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COMPACTNESS PROPERTIES OF AN OPERATOR WHICH IMPLY THAT IT IS AN INTEGRAL OPERATOR

BY

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ABSTRACT. In this paper we study necessary and (or) sufficient conditions on a given operator to be an integral operator. In particular we give another proof of a characterization of integral operators due to W. Schachermayer.

Introduction. This paper consists of two parts. In the first part we investigate sufficient conditions for operators, with domain $L_1(Y, \nu)$ or range in $L_\infty(X, \mu)$, to be integral operators. As a special case we get that weakly compact operators from $L_1(Y, \nu)$ into $L_1(X, \mu)$ are kernel operators and that they constitute an order ideal in $\mathfrak{L}(L_1, L_1)$. The corresponding results for weakly compact operators from $L_\infty(Y, \nu)$ into $L_\infty(X, \mu)$ also hold. In the second part we give an elementary proof of a generalization of the following result due to W. Schachermayer. If T is a linear operator from $L_2(Y, \nu)$ into $L_2(X, \mu)$, then T is an integral operator if and only if T maps order intervals into equimeasurable sets. We shall indicate the relation between this result and results of A. Grothendieck and of R. J. Nagel and U. Schlotterbeck.

1. Preliminaries. (Y, Σ, ν) and (X, Λ, μ) will always denote in this paper σ-finite measure spaces to which the Carathéodory extension procedure has been applied. The set of all realvalued μ-measurable functions will be denoted by $M(X, \mu)$, where functions equal a.e. are identified. A linear subspace M of $M(X, \mu)$ is called an (order) ideal of measurable functions, whenever it follows from $f \in M$ and $g \in M(X, \mu)$ with $|g| \leqslant |f|$ that $g \in M$. An order ideal $L \subset M(X, \mu)$ provided with an absolute and monotone norm ρ is called a normed function space and denoted by L_ρ. If the normed function space is norm complete it is called a Banach function space [**12**, Chapter 15]. Let L and M be ideals of measurable functions in $M(Y, \nu)$ and $M(X, \mu)$, respectively. Then a linear operator T: $L \to M$ is called an integral operator if there exists a $(\mu \times \nu)$-measurable $T(x, y) \in M(X \times Y, \mu \times \nu)$ such that, for every $f \in L$, $Tf(x) = \int T(x, y)f(y)\, d\nu(y)$ holds almost everywhere on X. An order bounded (absolutely bounded) integral operator T from L into M is here called an absolute integral operator. If T is an absolute integral operator, then $|T(x, y)|$ is also the kernel of an operator from L into M.

If L_ρ is a Banach function space, then L'_ρ denotes the first associate space and ρ' the first associate norm. If ρ has the weak Fatou property, then L_ρ and L''_ρ coincide

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as sets and ρ and ρ'' are equivalent norms (see **[12]**). The following theorem is due to W. A. J. Luxemburg **[5]**.

THEOREM 1.1. *Let $L_\rho = L_\rho(X, \mu)$ be a Banach function space and let $H \subset L_\rho$. If H is $\sigma(L_\rho, L'_\rho)$ relatively compact, then the following conditions hold.*

(i) *For all $g \in L'_\rho$ we have* $\sup(\int_X |fg|\, d\mu : f \in H) < \infty$.

(ii) *For all $\varepsilon > 0$ and all $g \in L'_\rho$ there exists $\delta > 0$ such that $A \in \Lambda$ with $\mu(A) < \delta$ implies that* $\sup(\int_A |gf|\, d\mu : f \in H) \leqslant \varepsilon$.

(iii) *For all $\varepsilon > 0$ and all $g \in L'_\rho$ there exists $X_0 \subset X$ with $\mu(X_0) < \infty$ such that for the complement $X_0^c = X \setminus X_0$ we have* $\sup(\int_{X_0^c} |fg|\, d\mu : f \in H) < \varepsilon$.

