RANDOM ERGODIC SEQUENCES ON LCA GROUPS

BY

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Abstract. Let \( \{X(t, \omega)\}_{t \in \mathbb{R}^+} \) be a stochastic process on a locally compact abelian group \( G \), which has independent stationary increments. We show that under mild restrictions on \( G \) and \( \{X(t, \omega)\} \) the random families of probability measures
\[
\mu_T(\cdot, \omega) = B_T^{-1} \int_0^T f(t) X(t, \omega) \, dt \quad \text{for } T > 0,
\]
where \( f(t) \) is a function from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \) of polynomial growth and \( B_T = \int_0^T f(t) \, dt \), converge weakly to Haar measure of the Bohr compactification of \( G \). As a consequence we obtain mean and individual ergodic theorems and asymptotic occupancy times for these processes.

0. Summary. Let \( G \) be an LCA group of the form \( \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{K} \) where \( \mathbb{K} \) is a closed subgroup of \( \mathbb{U}^\infty \), the countable product of the unit circle. Let \( \{X(t, \omega)\}_{t \in \mathbb{R}^+} \) be a stochastic process on a probability space \( (\Omega, \mathcal{F}, P) \) with independent, stationary increments and state space \( G \).

For \( \gamma \in \hat{G} \) let \( \phi_\gamma(\gamma) = E(\langle X(t, \omega), \gamma \rangle) \) be the characteristic function of the \( X(t)'s \). Call a function \( f: \mathbb{R}^+ \to \mathbb{R}^+ \) a weight function if it has polynomial growth, i.e., if there exist positive constants \( C_1, C_2 \) and a nonnegative \( p \) such that \( C_1 t^p < f(t) < C_2 t^p \). In this paper we show that for every weight function \( f \) there exists a set \( \Omega_f \subset \Omega \) with \( P(\Omega_f) = 1 \) such that for \( \omega \in \Omega_f \),
\[
\lim_{T \to \infty} B_T^{-1} \int_0^T f(t) \langle X(t, \omega), \gamma \rangle \, dt = 0 \quad (1)
\]
for all \( \gamma \in \hat{G} - \{0\} \), where \( B_T = \int_0^T f(t) \, dt \).

If for a given weight function \( f \) we define the random families of probability measures on \( G \) as
\[
\mu_T(dx, \omega) = B_T^{-1} \int_0^T f(t) X(dx, \omega) \, dt, \quad (2)
\]
then (1) says that for \( \omega \in \Omega_f \) the Fourier transforms \( \hat{\mu}_T(\gamma, \omega) \) satisfy
\[
\lim_{T \to \infty} |\hat{\mu}_T(\gamma, \omega)| = 0 \quad \text{for } \gamma \in \hat{G} - \{0\}. \quad (3)
\]
As a consequence we obtain mean ergodic theorems for unitary representations of \( G \) and weighted occupancy times for \( \{X(t, \omega)\} \).

1. Preliminaries. Let \( G \) be an LCA-group of the form \( \mathbb{R}^n \times \mathbb{Z}^m \times \mathbb{K} \) with dual \( \hat{G} = \mathbb{R}^n \times \mathbb{U}^m \times \mathbb{K} \). Since \( \mathbb{K} \) is a closed subgroup of \( \mathbb{U}^\infty \), \( \hat{\mathbb{K}} \) is countable. Let \( \hat{G} \) be the Bohr compactification of \( G \) and \( m \) Haar measure on \( G \). For details see [4].

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We say that a family \( \{ \mu_T \} \) of probability measures on \( G \) is ergodic if

\[
\lim_{T \to \infty} \mu_T(\gamma) = 0 \quad \text{for} \quad \gamma \in \hat{G} \setminus \{0\}.
\]

If we consider \( \mu_T \) as measures on \( \hat{G} \) this is equivalent to saying that weak

\[
\lim_{T \to \infty} \mu_T = m.
\]

As shown in [2] ergodic families of measures provide mean ergodic theorems for
unitary representations of \( G \) on a Hilbert space.

A measurable subset \( I \) of \( G \) is called a \( p \)-set if there exists \( p \in [0, 1] \) such that for

every ergodic family (or sequence) \( \{ \mu_T \} \), \( \lim_{T \to \infty} \mu_T(I) = p \). If \( \mathcal{B} \) is a continuity set in
\( \hat{G} \), i.e., its boundary has measure zero, then, by the Paul Lévy continuity

theorem, \( B = \mathcal{B} \cap G \) is a \( p \)-set with \( p = m(\mathcal{B}) \).

