t_j)\langle X(t_j, \omega), \gamma\rangle\, dt_j \\
&\quad \times \prod_{j=l+1}^{2l} B_T^{-1}\int_0^T f(t_j)\langle \overline{X(t_j, \omega), \gamma}\rangle\, dt_j \\
&= B_T^{-2l}\int_{[0, T]^{2l}} \prod_{j=1}^{2l} f(t_j) \prod_{j=1}^{2l} \langle \delta_j X(t_j, \omega), \gamma\rangle\, dt^{2l} \\
&= B_T^{-2l}\int_{[0, T]^{2l}} \prod_{j=1}^{2l} f(t_j)\left\langle \sum_{j=1}^{2l} \delta_j X(t_j, \omega), \gamma\right\rangle dt^{2l}
\end{aligned}$$

where

$$\delta_j = \begin{cases} 1 & \text{for } j = 1, \ldots, l, \\ -1 & \text{for } j = l + 1, \ldots, 2l. \end{cases}$$

Let $\mathcal{P}_{2l}$ be the permutations of $\{1, 2, \ldots, 2l\}$ and for $\sigma \in \mathcal{P}_{2l}$ define

$$S_\sigma = \left\{(t_1, \ldots, t_{2l}) \in [0, T]^{2l} \,|\, t_{\sigma(1)} \leqslant t_{\sigma(2)} \leqslant \cdots \leqslant t_{\sigma(2l)}\right\}.$$

Then $\{S_\sigma\}_{\sigma \in \mathcal{P}_{2l}}$ is an up to measure zero disjoint partition of $[0, T]^{2l}$ and therefore

$$\begin{aligned}
E|\hat{\mu}_T(\gamma, \omega)|^{2l} &= \sum_{\sigma \in \mathcal{P}_{2l}} B_T^{-2l} E\int_{S_\sigma} \prod_{j=1}^{2l} f(t_j)\left\langle \sum_{j=1}^{2l} \delta_j X(t_j, \omega), \delta\right\rangle dt^{2l} \\
&= \sum_{\sigma \in \mathcal{P}_{2l}} B_T^{-2l} E\int_{S_\sigma} \prod_{j=1}^{2l} f(t_{\sigma(j)})\left\langle \sum_{j=1}^{2l} \delta_{\sigma(j)} X(t_{\sigma(j)}, \omega), \gamma\right\rangle dt^{2l}. \quad (1)
\end{aligned}$$

From the definition of the δ_j's and $\delta_{\sigma(j)}$'s it follows that they satisfy the hypothesis of Lemma 1; therefore for each σ we can find integers k_j, $j = 1, 2, \ldots, 2l - 1$, such that in the last equality