Conversely, if ρ has the weak Fatou property, then (i)–(iii) *imply that H is $\sigma(L_\rho, L'_\rho)$ relatively compact.*

REMARKS. 1. The above theorem is a generalization of the well-known Dunford-Pettis theorem on relatively weakly compact subsets of $L_1(\mu)$ and can be reduced to it by observing that $gH = \{ gh : h \in H \}$ is a relatively weakly compact subset of $L_1(\mu)$ in case $g \in L'_\rho$ and H is $\sigma(L_\rho, L'_\rho)$ relatively compact.

2. It is not difficult to show that (ii) and (iii) together are equivalent with:

If $g_n \in L'_\rho$ and $g_n \downarrow 0$ a.e. then $\sup(\int |fg_n|\, d\mu : f \in H) \downarrow 0$.

3. If ρ is weakly Fatou, then it follows from the above theorem that in case $H \subset L_\rho$ is $\sigma(L_\rho, L'_\rho)$ relatively compact, then also the solid hull $\{ g \in L_\rho : |g| < h, h \in H \}$ is $\sigma(L_\rho, L'_\rho)$ relatively compact.

2. Operators with domain $L_1(Y, \nu)$ or range in $L_\infty(X, \mu)$. We first recall the following fundamental result (see **[10]** for an elementary proof and references).

THEOREM 2.1 (A. V. BUHVALOV). *Let L and M be ideals of measurable functions in $M(Y, \nu)$ and $M(X, \mu)$, respectively. Then a linear operator $T : L \to M$ is an integral operator if and only if it follows from $0 \leqslant u_n < u$ in L and $u_n \xrightarrow{*} 0$ that $Tu_n(x) \to 0$ a.e.*

The notation $u_n \xrightarrow{*} 0$ means here that every subsequence of $\{u_n\}$ contains a subsequence converging to zero a.e.

THEOREM 2.2. *Let L_ρ be a Banach function space and let $T : L_1(Y, \nu) \to L_\rho(X, \mu)$ be a norm bounded $\sigma(L_\rho, L'_\rho)$ compact operator. Then T is an integral operator.*

PROOF. Let $X = \bigcup_{n=1}^\infty X_n$ such that $\mu(X_n) < \infty$, $X_n \cap X_m = \varnothing$ for $n \neq m$ and $\chi_{X_n} \in L'_\rho$ for all n. Let $P_n f = f\chi_{X_n}$. Then $P_n \circ T$ is a $\sigma(L_\rho(X_n, \mu), L'_\rho(X_n, \mu))$ compact operator from $L_1(Y, \nu)$ into $L_\rho(X_n, \mu)$. If $P_n \circ T$ is an integral operator for all n, then T is also an integral operator. Therefore we may assume in the proof that $\chi_X \in L'_\rho$ and $\mu(X) < \infty$. Denote by T' the restriction of the adjoint operator to L'_ρ. We first prove that T' is a kernel operator from L'_ρ into $L_\infty(Y, \nu)$. Let $0 < g_n < g \in L'_\rho$ and assume $g_n \xrightarrow{*} 0$. We shall prove that $\|T'g_n\|_\infty \to 0$. Let $\varepsilon > 0$ be given. Then, by Theorem 1.1, there exists $\delta > 0$ such that $\mu(A) < \delta$ implies

$$\sup\left(\int_A g(x) |Tf(x)|\, d\mu(x) : \|f\|_1 < 1 \right) < \frac{\varepsilon}{2}.$$

Let $\eta = \varepsilon/2\|T'\|\rho'(\chi_X)$ and put $A_n = \{x : g_n(x) \geqslant \eta\}$. It follows from $g_n \xrightarrow{*} 0$ that $\mu(A_n) \to 0$. Let N be such that $\mu(A_n) \leqslant \delta$ for all $n \geqslant N$. Then, for all $n \geqslant N$, we have

$$\sup\left(\int_{A_n} g(x)|Tf(x)|\, d\mu(x) : \|f\|_1 \leqslant 1\right) \leqslant \frac{\varepsilon}{2};$$

so also

$$\sup\left(\left|\int_{A_n} g_n(x)Tf(x)\, d\mu(x)\right| : \|f\|_1 \leqslant 1\right) \leqslant \frac{\varepsilon}{2}$$

for $n \geqslant N$. From this we conclude that $\|T'(g_n\chi_{A_n})\|_\infty \leqslant \varepsilon/2$ for $n \geqslant N$. Hence