Reich constructed in [3] large classes of \( p \)-sets; the simplest construction can be
obtained as follows: let \( \gamma \in \hat{G} \) be of infinite order and \( I \) an interval in \( \mathbb{R} \). Then

\( \{ g \in \hat{G} \mid \langle g, \gamma \rangle \in I \} \) is a continuity set of measure \( |I| \) and therefore \( \{ g \in \hat{G} \mid \langle g, \gamma \rangle \in I \} \) is a \( p \)-set with \( p = |I| \).

2. The main results. Let \( X(t, \omega) = (X_1(t, \omega), \ldots, X_{n+m+1}(t, \omega)) \), i.e., the \( j \)th coordinate \( X_j \) has state space \( \mathbb{R} \), \( Z \), \( \mathbb{N} \) for \( 1 < j < n, n + 1 < j < n + m, j = n + m + 1 \) respectively.

By a well-known argument, using stationarity and independence of the increments, we can show that

\[
|\phi_1(\gamma)| = |\phi_1(\gamma)|'.
\]

Theorem 1. If \( |\phi(\gamma)| < 1 \) for \( \gamma \in \hat{G} \setminus \{0\} \) and \( E|X_j(t, \omega)| = O(t) \) for \( t > 0 \) and

\( j = 1, 2, \ldots, n + m \), then for every weight function \( f \) of polynomial growth, there

exists a set \( \Omega_f \subset \Omega \) with \( P(\Omega_f) = 1 \) such that for \( \omega \in \Omega_f \), \( \lim_{T \to \infty} |\hat{\mu}_T(\gamma, \omega)| = 0 \) for

all \( \gamma \in \hat{G} \setminus \{0\} \).

Remark. Note that \( |\phi_1(\gamma)| < 1 \) for \( \gamma \neq 0 \) is merely a condition to ensure that

\( X(t, \omega) \) is not distributed on a proper closed subgroup of \( G \).

3. Some lemmas. The first two lemmas are from [3].

Lemma 1. Let \( I \) be a positive integer and \( \delta_j = \pm 1, j = 1, 2, \ldots, 2l \), such that

\( \Sigma_{j=1}^{2l} \delta_j = 0 \). Define \( k_j = -\Sigma_{\delta_j = -1} \delta_j \) for \( j = 1, 2, \ldots, 2l - 1 \). Then for indeterminates

\( x_1, \ldots, x_{2l} \),

\[
\sum_{j=1}^{2l} \delta_j x_j = \sum_{j=1}^{2l-1} k_j(x_{j+1} - x_j).
\]

Furthermore, \( |k_j| < l \) for all \( j \) and \( k_{2j-1} \neq 0 \) for \( j = 1, \ldots, l \).

The proof is obvious.

Lemma 2. Let \( g \) be a continuous function from \( \mathbb{R}^n \times \mathcal{W}^m \) into the complex plane. Suppose \( K \) is a cube in \( \mathbb{R}^n \times \mathcal{W}^m \), i.e., \( K = \prod_{j=1}^{n+m} I_j \) where the \( I_j \)'s are intervals in \( \mathbb{R} \),
\( \mathcal{G} \), respectively. Suppose \( \max_{j=1, \ldots, n+m} |\partial g(\alpha)/\partial \alpha_j| < C \) for all \( \alpha \); then for any \( \alpha, \beta \in \mathcal{K} \),
\[
|g(\alpha)| < |g(\beta)| + C \sum_{j=1}^{n+m} |I_j|.
\]

**Proof.** By induction on \( n + m \), the case \( n + m = 1 \) follows from the mean value theorem applied to the real and imaginary part of \( g \).

**Lemma 3.** Let \( L \) be a positive integer, \( f \) a weight function of polynomial growth, \( 0 < r < 1 \),
\[
S = \left\{(t_1, \ldots, t_{2l}) \in [0, T]^{2l} | 0 < t_1 < t_2 < \cdots < t_{2l} < T \right\}
\]
and \( dt^{2l} \) Lebesgue measure on \( \mathbb{R}^{2l} \); then
\[
B_T^{-2l} \int_S \prod_{j=1}^{2l} f(t_j) \prod_{j=1}^{l} r^{t_{2j}-t_{2j-1}} dt^{2l} < C |\ln(r)|^{-l} T^{-l},
\]
where \( C \) only depends on \( f \) and \( l \).