$$\left|B_T^{-2l}E\int_{S_\sigma}\cdots\, dt^{2l}\right|$$

$$\begin{aligned}
&= \left|B_T^{-2l}E\int_{S_\sigma}\sum_{j=1}^{2l} f(t_{\sigma(j)})\left\langle \sum_{j=1}^{2l-1} k_j\left[X(t_{\sigma(j+1)},\omega)-X(T_{\sigma(j)},\omega)\right],\gamma\right\rangle dt^{2l}\right| \\
&= \left|B_T^{-2l}\int_{S_\sigma}\prod_{j=1}^{2l} f(t_{\sigma(j)})E\prod_{j=1}^{2l-1}\left\langle k_j\left[X(t_{\sigma(j+1)},\omega)-X(t_{\sigma(j)},\omega)\right],\gamma\right\rangle dt^{2l}\right| \\
&= B_T^{2l}\left|\int_{S_\sigma}\prod_{j=1}^{2l} f(t_{\sigma(j)})E\prod_{j=1}^{2l-1}\left\langle X(t_{\sigma(j+1)},\omega)-X(t_{\sigma(j)},\omega),k_j\gamma\right\rangle dt^{2l}\right| \\
&\leqslant B_T^{-2l}\int_{S_\sigma}\prod_{j=1}^{2l} f(t_{\sigma(j)})\prod_{j=1}^{2l-1}\left|E\left\langle X(t_{\sigma(j+1)},\omega)-X(t_{\sigma(j)},\omega),k_j\gamma\right\rangle\right| dt^{2l} \\
&= B_T^{2l}\int_{S_\sigma}\prod_{j=1}^{2l} f(t_{\sigma(j)})\prod_{j=1}^{2l-1}\left|\phi_1(k_j\gamma)\right|^{t_{\sigma(j+1)}-t_{\sigma(j)}} dt^{2l} \\
&\leqslant B_T^{-2l}\int_{S_\sigma}\prod_{j=1}^{2l} f(t_{\sigma(j)})\prod_{j=1}^{l}\left|\phi_1(k_{2j-1}\gamma)\right|^{t_{\sigma(2j)}-t_{\sigma(2j-1)}} dt^{2l} \\
&\leqslant B_T^{-2l}\int_{S_\sigma}\prod_{j=1}^{2l} f(t_{\sigma(j)})\prod_{j=1}^{l}\left(\max_{1\leqslant|k|\leqslant l}\left|\phi_1(k\gamma)\right|\right)^{t_{\sigma(2j)}-t_{\sigma(2j-1)}} dt^{2l} \\
&\leqslant C\left|\ln\left(\max_{1\leqslant|k|\leqslant l}\left|\phi_1(k\gamma)\right|\right)\right|^{-l}\cdot T^{-l}.
\end{aligned}$$

The first inequality follows from the fact that on S_σ, $t_{\sigma(1)} \leqslant t_{\sigma(2)} \leqslant \cdots \leqslant t_{\sigma(2l)}$ and independent increments of $X(t, \omega)$. The second and third inequalities follow from Lemma 1 since $|k_j| \leqslant l$ for all j and $k_{2j-1} \neq 0$ for $j = 1, 2, \ldots, l$. For the last inequality apply Lemma 3.

To finish the proof combine (1) and (2) to conclude that

$$E\left|\hat{\mu}_T(\gamma,\omega)\right|^{2l} \leqslant (2l)!\,CT^{-l}\left|\ln\left(\max_{1\leqslant|k|\leqslant l}\left|\phi_1(k\gamma)\right|\right)\right|^{-l}.$$

4. Proof of Theorem 1. Let $K = \prod_{j=1}^{n+m} I_j \times \{\alpha\}$, where the I_j's are closed intervals in $\mathbf{R}$, $\mathfrak{U}$ for $1 \leqslant j \leqslant n$, $n+1 \leqslant j \leqslant n+m$, respectively, and $\alpha \in \hat{\mathfrak{K}}$; we will call a set of this form a cube.

Fix $l = 3(n+m)+4$ and suppose for $\gamma \in K$, $k\gamma \neq 0$ for $1 \leqslant |k| \leqslant l$, i.e., K contains no roots of unity of order $\leqslant l$.

Define

$$r = \max_{1\leqslant|k|\leqslant l}\ \sup_{\gamma\in K}\left|\phi_1(k\gamma)\right|.$$

Then

$$r < 1. \tag{1}$$

This follows from the assumption $|\phi_1(\gamma)| < 1$ for $\gamma \neq 0$ and the fact that $|\phi_1(\gamma)|$ is continuous and K is compact and contains no roots of unity of order $\leqslant l$.