$$\begin{aligned}\|T'g_n\|_\infty &\leqslant \|T'(g_n\chi_{A_n})\|_\infty + \|T'(g_n\chi_{A_n^c})\|_\infty \\ &\leqslant \varepsilon/2 + \|T'\| \cdot \rho'(g_n\chi_{A_n^c}) \leqslant \varepsilon/2 + \|T'\| \cdot \eta \cdot \rho'(\chi_X) = \varepsilon\end{aligned}$$

for all $n \geqslant N$. It follows that $\|T'g_n\|_\infty \to 0$, so certainly $T'g_n(x) \to 0$ a.e. Hence T' is an integral operator from L'_ρ into $L_\infty(Y, \nu)$. Since, however, every norm bounded operator into $L_\infty(Y, \nu)$ is order bounded, we conclude that T is an absolute integral operator. This implies that T'' is an absolute integral operator into $L''_\rho(X, \mu)$, so T is an integral operator from $L_1(Y, \nu)$ into $L_\rho(X, \mu)$ (the modulus of its kernel defines an operator from L_1 into L''_ρ and need not map L_1 into L_ρ).

REMARKS. (i) If we add to the hypotheses in the above theorem that ρ has the weak Fatou property, then T is an absolute integral operator from L_1 into L_ρ.

(ii) If T is as in the above theorem and has kernel $T(x, y)$, then for ν-a.e. $y \in Y$ we have $T_y(x) = T(x, y) \in L''_\rho(X, \nu)$. This follows from the observation that $|T'f(y)| \leqslant \|T'\|\rho'(f)$ ν-a.e. and the methods used in **[11]**.

COROLLARY 2.3. *If $T : L_1(Y, \nu) \to L_1(X, \mu)$ is weakly compact, then T is an absolute integral operator.*

We shall now investigate the order structure of the set of $\sigma(L_\rho, L'_\rho)$ compact kernel operators from $L_1(Y, \nu)$ into $L_\rho(X, \mu)$. We first notice that for a proof that a positive operator T from $L_1(Y, \nu)$ into $L_\rho(X, \mu)$ is $\sigma(L_\rho, L'_\rho)$ compact, it suffices to show that $\{Tf : 0 \leqslant f \in L_1(Y, \nu), \|f\|_1 \leqslant 1\}$ is $\sigma(L_\rho, L'_\rho)$ relatively compact. We also notice that in case ρ has the weak Fatou property, then the space $\mathfrak{L}_b(L_1, L_\rho)$ of all order bounded operators is equal to the space $\mathfrak{L}(L_1, L_\rho)$ of all norm bounded operators.

THEOREM 2.4. *Let L_ρ be a Banach function space and assume that ρ has the weak Fatou property. Then the set of all $\sigma(L_\rho, L'_\rho)$ compact operators from $L_1(Y, \nu)$ into $L_\rho(X, \mu)$ is an order ideal in $\mathfrak{L}_b(L_1, L_\rho)$.*

PROOF. Assume $T : L_1 \to L_\rho$ is $\sigma(L_\rho, L'_\rho)$ compact. We show that the set $\{|T|f : 0 \leqslant f \in L_1, \|f\|_1 \leqslant 1\}$ satisfies the conditions of Theorem 1.1. It clearly satisfies condition (i), since it is a norm bounded set. To prove (ii), let $0 \leqslant g \in L'_\rho$ and $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $\mu(A) \leqslant \delta$ implies that

$$\sup\left(\int_A |Tf|\, g\, d\mu : f \in L_1, \|f\|_1 \leqslant 1\right) \leqslant \varepsilon. \tag{1}$$

Let $0 \leqslant u \in L_1$ with $\|u\|_1 \leqslant 1$. Then we have

$$|T|u = \sup\left(\sum_{i=1}^{n} |Tu_i| : u = \sum_{1}^{n} u_i, 0 \leqslant u_i \in L_1\right)$$

and we observe that the set, of which the supremum is taken, is directed upwards. Hence we have, for all $A \in \Lambda$,

$$\int_A |T|u \cdot g \, d\mu = \sup\left(\sum_{1}^{n} \int_A |Tu_i| g \, d\mu : u = \sum_{1}^{n} u_i, 0 \leqslant u_i \in L_1\right).$$

