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CHARACTERIZATIONS OF THE FISCHER GROUPS. I, II, III

BY

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ABSTRACT. B. Fischer, in his work on finite groups which contain a conjugacy class of 3-transpositions, discovered three new sporadic finite simple groups, usually denoted $M(22)$, $M(23)$ and $M(24)'$. In Part I two of these groups, $M(22)$ and $M(23)$, are characterized by the structure of the centralizer of a central involution. In addition, the simple groups $U_6(2)$ (often denoted by $M(21)$) and $P\Omega(7, 3)$, both of which are closely connected with Fischer's groups, are characterized by the same method.

The largest of the three Fischer groups $M(24)$ is not simple but contains a simple subgroup $M(24)'$ of index two. In Part II we give a similar characterization by the centralizer of a central involution of $M(24)$ and also a partial characterization of the simple group $M(24)'$.

The purpose of Part III is to complete the characterization of $M(24)'$ by showing that our abstract group G is isomorphic to $M(24)'$. We first prove that G contains a subgroup $X \cong M(23)$ and then we construct a graph (on the cosets of X) which is shown to be isomorphic to the graph for $M(24)$.

PART I

The results proved are:

THEOREM A. *Let G be a finite group, z an involution in G and $H = C_G(z)$. Suppose $J = O_2(H)$ is extra-special of order 2^9 and $C_H(J) \subseteq J$. Then*

(i) *if $H/J \cong PSp_4(3)$, $G \cong U_6(2)$ or $G = H \cdot O(G)$;*

(ii) *if $H/J \cong \operatorname{Aut} PSp_4(3)$, G contains a subgroup G_0 of index two, $G_0 \cong U_6(2)$ and $G \subset \operatorname{Aut} G_0$ or $G = H \cdot O(G)$.*

THEOREM B. *Let G be a finite group, z an involution in G and $H = C_G(z)$. Suppose that $J = O_2(H)$ is the direct product of a group of order two with an extra-special group of order 2^9 and that $J' = \langle z \rangle$. If $C_H(J) \subseteq J$ and $H/J \cong \operatorname{Aut} PSp_4(3)$, then one of the following holds:*

(i) *$G = H \cdot O(G)$;*

(ii) *there is an involution $z_1 \in Z(J) - \langle z \rangle$ with $\langle z_1 \rangle \lhd G$, G contains a subgroup G_0 of index two with $G_0/\langle z_1 \rangle \cong U_6(2)$ and $G \subset \operatorname{Aut} G_0$;*

(iii) *$G \cong M(22)$.*

THEOREM C. *Let G be a finite group which possesses an involution z such that $H = C_G(z)$ satisfies:*

(i) *$J = O_2(H)$ is the direct product of a four group and an extra-special group of order 2^9, and $C_H(J) \subseteq J$;*

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(ii) *H contains an element c of order three with $H/J\langle c\rangle \cong$* Aut $PSp_4(3)$ *and $C_H(c)/\langle c, z\rangle \cong PSp_4(3)$.*

Then either $G = H \cdot O(G)$ or $G \cong M(23)$ or G contains a normal four group $\langle z_1, z_2\rangle$ with $G/\langle z_1, z_2\rangle \cong$ Aut $U_6(2)$.

Previously, D. Hunt [**11**], [**12**] has characterized the groups $M(22)$ and $M(23)$ by the centralizer of an involution which is a 3-transposition. The proofs of Theorems B and C rely on his results as well as Theorem A. In Part II these results are used by the author to give a characterization of the larger group $M(24)$.

By using some of the lemmas in the proofs of Theorems B and C we are able to give a characterization of the simple group $P\Omega(7, 3) = B_3(3)$. (A certain element of order three in $M(23)$ has centralizer isomorphic to $Z_3 \times P\Omega(7, 3)$.)