For a positive integer N, divide K into $[N^{3/2}]^{n+m} = \bar{N}$ subcubes $\{K_j\}_{j=1}^{\bar{N}}$ of equal measure, which are disjoint up to measure zero ([ ] denotes the greatest integer part), i.e., divide each I_j into $[N^{3/2}]$ subintervals and take product sets. In each K_j fix a point γ_j and let

$$A_N = \left\{ \max_{j=1,\ldots,\bar{N}} |\hat{\mu}_N(\gamma_j, \omega)| < N^{-1/4} \right\}.$$

Then by Chebychev's inequality, Lemma 5 and (1),

$$\begin{aligned} P(A_N^c) &\leqslant \sum_{j=1}^{\bar{N}} N^{2l/4} E|\hat{\mu}_N(\gamma_j, \omega)|^{2l} \leqslant C\bar{N}N^{l/2}|\ln(r)|^{-l}N^{-l} \\ &\leqslant CN^{-l/2}N^{3/2(n+m)}|\ln(r)|^{-l} \leqslant CN^{-2} \cdot |\ln(r)|^{-l}. \end{aligned} \tag{2}$$

The constant C only depends on f and l by Lemma 5.

Let

$$B_N = \left\{ \max_{j=1,\ldots,n+m} \left| \frac{\partial}{\partial \gamma_j} \hat{\mu}_N(\gamma, \omega) \right| \leqslant N^{5/4} \right\}.$$

Then by Lemma 4 and Chebychev's inequality,

$$P(B_N^c) \leqslant \sum_{j=1}^{n+m} N^{-5/4}O(N) = O(N^{-1/4}). \tag{3}$$

Hence by (2) and (3),

$$\sum_{N=1}^{\infty} P\big((A_{N^8} \cap B_{N^8})^c\big) < \infty,$$

which by the Borel-Cantelli lemma implies that

$$P\{\omega|\omega \text{ is outside of at most finitely many of the } A_{N^8} \cap B_{N^8}\text{'s}\} = 1. \tag{4}$$

If $\omega \in A_{N^8} \cap B_{N^8}$, then for $\gamma \in K$ there is a subcube K_j such that $\gamma \in K_j$. Therefore by Lemma 2, Lemma 4 and the fact that to obtain the K_j's we divided each I_j into $[(N^8)^{3/2}]$ subintervals of equal length, we get

$$\begin{aligned} |\hat{\mu}_{N^8}(\gamma, \omega)| &\leqslant |\hat{\mu}_{N^8}(\gamma_j, \omega)| + \sum_{k=1}^{n+m} N^{10} \cdot |I_j| \cdot [N^{12}]^{-1} \\ &\leqslant N^{-2} + (n+m)\Big(\max_{j=1,\ldots,n+m} |I_j| \Big) 2N^{-2} = O(N^{-2}). \end{aligned}$$

Since this inequality does not depend on γ, we get for $\omega \in A_{N^8} \cap B_{N^8}$,

$$\sup_{\gamma \in K} |\hat{\mu}_{N^8}(\gamma, \omega)| \leqslant O(N^{-2}). \tag{5}$$

Therefore, by (4) and (5),

$$\lim_{N\to\infty} \sup_{\gamma \in K} |\hat{\mu}_{N^8}(\gamma, \omega)| = 0 \text{ with probability one.}$$

And since B_T grows geometrically with T by a well-known argument, we can conclude

$$\lim_{T\to\infty} \sup_{\gamma \in K} |\hat{\mu}_T(\gamma, \omega)| = 0 \text{ almost surely.}$$

From the structure of $\hat{G}$ we see that $\hat{G}$-{roots of unity of order $< l$} is a countable union of such cubes K and that there are at most countably many roots of unity of order $\leqslant l$. If γ is a root of unity of order $\leqslant l$ and $\gamma \neq 0$, then letting

$$A_N = \{\omega \mid |\hat{\mu}_N(\gamma, \omega)| < N^{-1/4}\},$$

it follows from Lemma 5 with $l = 1$ that

$$P(A_N^c) \leqslant N^{1/2} E|\hat{\mu}_N(\gamma, \omega)|^2 \leqslant CN^{-1/2}$$

and therefore $\Sigma_{N=1}^{\infty} P(A_{N^4}^c) < \infty$. Now by an argument as above using the Borel-Cantelli lemma,

$$\lim_{T\to\infty} |\hat{\mu}_T(\gamma, \omega)| = 0 \text{ almost surely.}$$

Taking the intersection of this countable collection of sets of probability one, gives us the desired result.