Let $\mu(A) \leqslant \delta$ and let $0 \leqslant u_i \in L_1$ with $u = \Sigma_1^n u_i$. Then it follows from (1) that $\int_A |Tu_i| g \, d\mu \leqslant \varepsilon \|u_i\|_1$ for $i = 1, 2, \ldots, n$, so

$$\sum_{1}^{n} \int_A |Tu_i| g \, d\mu \leqslant \varepsilon \sum_{1}^{n} \|u_i\|_1 = \varepsilon \|u\|_1 \leqslant \varepsilon.$$

We conclude that for $A \in \Lambda$, with $\mu(A) \leqslant \delta$,

$$\sup\left(\int_A |T|u \cdot g \, d\mu : u \in L_1^+, \|u\|_1 \leqslant 1\right) \leqslant \varepsilon,$$

i.e., condition (ii) of Theorem 1.1 is satisfied. The proof of condition (iii) is completely similar and therefore omitted. It follows now that $|T|$ is $\sigma(L_\rho, L'_\rho)$ compact. If now $0 \leqslant S \leqslant T$ and T is $\sigma(L_\rho, L'_\rho)$ compact, then it follows immediately from Theorem 1.1 that S is $\sigma(L_\rho, L'_\rho)$ compact. Hence the proof of the theorem is complete.

EXAMPLE. Let $X = Y = [0, 1]$ and $\mu = \nu$ be Lebesgue measure. Let $g_n(x)$ be the characteristic function of the interval $(1/2^{n+1}, 1/2^n)$ and let ϕ_n be the nth Rademacher function. We define $T : L_1(Y, \nu) \to L_\infty(X, \mu)$ by means of

$$Tf(x) = \sum_{n=0}^{\infty} \left(\int_0^1 f(y)\phi_n(y) \, dy\right) g_n(x).$$

Then T is not weakly compact, but T is $\sigma(L_\infty, L_1)$ compact. Since $|T|$ is a rank one operator, it follows that the above theorem is false for weakly compact operators.

We now dualize the above results. The following theorem singles out the right notion of duality for $\sigma(L_\rho, L'_\rho)$ compact operators from L_1 into L_ρ.

THEOREM 2.5. *Let $L_\rho = L_\rho(Y, \nu)$ be a Banach function space and let $T : L_\rho(Y, \nu) \to L_\infty(X, \mu)$ be a bounded linear operator. Then the following are equivalent.*

(i) *T is order continuous and $T[0, g]$ is $\sigma(L_\infty, L_\infty^*)$ relatively compact for all $0 \leqslant g \in L_\rho$.*

(ii) *For all $0 \leqslant g_n \leqslant g \in L_\rho$ with $g_n(x) \to 0$ a.e. we have $\|Tg_n\|_\infty \to 0$.*

(iii) *T is order continuous and for every disjoint sequence $g_n \in L_\rho$ with $0 \leqslant g_n \leqslant g \in L_\rho$ we have $\|Tg_n\|_\infty \to 0$.*

(iv) *T is order continuous and for every disjoint sequence $g_n \in L_\rho$ with $0 \leqslant g_n \leqslant g \in L_\rho$ we have $\| |T| g_n\|_\infty \to 0$.*

PROOF. (i) $\Rightarrow$ (ii) Let $0 \leqslant g_n \leqslant g \in L_\rho$ with $g_n \to 0$ a.e. Then $A_\lambda = \{f \in L_\rho : |f| \leqslant c \cdot g\}$ is a Banach function space with respect to the norm

$$\lambda(f) = \inf(c : |f| \leqslant c \cdot g).$$

We observe also that A'_λ can be identified with $L_1(g\,d\nu)$. Then T considered as an operator from A_λ into L_∞ is by assumption weakly compact. Hence $T^* : L^*_\infty \to A^*_\lambda$ is $\sigma(A^*_\lambda, A^{**}_\lambda)$ compact. From the order continuity of T it follows that $T^*(L_1) \subset A'_\lambda = L_1(g\,d\nu)$. Denote by T' the restriction of T^* to $L_1(X, \mu)$. Then $T' : L_1(X, \mu) \to L_1(g\,d\nu)$ is weakly compact. It follows now from Remark 2, following Theorem 1.1., that