**Proof.** From \( C^{\ell_p} < f(t) < \overline{C} \ell^p \) we obtain
\[
\mathcal{C} T^p + 1 / (p + 1) < B_T < \overline{C} T^{p+1} / (p + 1).
\]
Now by induction on \( l \), let \( l = 1 \) and \( p > 0 \). Then
\[
\int_0^T \int_{t_1}^T f(t_1) f(t_2) r^{t_2-t_1} dt_2 dt_1 < \overline{C}^2 \int_0^T \int_{t_1}^T t^p r^{t_2-t_1} dt_2 dt_1
\]
\[
= \overline{C}^2 \int_0^T t^p \left[ \frac{t^p r^{t_2-t_1}}{|\ln(r)|} \right]_{t_1}^T - \frac{p}{|\ln(r)|} \int_{t_1}^T t^p r^{t_2-t_1} dt_2 dt_1
\]
\[
< \overline{C}^2 \int_0^T t^p \left[ \frac{T^p}{|\ln(r)|} \right] dt_1 < 2 \overline{C}^2 T^{2p+1} |\ln(r)|^{-1}.
\]
Now divide both sides by the lower bound in (1) to obtain the inequality.

For the case \( p = 0 \) we can compute the iterated integral directly.

Now assume true for \( l \), to prove the inequality for \( l + 1 \). Write \( \int_S \ldots dt^{2l} \) as an iterated integral, split off the two innermost integrals which are handled as for \( l = 1 \), then apply the induction hypothesis.

**Lemma 4.**
\[
E \left( \sup_{\gamma \in \mathcal{G}} \max_{j=1, \ldots, n+m} \left| \frac{\partial}{\partial \gamma_j} \hat{\mu}_T(\gamma, \omega) \right| \right) = O(T).
\]

**Proof.** By hypothesis there is some positive \( C \) such that
\[
\max_{j=1, \ldots, n+m} E[X_j(t, \omega)] < C \cdot t.
\]
(1)

For \( \gamma \in \mathcal{G}, \gamma = (\gamma_1, \ldots, \gamma_{n+m}, \gamma_{n+m+1}) \), hence
\[
\langle X(t, \omega), \gamma \rangle = \prod_{j=1}^{n+m+1} \langle X_j(t, \omega), \gamma_j \rangle
\]
and, therefore,
\[ \frac{\partial}{\partial y_j} \langle X(t, \omega), \gamma \rangle = iX_j(t, \omega)\langle X(t, \omega), \gamma \rangle \quad \text{for } j = 1, \ldots, n + m. \]

From the last equation it follows that
\[ \left| \frac{\partial}{\partial y_j} \mu_T(\gamma, \omega) \right| = \left| B_T^{-1} \int_0^T f(t) \frac{\partial}{\partial y_j} \langle X(t, \omega), \gamma \rangle \, dt \right| < CB_T^{-1} \int_0^T f(t)|X(t, \omega)| \, dt. \]

Taking expectations on both sides, using (1) and the fact that \( f \) has polynomial growth finishes the proof.

**Lemma 5.** Let \( l \) be a positive integer and \( \gamma \in \hat{G} \) such that \( k \gamma \neq 0 \) for \( 1 < |k| < l \). Then
\[ E|\mu_T(\gamma, \omega)|^{2l} < C \cdot \left( \ln \left( \max_{1 < |k| < l} |\phi_1(k\gamma)| \right) \right)^{-l} T^{-l} \]
where \( C \) is independent of \( T \) and \( \gamma \).

**Proof.**

\[ |\mu_T(\gamma, \omega)|^{2l} = \prod_{j=1}^l B_T^{-1} \int_0^T f(t_j)\langle X(t_j, \omega), \gamma \rangle \, dt_j \]
\[ \times \prod_{j=l+1}^{2l} B_T^{-1} \int_0^T f(t_j)\overline{\langle X(t_j, \omega), \gamma \rangle} \, dt_j \]
\[ = B_T^{-2l} \int_{[0, T]^2l} \prod_{j=1}^{2l} f(t_j) \prod_{j=1}^{2l} \langle \delta_j X(t_j, \omega), \gamma \rangle \, dt^{2l} \]
\[ = B_T^{-2l} \int_{[0, T]^2l} \prod_{j=1}^{2l} f(t_j) \left( \sum_{j=1}^{2l} \delta_j X(t_j, \omega), \gamma \right) \, dt^{2l} \]

where
\[ \delta_j = \begin{cases} 1 & \text{for } j = 1, \ldots, l, \\ -1 & \text{for } j = l + 1, \ldots, 2l. \end{cases} \]