5. Some examples. Let $X_1(t, \omega), \ldots, X_n(t, \omega)$ be Brownian motions on $\mathbf{R}$ such that:

(i) the random variables $X_1(1, \omega), \ldots, X_n(1, \omega)$ are linearly independent, i.e., $P\{\Sigma_{j=1}^{n} r_j X_j(1, \omega) = 0\} = 1$ iff $r_1 = \cdots = r_n = 0$; and

(ii) for $0 \leqslant r \leqslant s \leqslant t$, $X_j(t, \omega) - X_j(s, \omega)$ is independent of $X_k(r, \omega)$ for all j, k.

Then the process $X(t, \omega) = (X_1(t, \omega), \ldots, X_n(t, \omega))$ on $\mathbf{R}^n$ has independent stationary increments by (ii) and the characteristic function satisfies the hypothesis of Theorem 1 by (i). In particular, (ii) is satisfied if the processes X_j are independent. Similarly, using Poisson processes, we can construct a process on Z^m, which satisfies the conditions of Theorem 1. Combining these processes we obtain a process on $\mathbf{R}^n \times Z^m$ with the desired properties.

6. Applications to unitary representations. Let $\{U_g\}_{g\in G}$ be a weakly continuous unitary representation of G on a Hilbert space $\mathcal{H}$. Denote by $P_{\mathfrak{I}}$ the orthogonal projection onto the closed subspace $\mathfrak{I}$ of invariant elements under $\{U_g\}$.

THEOREM 2. *Let $\{X(t, \omega)\}, f, \Omega_f$ be as in Theorem 1, and $\{U_g\}_{g\in G}$ any weakly continuous unitary representation of G on a Hilbert space. Then for $\omega \in \Omega_f$,*

$$\lim_{T\to\infty} \left\| B_T^{-1} \int_0^T f(t)(U_{X(t,\omega)}h)\, dt - P_{\mathfrak{I}}h \right\| = 0$$

for all $h \in \mathcal{H}$.

PROOF. Since

$$B_T^{-1} \int_0^T f(t)(U_{X(t,\omega)}h)\, dt = \int_G (U_g h)\mu_T(dg, \omega)$$

and $\hat{\mu}_T(\gamma, \omega) \to 0$ for $\gamma \in \hat{G} - \{0\}$, the result follows from a theorem in **[2]**.

THEOREM 3. *Let $\{X(t, \omega)\}, f, \Omega_f$ be as in Theorem 1. Let $\{U_g\}_{g\in G}$ be a weakly continuous representation on some L^2 space. Then there exists a dense set $\mathfrak{D} \subset L^2$ such that for $\omega \in \Omega_f$,*

$$\lim_{N\to\infty} B_{N^8}^{-1} \int_0^{N^8} f(t)\, U_{X(t,\omega)}h(y)\, dt = P_{\mathfrak{I}}h$$

for almost every y and all $h \in \mathfrak{D}$.

If, in addition, the U_g's are uniformly bounded on L^∞ and the set of eigenvalues does not have any limit points, then we can find a dense $\mathfrak{D} \subset L^2$ such that

$$\lim_{T\to\infty} B_T^{-1}\int_0^T f(t)\big(U_{X(t,\omega)}h(y)\big)\,dt = P_{\mathfrak{J}}h$$

for almost every y and all $h \in \mathfrak{D}$.

Remark. Note that the two statements of the theorem hold for all $\omega \in \Omega_f$, i.e., the set of probability one does not depend on the unitary representation nor the particular function selected from $\mathfrak{D}$.