$$\sup\left(\int |T'f| \cdot g_n\,d\nu : \|f\|_1 \leqslant 1\right) \to 0,$$

so also

$$\sup\left(\left|\int T'f \cdot g_n\,d\nu\right| : \|f\|_1 \leqslant 1\right) \to 0,$$

i.e., $\|Tg_n\|_\infty \to 0$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (iv) Assume (iv) is false. Then there exist $\varepsilon > 0$, $0 \leqslant g_n \leqslant g \in L_\rho$ with $g_n \wedge g_m = 0$ for $n \neq m$ such that $\| |T| g_n\|_\infty > \varepsilon$. From the formula $|T| g_n = \sup(|Tf| : |f| \leqslant g_n)$ we conclude that

$$\| |T| g_n\|_\infty = \sup(\|Tf\|_\infty : |f| \leqslant g_n).$$

Hence for all n there exist $f_n \in L_\rho$, $|f_n| \leqslant g_n$ such that $\|Tf_n\|_\infty \geqslant \| |T| g_n\| - \varepsilon/2$, so $\|Tf_n\|_\infty \geqslant \varepsilon/2$ for all n. By applying (iii) to the sequence $\{f_1^+, f_1^-, f_2^+, f_2^-, \ldots\}$ we get a contradiction.

(iv) $\Rightarrow$ (i) Let $0 \leqslant g \in L_\rho$ and, as in the proof of (i) $\Rightarrow$ (ii), denote by A_λ the Banach function space $\{f \in L_\rho : |f| \leqslant c \cdot g\}$ with the AM-norm λ. Then $|T|^*$ can be considered as a mapping from L^*_∞ into A^*_λ and $|T|'$ denotes its restriction to $L_1(X, \mu)$. Then $|T|' : L_1(X, \mu) \to A'_\lambda \cong L_1(g\,d\nu)$. For $h \in A_\lambda$ we define the semi-norm λ_1 by

$$\lambda_1(h) = \sup\left(\int |T|'f \cdot |h|\,d\nu : \|f\|_1 \leqslant 1\right).$$

From (iv) it follows that λ_1 has the property that if $0 \leqslant h_n \leqslant h \in A_\lambda$, $h_n \wedge h_m = 0$ for $n \neq m$, then $\lambda_1(h_n) \to 0$. It follows now from Meyer-Nieberg's result on disjoint sequences (see [**9**, Chapter II]) that λ_1 is order continuous, i.e. if $f_n \in A_\lambda$, $f_n \downarrow 0$ a.e., then $\lambda_1(f_n) \downarrow 0$. It follows from Theorem 1.1 that $\{|T|'f : \|f\|_1 \leqslant 1\}$ is a $\sigma(A'_\lambda, A''_\lambda)$ relatively compact subset of A'_λ. If we denote by T' the restriction of T^* to $L_1(X, \mu)$, then $|T'| \leqslant |T|'$ (actually $|T'| = |T|'$ here, but we do not need this stronger result). Hence, $\{T'f : \|f\|_1 \leqslant 1\}$ is also a $\sigma(A'_\lambda, A''_\lambda)$ relatively compact subset of A'_λ, i.e., $T' : L_1 \to A'_\lambda$ is weakly compact. It follows that $(T')^* : A''_\lambda = A_\lambda \to L_\infty(X, \mu)$ is weakly compact, i.e., $T[0, f]$ is relatively weakly compact in $L_\infty(X, \mu)$. This completes the proof of the theorem.

Remark. The equivalence (i) $\Leftrightarrow$ (iii) could have been proved using some results of Grothendieck (see [**8**, Lemma 3.1]). The above result could also partly be derived from Theorem 4.2 of [**2**]. Moreover, from Theorem 6.4 of [**2**] it follows that the set of operators satisfying (i)–(iv) is strongly closed.