THEOREM D.[1] *Let G be a finite group, z an involution in G and suppose $H = C_G(z)$ satisfies:*

(i) *H contains a normal subgroup $K_1 * K_2 \times K_3$ of index four with $K_1 \cong K_2 \cong$* SL(2, 3) *and $K_3 \cong \mathfrak{A}_4$;*

(ii) *$H = (K_1 * K_2 \times K_3)\langle t, u\rangle$ where $\langle t, u\rangle \cong E_4$, $K_1^t = K_2$, $[K_3, t] = 1$ and ut acts fixed-point-free on a Sylow 3-subgroup of H.*

Then either $G = H \cdot O(G)$; $O_2(K_3) = \langle z_1, z_2\rangle \lhd G$ with $G/\langle z_1, z_2\rangle \cong Z_3 \cdot$ Aut $PSp_4(3)$; *or $G \cong P\Omega(7, 3)$.*

1. Notation and preliminary results. As far as notation is concerned, we will in general follow Gorenstein [**3**]. In addition, we will use:

$X * Y$: the central product of the groups X, Y;

$x \sim_X y$: x is conjugate to y in X;

x^X: $\{y^{-1}xy \mid y \in X\}$ = conjugacy class of x in X;

Z_n, D_n: the cyclic, dihedral groups of order n, respectively;

E_{p^n}: the elementary abelian group of order p^n (p prime);

Q_8: the quaternion group of order 8;

Σ_n, $\mathfrak{A}_n$ the symmetric, alternating groups of degree n.

PROPOSITION 1 (SUZUKI [**3**, pp. 328, 105]). *If x is an involution in the finite group G and $x \notin O_2(G)$, then x inverts an element of odd order in $G^{\#}$.*

PROPOSITION 2 (GLAUBERMAN [**2**]). *Let x be an involution in a finite group G. If $G \neq C_G(x) \cdot O(G)$ then x is conjugate (in G) to an involution in $C_G(x) - \langle x\rangle$.*

PROPOSITION 3 (THOMPSON [**10**, COROLLARY 1]). *Let S be a Sylow 2-subgroup of the finite group G, S_0 a maximal subgroup of S and x an involution in $S - S_0$. If x is not conjugate (in G) to any involution in S_0 then G contains a subgroup G_0 of index two with $x \in G - G_0$.*

PROPOSITION 4. *Let G be a finite group, z an involution in G and $H = C_G(z)$. Suppose that $P \neq 1$ is a p-subgroup of H (p an odd prime) which satisfies ($*$) if $g^{-1}Pg \subset H$, $g \in G$, then there exists $h \in H$ so that $g^{-1}Pg = h^{-1}Ph$.*

[1]In the case $G = O^2(G)$, Theorem D is a special case of a result of Olsson [**21**] and Solomon [**22**].

Then for any involution $t \in C_H(P)$, $t \sim_G z$ if and only if $t \sim_{N(P)} z$. If, in addition, $N_G(P) = N_H(P) \cdot C_G(P)$ then $t \sim_G z$ if and only if $t \sim_{C(P)} z$.

PROPOSITION 5. *Let H be a finite group with $J = O_2(H)$ extra-special (i.e. $J \cong D_8 * \dots * D_8$ or $D_8 * \dots * D_8 * Q_8$). If $P \neq 1$ is a p-subgroup of H (p an odd prime) then each of $C_J(P)$, $[P, J]$ is extra-special or equal to $Z(J)$.*

PROOF. This follows immediately from $C_J(P)[P, J] = J$ [**3**, Theorem 5.3.5] and the Three Subgroups Lemma [**3**, Theorem 2.2.3].

The next result is certainly a consequence of Gorenstein and Harada's main theorem [**6**]. However we will give a short proof which essentially relies on the original characterizations of the groups involved.

PROPOSITION 6. *Let G be a finite group which contains an involution z. Suppose that $H = C_G(z)$ satisfies*:

(i) $O_{2,3}(H) = L_1 * L_2$, $L_i \cong \mathrm{SL}(2, 3)$, $i = 1, 2$;

(ii) $H/O_{2,3}(H) \cong E_4$;

(iii) $L_i \ntriangleleft H$, $i = 1, 2$.

Then one of the following holds:

(a) $G = H \cdot O(G)$ *or*

(b) $G \cong L_4(3)$, $U_4(3)$, Aut $PSp_4(3)$ *or* Aut $G_2(3)$.