Let \( \mathcal{P}_{2l} \) be the permutations of \( \{1, 2, \ldots, 2l\} \) and for \( \sigma \in \mathcal{P}_{2l} \) define
\[ S_\sigma = \{(t_1, \ldots, t_{2l}) \in [0, T]^{2l} | t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(2l)} \}. \]

Then \( \{S_\sigma \}_{\sigma \in \mathcal{P}_{2l}} \) is an up to measure zero disjoint partition of \( [0, T]^{2l} \) and therefore
\[ E|\mu_T(\gamma, \omega)|^{2l} = \sum_{\sigma \in \mathcal{P}_{2l}} B_T^{-2l} E \int_{S_\sigma} \prod_{j=1}^{2l} f(t_j) \left( \sum_{j=1}^{2l} \delta_j X(t_j, \omega), \delta \right) \, dt^{2l} \]
\[ = \sum_{\sigma \in \mathcal{P}_{2l}} B_T^{-2l} E \int_{S_\sigma} \prod_{j=1}^{2l} f(t_{\sigma(j)}) \left( \sum_{j=1}^{2l} \delta_{\sigma(j)} X(t_{\sigma(j)}, \omega), \gamma \right) \, dt^{2l}. \quad (1) \]

From the definition of the \( \delta_j \)'s and \( \delta_{\sigma(j)} \)'s it follows that they satisfy the hypothesis of Lemma 1; therefore for each \( \sigma \) we can find integers \( k_j, j = 1, 2, \ldots, 2l - 1, \)
such that in the last equality

\[
\left| B_T^{-2l} E \int_{S_0} \cdots dt^2 \right|
\]

\[
= \left| B_T^{-2l} E \int_{S_0} \sum_{j=1}^{2l} f(t_{\sigma(j)}) \left( \sum_{j=1}^{2l-1} k_j \left[ X(t_{\sigma(j+1)}), \omega \right] - \left[ X(t_{\sigma(j)}, \omega) \right], \gamma \right) dt^2 \right|
\]

\[
= \left| B_T^{-2l} E \int_{S_0} \prod_{j=1}^{2l} f(t_{\sigma(j)}) E \prod_{j=1}^{2l-1} \left[ k_j \left[ X(t_{\sigma(j+1)}), \omega \right] - \left[ X(t_{\sigma(j)}, \omega) \right], \gamma \right] dt^2 \right|
\]

\[
= B_T^{2l} \left| \int_{S_0} \prod_{j=1}^{2l} f(t_{\sigma(j)}) E \prod_{j=1}^{2l-1} \left[ X(t_{\sigma(j+1)}), \omega \right] - \left[ X(t_{\sigma(j)}, \omega) \right], k_j \gamma \right| dt^2
\]

\[
< B_T^{-2l} \left| \int_{S_0} \prod_{j=1}^{2l} f(t_{\sigma(j)}) \prod_{j=1}^{2l-1} \left[ E \left( X(t_{\sigma(j+1)}), \omega \right) - X(t_{\sigma(j)}, \omega), k_j \gamma \right] dt^2 \right|
\]

\[
= B_T^{2l} \left| \int_{S_0} \prod_{j=1}^{2l} f(t_{\sigma(j)}) \prod_{j=1}^{2l-1} \left| \phi_1(k_j \gamma) \right| dt^2 \right|
\]

\[
< B_T^{-2l} \left| \int_{S_0} \prod_{j=1}^{2l} f(t_{\sigma(j)}) \prod_{j=1}^{2l-1} \left( \max_{1 < |k| < l} \left| \phi_1(k \gamma) \right| \right) dt^2 \right|
\]

\[
< C \left| \ln \left( \max_{1 < |k| < l} \left| \phi_1(k \gamma) \right| \right) \right|^{-l} \cdot T^{-l}.
\]

The first inequality follows from the fact that on \( S_o \), \( t_{\sigma(1)} < t_{\sigma(2)} < \cdots < t_{\sigma(2l)} \) and independent increments of \( X(t, \omega) \). The second and third inequalities follow from Lemma 1 since \( |k_j| < l \) for all \( j \) and \( k_{2j-1} \neq 0 \) for \( j = 1, 2, \ldots, l \). For the last inequality apply Lemma 3.