THEOREM 2.6. *Let L_ρ be a Banach function space and let $T : L_\rho \to L_\infty$ be a norm bounded linear operator satisfying one of the four equivalent conditions of Theorem 2.5. Then T is a kernel operator.*

PROOF. Condition (ii) implies, by Theorem 2.1, immediately that T is a kernel operator.

THEOREM 2.7. *Let L_ρ be a Banach function space. Then the set of all bounded operators from L_ρ into $L_\infty(X, \mu)$ satisying one of the four conditions of Theorem 2.5 is an order ideal in $\mathfrak{L}_b(L_\rho, L_\infty)$.*

PROOF. Immediate from Theorem 2.5.

COROLLARY 2.8. *The set of order continuous weakly compact operators from $L_\infty(Y, \nu)$ into $L_\infty(X, \mu)$ is an order ideal in $\mathfrak{L}_b(L_\infty, L_\infty)$, consisting of kernel operators.*

EXAMPLE. Let $Y = \mathbf{N}$, ν counting measure and $X = [0, 1]$, μ Lebesgue measure. Let g_n be the characteristic functon of the interval $(1/2^n, 1/2^{n-1})$. Then $T : l_\infty \to L_\infty[0, 1]$ is defined by $T(\{\alpha_n\}) = \sum_{n=1}^\infty \alpha_n g_n$. Then T is a positive kernel operator, but T is not weakly compact, since $\|T(e_n)\|_\infty = 1$ for all n, where $e_n = (0, \ldots, 1, 0, \ldots)$. This example shows that the weakly compact operators from l_∞ into $L_\infty([0, 1])$ do not form a band in $\mathfrak{L}_b(l_\infty, L_\infty)$.

3. A characterization of integral operators in terms of equimeasurable sets. We begin with recalling a notion due to A. Grothendieck.

DEFINITION [**3**, 20]. A set $H \subset M(X, \mu)$ is called equimeasurable if for all $\varepsilon > 0$ and all $X_0 \subset X$ with $\mu(X_0) < \infty$ there exists $X_1 \subset X_0$ with $\mu(X_0 \setminus X_1) < \varepsilon$ such that $\{h\chi_{X_1} : h \in H\}$ is a relatively compact subset of $L_\infty(X, \mu)$.

We start with two lemmata.

LEMMA 3.1. *Let $\{g_n : n = 1, 2, \ldots\} \subset M(X, \mu)$ be equimeasurable and assume $g_n \xrightarrow{*} 0$. Then $g_n(x) \to 0$ a.e.*

PROOF. The proof is straightforward and therefore omitted.

LEMMA 3.2. *Let $T : L_\infty(Y, \nu) \to L_\infty(X, \mu)$ be an integral operator and assume $\mu(X) < \infty$. Then for all $\varepsilon > 0$ there exists $X_0 \subset X$ with $\mu(X \setminus X_0) < \varepsilon$ such that $P_0 \circ T : L_\infty(Y, \nu) \to L_\infty(X, \mu)$ is compact, where P_0 denotes the operator $P_0 f = \chi_{X_0} f$.*

PROOF. First we observe that T is norm bounded. A proof of this can be given by means of the closed graph theorem, similar to the proof for L_2-space as in [**4**]. Hence T is order bounded, so if T has kernel $T(x, y)$, then $|T(x, y)|$ is the kernel of $|T|$ (see [**6**] and [**10**] for a simpler proof). It follows that $\int\int |T(x, y)|\, d\nu(y)\, d\mu(x) < \infty$. Hence there exist $t_n(x, y) \in L_1(X \times Y, \mu \times \nu)$ of the form $\sum_1^m a_k \chi_{A_k}(x) \chi_{B_k}(y)$ such that $t_n(x, y) \to T(x, y)$ in $L_1(X \times Y, \mu \times \nu)$. Passing to a subsequence we may assume that the convergence is also pointwise. From Fubini's theorem it follows that we may assume that $h_n(x) = \int |t_n(x, y) - T(x, y)|\, d\nu(y) \to 0$ a.e. on X.