PROOF. Let $L_i = \langle \alpha_i, \beta_i, \sigma_i \rangle$ where $\langle \alpha_i, \beta_i \rangle \cong Q_8$ and σ_i is of order three for $i = 1, 2$, and let $V = \langle u, v, z \rangle$ be a Sylow 2-subgroup of $N_H(\langle \sigma_1, \sigma_2 \rangle)$. Note that $V \cdot L_1 \cdot L_2 = H$ and $V \cap O_2(H) = \langle z \rangle$. From the assumptions listed above we can choose the generators of H in such a way that the following relations hold:

$$\sigma_1^u = \sigma_2, \quad \alpha_1^u = \alpha_2, \quad \beta_1^u = \beta_2;$$
$$\sigma_i^v = \sigma_i^{-1}, \quad \alpha_i^v = \alpha_i^{-1}, \quad \beta_i^v = \alpha_i\beta_i, \qquad i = 1, 2.$$

Suppose now that (a) does not hold. By Proposition 2, $z \sim_G h$ for some $h \in H - \langle z \rangle$. As $O_2(H) - \langle z \rangle$ has only one class of involutions in H with representative $\alpha_1\alpha_2$ say, and as $C(\alpha_1\alpha_2) \cap O_2(H) \cong Z_2 \times D_8$, Sylow's theorem yields that $z \sim_G h$, for some $h \in H - O_2(H)$. Thus there are involutions in $V - \langle z \rangle$ by Proposition 1 so $V \cong E_8$, D_8 or $Z_4 \times Z_2$.

We first consider the case when G does not contain a subgroup of index two.

If $V \cong E_8$ then $G \cong L_4(3)$ by Phan's result [**17**]. In this case G has two classes of involutions with $z \sim_G v \sim_G u \sim_G uv$ say and $\alpha_1\alpha_2 \sim_G uz \sim_G uvz \nsim_G z$.

If $V \cong D_8$ and v is of order four then $G \cong U_4(3)$ by another result of Phan [**18**]. We note that $U_4(3)$ has only one class of involutions. Suppose that $V \cong D_8$ and $v^2 = 1$. Without loss we take $(uv)^2 = z$. As $\Omega_1(C_H(v)) = \langle v, \alpha_1\alpha_2, z \rangle$, $z \sim_G v$ forces $z \sim_G \alpha_1\alpha_2$, since all involutions in $vO_2(H)$ are conjugate in H. Let $S = \langle u, \alpha_1\alpha_2, \beta_1\beta_2, z \rangle$, a Sylow 2-subgroup of $C_H(u)$. Clearly S is the unique (normal) elementary abelian subgroup of order 16 in $V \cdot O_2(H)$, a Sylow 2-subgroup of H (and hence of G). Since $N_H(S)/S \cong \Sigma_4$, $C_G(S) = S$ and z is conjugate to an involution in $S - \langle z \rangle$ in G, we have $N_G(S)/S \cong L_2(7)$ or $\mathcal{Q}_6$. In either case $z \sim_G \alpha_1\alpha_2$ and $vS \sim_{N(E)} \alpha_1\beta_2 S$ so that $z \sim_G v$ also. Finally, Proposition 3 yields

$z \sim_G u \sim_G uz$ and G has one class of involutions. As $C_H(\sigma_1)$, $C_H(\sigma_1\sigma_2^{-1})$ and $C_H(\sigma_1\sigma_2)$ have quaternion, cyclic and elementary abelian Sylow 2-subgroups, respectively, it follows that $\langle\sigma_1\rangle \nsim_G \langle\sigma_1\sigma_2\rangle \nsim_G \langle\sigma_1\sigma_2^{-1}\rangle$. Proposition 4 now yields that $z \sim_{C(\sigma_1\sigma_2)} u$. We have $C_H(\sigma_1\sigma_2)/\langle\sigma_1\sigma_2\rangle \cong D_{12}$ whence Gorenstein and Walter's result [**3**, Theorem 15.2.1] gives $C_G(\sigma_1\sigma_2)/\langle\sigma_1\sigma_2\rangle \cong L_2(11)$ or $L_2(13)$. It follows that either $5|\,|C_G(v)|$ or $7|\,|C_G(v)|$ as $v \in N_H(\langle\sigma_1\sigma_2\rangle)$. This contradicts $|C_G(v)| = |H|$, so this case cannot occur.