To finish the proof combine (1) and (2) to conclude that

\[
E \left| \hat{p}_T(\gamma, \omega) \right|^{2l} < (2l)! C T^{-l} \left| \ln \left( \max_{1 < |k| < l} \left| \phi_1(k \gamma) \right| \right) \right|^{-l}.
\]

4. Proof of Theorem 1. Let \( K = \bigotimes_{j=1}^{n+m} I_j \times \{ \alpha \} \), where the \( I_j \)'s are closed intervals in \( \mathbb{R} \), \( \mathfrak{U} \) for \( 1 < j < n, n + 1 < j < n + m \), respectively, and \( \alpha \in \mathfrak{C} \); we will call a set of this form a cube.

Fix \( l = 3(n + m) + 4 \) and suppose for \( \gamma \in K \), \( k \gamma \neq 0 \) for \( 1 < |k| < l \), i.e., \( K \) contains no roots of unity of order \( \leq l \).

Define

\[
r = \max_{1 < |k| < l} \sup_{\gamma \in K} \left| \phi_1(k \gamma) \right|.
\]

Then

\[
r < 1.
\]

This follows from the assumption \( |\phi_1(\gamma)| < 1 \) for \( \gamma \neq 0 \) and the fact that \( |\phi_1(\gamma)| \) is continuous and \( K \) is compact and contains no roots of unity of order \( < l \).
For a positive integer $N$, divide $K$ into $[N^{3/2}]_{n+m} = N$ subcubes $(K_j)_{j=1}^N$ of equal measure, which are disjoint up to measure zero ($[\cdot]$ denotes the greatest integer part), i.e., divide each $I_j$ into $[N^{3/2}]$ subintervals and take product sets. In each $K_j$ fix a point $\gamma_j$ and let

$$A_N = \left\{ \max_{j=1, \ldots, N} |\hat{\mu}_N(\gamma_j, \omega)| < N^{-1/4} \right\}.$$ 

Then by Chebychev's inequality, Lemma 5 and (1),

$$P(A_N^c) \leq \sum_{j=1}^N N^{2/4} E|\hat{\mu}_N(\gamma_j, \omega)|^{2/4} < C N^{1/2} |\ln(r)|^{-1} N^{-1}$$

$$< C N^{-1/2} N^{3/2(n+m)} |\ln(r)|^{-1} < C N^{-2} |\ln(r)|^{-1}.$$ (2)

The constant $C$ only depends on $f$ and $l$ by Lemma 5.

Let

$$B_N = \left\{ \max_{j=1, \ldots, n+m} \left| \frac{\partial}{\partial \gamma_j} \hat{\mu}_N(\gamma, \omega) \right| < N^{5/4} \right\}.$$ 

Then by Lemma 4 and Chebychev's inequality,

$$P(B_N^c) \leq \sum_{j=1}^{n+m} N^{-5/4} O(N) = O(N^{-1/4}).$$ (3)

Hence by (2) and (3),

$$\sum_{N=1}^\infty P((A_N^c \cap B_N^c)^c) < \infty,$$

which by the Borel-Cantelli lemma implies that

$$P\{\omega|\omega \text{ is outside of at most finitely many of the } A_N^c \cap B_N^c\}'s \} = 1.$$ (4)

If $\omega \in A_N^c \cap B_N^c$, then for $\gamma \in K$ there is a subcube $K_j$ such that $\gamma \in K_j$. Therefore by Lemma 2, Lemma 4 and the fact that to obtain the $K_j$'s we divided each $I_j$ into $[(N^8)^{3/2}]$ subintervals of equal length, we get

$$|\hat{\mu}_N(\gamma, \omega)| < |\hat{\mu}_N(\gamma_j, \omega)| + \sum_{k=1}^{n+m} N^{10} |I_j| [N^{12}]^{-1}$$

$$< N^{-2} + (n + m) \left( \max_{j=1, \ldots, n+m} |I_j| \right) 2N^{-2} = O(N^{-2}).$$

Since this inequality does not depend on $\gamma$, we get for $\omega \in A_N^c \cap B_N^c$,

$$\sup_{\gamma \in K} |\hat{\mu}_N(\gamma, \omega)| < O(N^{-2}).$$ (5)