Let $\varepsilon > 0$. Then, by Egoroff's theorem, there exists $X_0 \subset X$ with $\mu(X \setminus X_0) < \varepsilon$ such that $h_n(x) \to 0$ uniformly on X_0. Define S_n from $L_\infty(Y, \mu)$ into $L_\infty(X, \mu)$ by means of

$$S_n f(x) = \int \chi_{X_0}(x) t_n(x, y) f(y)\, d\nu(y).$$

Then S_n is a finite rank operator and we shall show that $\|S_n - P_0 \circ T\| \to 0$, where P_0 denotes the operator $P \circ f = \chi_{X_0} \cdot f$. Let $f \in L_\infty(Y, \nu)$ with $\|f\|_\infty < 1$. Then

$$|S_m f(x) - P_0 \circ Tf(x)| \leqslant \chi_{X_0}(x) \cdot \int |t_m(x, y) - T(x, y)|\, |f(y)|\, d\nu(y)$$

$$\leqslant \chi_{X_0}(x)\left(\int |t_m(x, y) - T(x, y)|\, d\nu(y)\right)\|f\|_\infty \leqslant \chi_{X_0}(x) h_m(x).$$

Hence

$$\sup(\|S_m f - P_0 \circ Tf\|_\infty : \|f\|_\infty \leqslant 1) \leqslant \|\chi_{X_0} h_m\|_\infty \to 0$$

as $m \to \infty$, i.e., $\|S_m - P_0 \circ T\|_\infty \to 0$ as $m \to \infty$. We conclude from this that $P_0 \circ T$ is compact.

REMARK. As noted by the referee, one could also prove the above theorem by some well-known vector measure theory. We outline the proof. Denote by g_x the L_1-valued function $g_x(y) = T(x, y)$. Then $Tf(x) = \langle g_x, f \rangle$. By a theorem of D. R. Lewis [**1**, p. 88] we conclude, since L_1 is weakly compactly generated, that we may assume that g_x is strongly measurable. The set X_0 is then found via the Egoroff theorem.

We note that the above lemma is, of course, false in case $\mu(X) = \infty$, since we can take as a counterexample the identity operator from l_∞ into l_∞. Let now L and M be ideals of measurable functions in $M(Y, \nu)$ and $M(X, \mu)$, respectively. Then a linear operator $T : L \to M$ is called **-continuous* if it follows from $0 \leqslant u_n \leqslant u \in L$ and $u_n \xrightarrow{*} 0$ that $Tu_n \xrightarrow{*} 0$. With these notations we have the following theorem.

THEOREM 3.3. *The linear operator $T : L \to M$ is an integral operator if and only if T is *-continuous and maps order intervals into equimeasurable sets.*

PROOF. Let $T : L \to M$ be an integral operator. Then T is certainly *-continuous (see Theorem 2.1). Let $0 \leqslant f \in L$, $X_0 \subset X$ with $\mu(X_0) < \infty$ and $\varepsilon > 0$. Then there exist $M > 0$ and $X_1 \subset X_0$ with $\mu(X_0 \setminus X_1) \leqslant \varepsilon/2$ such that $\int |T(x, y) f(y)|\, d\nu(y) \leqslant M$ on X_1. Denote $A_f = \{ g \in L : |g| \leqslant cf \}$ and by P_1 the operator $P_1 f = \chi_{X_1} f$. Then $P_1 \circ T$ is an integral operator from A_f into $L_\infty(X_1, \mu)$. It follows from the above lemma that there exists $X_2 \subset X_1$ with $\mu(X_1 \setminus X_2) \leqslant \varepsilon/2$ such that $P_2 P_1 T$ is a compact operator from A_f into $L_\infty(X_1, \mu)$, where P_2 denotes the operator $P_2 f = \chi_{X_2} f$. Hence $\{\chi_{X_2} Tg : |g| \leqslant f\}$ is a relatively compact subset of $L_\infty(X, \mu)$, and since $\mu(X_0 \setminus X_2) \leqslant \varepsilon$ it follows that $T([-f, f])$ is an equimeasurable subset of $M(X, \mu)$. Assume now that T is *-continuous and maps order intervals into equimeasurable sets. Let $0 \leqslant u_n \leqslant u$ in L with $u_n \xrightarrow{*} 0$. Then by assumption $\{ Tu_n : n = 1, 2, \ldots \}$ is equimeasurable and $Tu_n \xrightarrow{*} 0$. It follows now from Lemma 3.1 that $Tu_n(x) \to 0$ and from Theorem 2.1 we conclude that T is an integral operator.