Proof. Let $E(\cdot)$ denote the resolution of the identity for $\{U_g\}$ on $\hat{G}$. Let $h \in L^2$ and $\{\gamma_j\}$ be the nonzero eigenvalues such that $E(\gamma_j)h = h\gamma_j \neq 0$. Assume first

$$h = \sum_{j=1}^{\infty} h\gamma_j + P_{\mathfrak{J}}h. \tag{1}$$

Then for $\varepsilon > 0$ and N sufficiently large,

$$\tilde{h} = \sum_{j=1}^{N} h\gamma_j + P_{\mathfrak{J}}h \text{ is } \varepsilon\text{-closed to } h. \tag{2}$$

For $\tilde{h}$ we get for $\omega \in \Omega_f$,

$$\lim_{T\to\infty}\int_G U_g\tilde{h}\mu_T(dg,\omega) = \lim_{T\to\infty}\sum_{j=1}^{N}\hat{\mu}_T(\gamma_j,\omega)h\gamma_j + P_{\mathfrak{J}}h = P_{\mathfrak{J}}h$$

since the γ_j's are nonzero.

Assume now that $h \in L^2$ such that

$$E(\gamma)h = 0 \quad \text{for all } \gamma \in \hat{G}. \tag{3}$$

This implies the Borel measure $(E(d\gamma)h, h)$ is continuous on $\hat{G}$. Therefore, for $\varepsilon > 0$ by the σ-compactness of $\hat{G}$ we can find a compact cube $\tilde{K}$ such that

$$\|E(\tilde{K})h - h\|_2 < \varepsilon/2. \tag{4}$$

From the structure of $\hat{G}$ one sees that a compact cube K only can contain finitely many roots of unity of order $\leqslant l$. Deleting sufficiently small cubical open neighborhoods around each root of order $\leqslant l$ from $\tilde{K}$ gives us a compact set K such that

$$\begin{aligned} &\text{(i)} \quad \|E(K)h - E(\tilde{K})h\|_2 < \varepsilon/2;\\ &\text{(ii)} \quad K = \bigcup_{j=1}^{M} K_j; \end{aligned} \tag{5}$$

the K_j's are disjoint and each K_j is of the form $\prod_{j=1}^{n+m} I_j \times \{\alpha\}$ where the I_j's are intervals (not necessarily closed) and $\alpha \in \hat{\mathfrak{K}}$. Also note that the closure of K_j does not contain any roots of order $\leqslant l$.

Since $E(K)h = \sum_{j=1}^{M} E(K_j)h$, it is sufficient to prove pointwise convergence for each function $E(K_j)h$.

From (5) in the proof of Theorem 1 it follows that for $\omega \in \Omega_f$ and N sufficiently large

$$\sup_{\gamma\in K_j}|\hat{\mu}_{N^8}(\gamma,\omega)| \leqslant O(N^{-2}). \tag{6}$$

Therefore for $\lambda > 0$, letting

$$F_N = \left\{ y \mid \left| \int_G U_g[E(K_j)h](y)\mu_{N^8}(dg, \omega) \right| < \lambda \right\},$$

we obtain the estimate

$$\begin{aligned} |F_N^c| &\leqslant \lambda^{-2} \left\| \int_G U_g[E(K_j)h]\, \mu_{N^8}(dg, \omega) \right\|_2^2 \\ &= \lambda^{-2} \int_{K_j} |\hat{\mu}_{N^8}(\gamma, \omega)|^2 (E(d\gamma)h, h) \leqslant \lambda^{-2} N^{-4} \|h\|_2^2. \end{aligned} \tag{7}$$

The last inequality follows from (6). From (7) and the Borel-Cantelli lemma it follows that except for a set of measure zero all y's are at most in finitely many of the F_N^c's; since λ can be made arbitrarily small, we deduce pointwise convergence a.e. to 0 for $E(K_j)h$ and therefore also for $E(K)h$. Finally, each function in L^2 is a sum of two functions of the form given in (1) and (3).