Therefore, by (4) and (5),

$$\lim_{N \to \infty} \sup_{\gamma \in K} |\hat{\mu}_N(\gamma, \omega)| = 0 \text{ with probability one.}$$

And since $B_T$ grows geometrically with $T$ by a well-known argument, we can conclude

$$\lim_{T \to \infty} \sup_{\gamma \in K} |\hat{\mu}_T(\gamma, \omega)| = 0 \text{ almost surely.}$$
From the structure of \( \hat{G} \) we see that \( \hat{G} - \{ \text{roots of unity of order }< l \} \) is a countable union of such cubes \( K \) and that there are at most countably many roots of unity of order \( < l \). If \( \gamma \) is a root of unity of order \( < l \) and \( \gamma \neq 0 \), then letting
\[
A_N = \{ \omega | |\hat{\mu}_N(\gamma, \omega)| < N^{-1/4} \},
\]
it follows from Lemma 5 with \( l = 1 \) that
\[
P(A_N) < N^{1/2}E|\hat{\mu}_N(\gamma, \omega)|^2 \leq CN^{-1/2}
\]
and therefore \( \sum_{N=1}^{\infty} P(A_N) < \infty \). Now by an argument as above using the Borel-Cantelli lemma,
\[
\lim_{T \to \infty} |\hat{\mu}_T(\gamma, \omega)| = 0 \text{ almost surely.}
\]
Taking the intersection of this countable collection of sets of probability one, gives us the desired result.

5. Some examples. Let \( X_1(t, \omega), \ldots, X_n(t, \omega) \) be Brownian motions on \( \mathbb{R} \) such that:

(i) the random variables \( X_1(1, \omega), \ldots, X_n(1, \omega) \) are linearly independent,

i.e., \( P(\sum_{j=1}^n r_j X_j(1, \omega) = 0) = 1 \) iff \( r_1 = \cdots = r_n = 0 \); and

(ii) for \( 0 < r < s < t \), \( X_j(t, \omega) - X_j(s, \omega) \) is independent of \( X_k(r, \omega) \) for all \( j, k \).

Then the process \( X(t, \omega) = (X_1(t, \omega), \ldots, X_n(t, \omega)) \) on \( \mathbb{R}^n \) has independent stationary increments by (ii) and the characteristic function satisfies the hypothesis of Theorem 1 by (i). In particular, (ii) is satisfied if the processes \( X_j \) are independent. Similarly, using Poisson processes, we can construct a process on \( \mathbb{Z}^m \), which satisfies the conditions of Theorem 1. Combining these processes we obtain a process on \( \mathbb{R}^n \times \mathbb{Z}^m \) with the desired properties.

6. Applications to unitary representations. Let \( \{ U_g \}_{g \in G} \) be a weakly continuous unitary representation of \( G \) on a Hilbert space \( \mathcal{H} \). Denote by \( P_{\mathcal{H}} \) the orthogonal projection onto the closed subspace \( \mathcal{F} \) of invariant elements under \( \{ U_g \} \).

**Theorem 2.** Let \( \{ X(t, \omega) \}, f, \Omega_f \) be as in Theorem 1, and \( \{ U_g \}_{g \in G} \) any weakly continuous unitary representation of \( G \) on a Hilbert space. Then for \( \omega \in \Omega_f \),
\[
\lim_{T \to \infty} \left\| B_T^{-1} \int_0^T f(t)(U_{X(t, \omega)} h) \, dt - P_{\mathcal{H}} h \right\| = 0
\]
for all \( h \in \mathcal{H} \).

**Proof.** Since
\[
B_T^{-1} \int_0^T f(t)(U_{X(t, \omega)} h) \, dt = \int_G (U_g h) \mu_T(dg, \omega)
\]
and \( \hat{\mu}_{\mathcal{H}}(\gamma, \omega) \to 0 \) for \( \gamma \in \hat{G} - \{0\} \), the result follows from a theorem in [2].