COROLLARY 3.4 (W. SCHACHERMAYER [7]). *The linear operator* $T : L_p(Y, \nu) \to L_q(X, \mu)$, *with* $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, *is an integral operator if and only if* T *maps order intervals into equimeasurable sets.*

PROOF. If T maps L_p into L_q and order intervals into equimeasurable sets, then a closed graph argument shows that T is norm bounded (see [7]). The corollary now follows immediately, since norm bounded operators from L_p $(1 \leqslant p < \infty)$ into L_q are *-continuous.

REMARK. The above corollary was first observed by Schachermayer, who proved it by using vectorvalued integration.

We note another corollary of Theorem 3.3.

COROLLARY 3.5. *Let* L *be an ideal in* $M(Y, \nu)$ *and let* $L_\rho = L_\rho(X, \mu)$ *be a Banach function space. If* $T : L \to L_\rho$ *is an integral operator, then it follows from* $0 \leqslant u_n \leqslant u$ *in* L *and* $\rho(Tu_n) \to 0$ *that* $Tu_n(x) \to 0$ *a.e.*

PROOF. From $\rho(Tu_n) \to 0$ it follows that $Tu_n \xrightarrow{*} 0$. From Theorem 3.3 we know that $\{Tu_n\}$ is equimeasurable, so by Lemma 3.1 we have $Tu_n(x) \to 0$ a.e.

The converse of the above corollary is false in general, as can be seen by taking $L = L_\rho = L_\infty([0, 1])$ and $T =$ the identity operator. There are, however, two partial converses, which follow immediately from Theorem 2.1.

(1) Let L_{ρ_1}, L_{ρ_2} be Banach function spaces and assume ρ_2 is order continuous. If $T : L_{\rho_2} \to L_{\rho_1}$ is norm bounded and has the property that it follows from $0 \leqslant u_n \leqslant u$ in L_{ρ_2} and $\rho_1(Tu_n) \to 0$ that $Tu_n(x) \to 0$, then T is an integral operator.

(2) Let L be an ideal in $M(Y, \nu)$ and L_ρ a Banach function space with order continuous norm. If $T : L \to L_\rho$ is an order bounded order continuous operator with the property that it follows from $0 \leqslant u_n \leqslant u$ in L and $\rho(Tu_n) \to 0$ that $Tu_n(x) \to 0$ a.e., then T is an absolute integral operator.

We now indicate the relation between Theorem 3.3 and results of Grothendieck and of Nagel and Schlotterbeck. The conclusions drawn from these results are of a less elementary nature than some of the results above. To distinguish integral operators in our sense and in the sense of Grothendieck, we call the latter G-integral operators. With this notation A. Grothendieck [3] proved the following result.

THEOREM 3.6 (A. GROTHENDIECK). *Let* E *be a Banach space with unit ball* B *and* $T : E \to L_1(X, \mu)$ *be a bounded operator. Then*

(1) T *is* G*-integral if and only if* $T(B)$ *is order bounded in* L_1.

(2) T *is nuclear if and only if* $T(B)$ *is order bounded in* L_1 *and equimeasurable.*

For a proof in case $\mu(X) < \infty$, see [1, pp. 258–259]. For the sake of convenience we only consider now the case $L_2(Y, \nu)$ and $L_2(X, \mu)$. Then the result of Corollary 3.4 for absolute integral operators (i.e. order bounded integral operators) can be phrased as follows. Let $T : L_2(Y, \nu) \to L_2(X, \mu)$ be an order bounded operator. Then T is an absolute integral operator if and only if for all $0 \leqslant f \in L_2(Y, \nu)$ and all $0 \leqslant g \in L_2(X, \mu)$ the mapping induced by T from $A_f = \{g \in L_2, |g| \leqslant cf\}$

into $L_1(X, g\, d\mu)$ is nuclear. This is precisely the result of Nagel and Schlotterbeck obtained in the context of Banach lattices (see [**9**, Chapter IV]). It is for this reason that one can consider Theorem 3.3 as a nonorder bounded extension of their results.

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