For the second part, for $h \in L^2 \cap L^\infty$ and $\varepsilon > 0$ find first a compact cube $\tilde{K}$ such that

$$\|E(\tilde{K})h - h\|_2 < \varepsilon/2. \tag{8}$$

Then as before delete sufficiently small neighborhoods around all roots of order $\leqslant l$ and all eigenvalues in $\tilde{K}$ to obtain a compact set K such that

$$\left\| E(\tilde{K})h - \left(E(K)h + \sum_{\gamma \in \tilde{K}} E(\{\gamma\})h \right) \right\|_2 < \varepsilon/2. \tag{9}$$

From the assumption that the e-values have no limit points we conclude that there are only finitely many e-values in $\tilde{K}$ and therefore

$$\sum_{\gamma \in \tilde{K}} E(\{\gamma\})h \text{ is a finite sum.} \tag{10}$$

Let $\mathcal{O}$ be an open cover of K which has compact closure such that all roots of unity of order $\leqslant l$ and all e-values are in the interior of $\mathcal{O}^c$ and let σ be a finite measure on G such that

$$\begin{aligned} &\text{(i)} \quad 0 \leqslant \hat{\sigma}(\gamma) \leqslant 1, \qquad \gamma \in \hat{G}, \\ &\text{(ii)} \quad \hat{\sigma}(\gamma) = \begin{cases} 1 & \text{for } \gamma \in K, \\ 0 & \text{for } \gamma \in \mathcal{O}^c. \end{cases} \end{aligned} \tag{11}$$

We define

$$h^* = \int_G U_g h \sigma(dg).$$

From the assumption of uniform boundedness of $\{U_g\}$ on L^∞ it follows that

$$h^* \in L^\infty \cap L^2. \tag{12}$$

Finally, define

$$h_\varepsilon = h^* + \sum_{\gamma \in \tilde{K}} E(\{\gamma\})h. \tag{13}$$

From (8) and (9) conclude that h_ε is ε-closed to h, and from (10) we see that $\Sigma_{\gamma \in \tilde{K}} E(\{\gamma\})h$ converges pointwise.

For h^* we obtain

$$\left\| \int_G U_g h^* \mu_{N^8}(dg, \omega) \right\|_2^2 = \int_{\hat{G}} |\hat{\sigma}(\gamma)|^2 \, |\hat{\mu}_{N^8}(\gamma, \omega)|^2 (E(d\gamma)h, h)$$

$$\leqslant \sup_{\gamma \in \Theta} |\hat{\mu}_{N^8}(\gamma, \omega)|^2 \, \|h\|^2 < N^{-4} \|h\|^2 \tag{14}$$

for all $\omega \in \Omega_f$. The last inequality follows as in (6).

Now we argue as in (7) to obtain

$$\lim_{N \to \infty} B_{N^8}^{-1} \int_0^{N^8} f(t) U_{X(t, \omega)} h^* \, dt = 0 \quad \text{a.e.}$$

Then

$$\lim_{T \to \infty} B_T^{-1} \int_0^T f(t) U_{X(t, \omega)} h^* \, dt = 0 \quad \text{a.e.}$$

follows from the fact that $h^* \in L^\infty$, $\{U_g\}$ is uniformly bounded on L^∞ and the B_T's grow geometrically.

7. p-occupancy. Let $\{X(t, \omega)\}$ be a process as in Theorem 1; then for $\omega \in \Omega_f$, $\{\mu_T(dg, \omega)\}$ is an ergodic family of measures on G (as defined in §1). Hence for I_p a p-set,

$$\lim_{T \to \infty} \mu_T(I_p, \omega) = p \quad \text{for all } \omega \in \Omega_f; \tag{1}$$

in particular, if $\gamma \in \hat{G}$ of infinite order and I an interval in $\mathfrak{U}$,

$$\lim_{T \to \infty} \frac{1}{B_T} \int_0^T f(t) \chi_{\{g | \langle g, \gamma \rangle \in I\}}(X(t, \omega)) \, dt = |I| \tag{2}$$

for all $\omega \in \Omega_f$.

It should be noted that for $f \equiv 1$, (1) and (2) are the limit of the average amount of time the process spends in the given set up to time T; this case is a generalization of a result on random walks in [**1**].

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Department of Mathematics, Baruch College, City University of New York, New York, New York 10010