**Theorem 3.** Let \( \{ X(t, \omega) \}, f, \Omega_f \) be as in Theorem 1. Let \( \{ U_g \}_{g \in G} \) be a weakly continuous representation on some \( L^2 \) space. Then there exists a dense set \( \mathcal{D} \) \( \subset L^2 \) such that for \( \omega \in \Omega_f \),
\[
\lim_{N \to \infty} B_{N^4}^{-1} \int_0^{N^8} f(t) U_{X(t, \omega)} h(y) \, dt = P_{\mathcal{H}} h
\]
for almost every \( y \) and all \( h \in \mathcal{D} \).
If, in addition, the $U_g$'s are uniformly bounded on $L^\infty$ and the set of eigenvalues does not have any limit points, then we can find a dense $D \subset L^2$ such that

$$\lim_{T \to \infty} B_T^{-1} \int_0^T f(t)(U_{X(t, \omega)}h(y)) \, dt = P_\varnothing h$$

for almost every $y$ and all $h \in D$.

**Remark.** Note that the two statements of the theorem hold for all $\omega \in \Omega_f$, i.e., the set of probability one does not depend on the unitary representation nor the particular function selected from $D$.

**Proof.** Let $E(\cdot)$ denote the resolution of the identity for $\{ U_g \}$ on $\hat{G}$. Let $h \in L^2$ and $\{ \gamma_j \}$ be the nonzero eigenvalues such that $E(\gamma_j)h = h\gamma_j \neq 0$. Assume first

$$h = \sum_{j=1}^\infty h\gamma_j + P_\varnothing h. \tag{1}$$

Then for $\varepsilon > 0$ and $N$ sufficiently large,

$$\tilde{h} = \sum_{j=1}^N h\gamma_j + P_\varnothing h \text{ is } \varepsilon\text{-closed to } h. \tag{2}$$

For $\tilde{h}$ we get for $\omega \in \Omega_f$,

$$\lim_{T \to \infty} \int_G U_g \tilde{h}\mu_T(dg, \omega) = \lim_{T \to \infty} \sum_{j=1}^N \mu_T(\gamma_j, \omega)h\gamma_j + P_\varnothing h = P_\varnothing h$$

since the $\gamma_j$'s are nonzero.

Assume now that $h \in L^2$ such that

$$E(\gamma)h = 0 \text{ for all } \gamma \in \hat{G}. \tag{3}$$

This implies the Borel measure $(E(d\gamma)h, h)$ is continuous on $\hat{G}$. Therefore, for $\varepsilon > 0$ by the $\sigma$-compactness of $\hat{G}$ we can find a compact cube $\tilde{K}$ such that

$$\|E(\tilde{K})h - h\|_2 < \varepsilon/2. \tag{4}$$

From the structure of $\hat{G}$ one sees that a compact cube $K$ only can contain finitely many roots of unity of order $< l$. Deleting sufficiently small cubical open neighborhoods around each root of order $< l$ from $\tilde{K}$ gives us a compact set $K$ such that

(i) $\|E(K)h - E(\tilde{K})h\|_2 < \varepsilon/2;$

(ii) $K = \bigcup_{j=1}^M K_j; \tag{5}$

the $K_j$'s are disjoint and each $K_j$ is of the form $\prod_{j=1}^{\alpha} I_j \times \{ \alpha \}$ where the $I_j$'s are intervals (not necessarily closed) and $\alpha \in \hat{K}$. Also note that the closure of $K_j$ does not contain any roots of order $< l$.

Since $E(K)h = \sum_{j=1}^M E(K_j)h$, it is sufficient to prove pointwise convergence for each function $E(K_j)h$.

From (5) in the proof of Theorem 1 it follows that for $\omega \in \Omega_f$ and $N$ sufficiently large

$$\sup_{\gamma \in K_j} |\tilde{\mu}(\gamma, \omega)| < O(N^{-2}). \tag{6}$$
Therefore for $\lambda > 0$, letting
\[
F_N = \left\{ y | \left| \int G U_y \left[ E(K_j)h \right](y) \mu_N(dy, \omega) \right| < \lambda \right\},
\]
we obtain the estimate
\[
|F_N| \leq \lambda^{-2} \left\| \int G U_y \left[ E(K_j)h \right] \mu_N(dy, \omega) \right\|_2^2 = \lambda^{-2} \int K |\tilde{\mu}_N(y, \omega)|^2 (E(dy)h, h) \leq \lambda^{-2} N^{-4} ||h||^2_2.
\]
(7)

The last inequality follows from (6). From (7) and the Borel-Cantelli lemma it follows that except for a set of measure zero all $y$'s are at most in finitely many of the $F_N$'s; since $\lambda$ can be made arbitrarily small, we deduce pointwise convergence a.e. to 0 for $E(K_j)h$ and therefore also for $E(K)h$. Finally, each function in $L^2$ is a sum of two functions of the form given in (1) and (3).

For the second part, for $h \in L^2 \cap L^\infty$ and $\varepsilon > 0$ find first a compact cube $\tilde{K}$ such that
\[
\|E(\tilde{K})h - h\|_2 < \varepsilon/2.
\]
(8)

Then as before delete sufficiently small neighborhoods around all roots of order $< l$ and all eigenvalues in $\tilde{K}$ to obtain a compact set $K$ such that
\[
\|E(K)h - h + \sum_{\gamma \in \hat{K}} E(\{\gamma\})h\|_2 < \varepsilon/2.
\]
(9)

From the assumption that the $\varepsilon$-values have no limit points we conclude that there are only finitely many $\varepsilon$-values in $\tilde{K}$ and therefore
\[
\sum_{\gamma \in \hat{K}} E(\{\gamma\})h \text{ is a finite sum.}
\]
(10)

Let $\Theta$ be an open cover of $K$ which has compact closure such that all roots of unity of order $< l$ and all $\varepsilon$-values are in the interior of $\Theta^c$ and let $\sigma$ be a finite measure on $G$ such that
\[
(i) \quad 0 < \sigma(\gamma) < 1, \quad \gamma \in \hat{G},
\]
\[
(ii) \quad \sigma(\gamma) = \begin{cases} 1 & \text{for } \gamma \in K, \\ 0 & \text{for } \gamma \in \Theta^c. \end{cases}
\]
(11)

We define
\[
h^* = \int G U_y h\sigma(dy).
\]

From the assumption of uniform boundedness of $\{U_y\}$ on $L^\infty$ it follows that
\[
h^* \in L^\infty \cap L^2.
\]
(12)

Finally, define
\[
h_\varepsilon = h^* + \sum_{\gamma \in \hat{K}} E(\{\gamma\})h.
\]
(13)
From (8) and (9) conclude that \( h \) is \( \varepsilon \)-closed to \( h \), and from (10) we see that 
\[ \sum_{\gamma \in \hat{G}} E(\{ \gamma \}) h \] converges pointwise.
For \( h^* \) we obtain
\[
\left\| \int_{\hat{G}} U_\gamma h^* \mu_{N^*}(dg, \omega) \right\|_2^2 = \int_{\hat{G}} |\hat{\sigma}(\gamma)|^2 |\hat{\mu}_{N^*}(\gamma, \omega)|^2 (E(\delta_{\gamma})h, h) \\
\leq \sup_{\gamma \in \theta} |\hat{\mu}_{N^*}(\gamma, \omega)|^2 \|h\|^2 < N^{-4} \|h\|^2
\] (14)
for all \( \omega \in \Omega_f \). The last inequality follows as in (6).

Now we argue as in (7) to obtain
\[
\lim_{N \to \infty} B_{N^*}^{-1} \int_0^{N^*} f(t) U_{X(t, \omega)} h^* dt = 0 \quad \text{a.e.}
\]
Then
\[
\lim_{T \to \infty} B_{T}^{-1} \int_0^{T} f(t) U_{X(t, \omega)} h^* dt = 0 \quad \text{a.e.}
\]
follows from the fact that \( h^* \in L^\infty \), \( \{ U_\gamma \} \) is uniformly bounded on \( L^\infty \) and the \( B_T \)'s grow geometrically.

7. \( p \)-occupancy. Let \( \{ X(t, \omega) \} \) be a process as in Theorem 1; then for \( \omega \in \Omega_f \), 
\( \{ \mu_T(dg, \omega) \} \) is an ergodic family of measures on \( G \) (as defined in §1). Hence for \( I_p \) 
a \( p \)-set,
\[
\lim_{T \to \infty} \mu_T(I_p, \omega) = p \quad \text{for all } \omega \in \Omega_f;
\] (1)
in particular, if \( \gamma \in \hat{G} \) of infinite order and \( I \) an interval in \( \mathbb{G} \),
\[
\lim_{T \to \infty} \frac{1}{B_T} \int_0^T f(t) \chi_{\{ \delta < \gamma \in I \}}(X(t, \omega)) dt = |I|
\] (2)
for all \( \omega \in \Omega_f \).

It should be noted that for \( f \equiv 1 \), (1) and (2) are the limit of the average amount of time the process spends in the given set up to time \( T \); this case is a generalization of a result on random walks in [1].

REFERENCES


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