In the case when $V \cong Z_4 \times Z_2$ we show that G has a subgroup of index two. If $v^2 = 1$ then $\langle v, O_2(H)\rangle$ contains two classes of elements of order four in H, with representatives α_1, $v\alpha_1\alpha_2$. Obviously $u \nsim_G \alpha_1$, and as $C_H(u)$ has Sylow 2-subgroup isomorphic to $Z_4 \times Z_2 \times Z_2$ while $C_H(v\alpha_1\alpha_2) \cong Z_4 \times Z_4$, $u \nsim_G v\alpha_1\alpha_2$ either. By Harada's transfer lemma [**10**], G has a subgroup of index two. If $u^2 = 1$, $\langle u\rangle \cdot O_2(H)$ has two classes of elements of order four in H with representatives α_1, $u\alpha_1$. Obviously $v \nsim_G \alpha_1$ and as $C_H(v) \cong Z_4 \times Z_2 \times Z_2$ while $C_H(u\alpha_1) \cong Z_4 \times Z_4$, we also have $v \nsim_G u\alpha_1$. Harada's lemma again yields that G has a subgroup of index two. (Clearly, when $(uv)^2 = 1$, an identical argument gives the same result.)

Finally we consider the case when G has a subgroup G_0 of index two. Without loss we may assume $G_0 \cap H = \langle u\rangle O_2(H)$ or $\langle v\rangle O_2(H)$. In the first case, $G_0 \cong PSp_4(3)$ by Janko [**14**], and in the second case $G_0 \cong G_2(3)$ by another result of Janko [**15**]. It follows that $G \cong \operatorname{Aut} PSp_4(3)$, in which case $V \cong E_8$, or $G \cong \operatorname{Aut} G_2(3)$ (and $V \cong D_8$).

We conclude this section by listing various properties of the groups $PSp_4(3)$ and $\operatorname{Aut} PSp_4(3)$ which will be needed in the proofs of the theorems.

Some properties of $PSp_4(3)$. The group $PSp_4(3)$ is a simple group of order $2^6 \cdot 3^4 \cdot 5$. A Sylow 2-subgroup of $PSp_4(3)$ has centre of order two, and if t is an involution in the centre of a Sylow 2-subgroup, $C(t) = L_1 * L_2\langle u\rangle$, where $u^2 = 1$, $L_1 \cong \mathrm{SL}(2, 3)$ and $L_1^u = L_2$. If we take $L_i = \langle\alpha_i, \beta_i, \sigma_i\rangle$ (as above) we have the following relations for $C(t)$: $\sigma_1^u = \sigma_2$, $\alpha_1^u = \alpha_2$, $\beta_1^u = \beta_2$.

Note that $O_2(C(t)) = \langle\alpha_i, \beta_i | i = 1, 2\rangle \cong Q_8 * Q_8$ has one noncentral class of involutions in $C(t)$ with representative $\alpha_1\alpha_2$ say.

In $PSp_4(3)$ there are two classes of involutions with representatives t, ut where $t \sim u$ and $ut \sim \alpha_1\alpha_2$ (the coset $uO_2(C(t))$ contains two classes of involutions in $C(t)$ with representatives u, ut). The Sylow 2-subgroup $\langle u, O_2(C(t))\rangle$ contains precisely one elementary abelian subgroup of order 16, namely $S = \langle u, \alpha_1\alpha_2, \beta_1\beta_2, t\rangle$. We note that $C(S) = S$, $N(S)/S \cong \mathfrak{A}_5$ and that $C(ut)$ $(\subset N(S))$ is isomorphic to the centralizer of a noncental involution in $\mathfrak{A}_8$.

We have $\langle\sigma_1, \sigma_2\rangle$ is a Sylow 3-subgroup of $C(t)$ and $C(\langle\sigma_1, \sigma_2\rangle) = M\langle t\rangle$ where M is elementary of order 27, and $N(M)/M \cong \Sigma_4$. Let $\langle u, t, \beta\rangle$ be a Sylow 2-subgroup of $N(M)$ with $\langle u, t, \beta\rangle \subset N(\langle\sigma_1\sigma_2\rangle)$ and $Z(\langle u, t, \beta\rangle) = \langle ut\rangle$. $PSp_4(3)$ has four classes of elements of order three; σ_1, σ_1^{-1}, $\sigma_1\sigma_2$, $\sigma_1\sigma_2^{-1}$, and two classes of elements of order nine with cube conjugate to σ_1 or σ_1^{-1}. Further, $C(\sigma_1\sigma_2^{-1}) = M\langle t\rangle$ and $N(\langle\sigma_1\sigma_2^{-1}\rangle) = M\langle u, t\rangle$; $C(\sigma_1\sigma_2) = M\langle u, t\rangle$, $N(\langle\sigma_1\sigma_2\rangle) = M\langle u, t, \beta\rangle$; $C(\sigma_1) = N(\langle\sigma_1\rangle)$ is a split extension of a nonabelian group $O_3(C(\sigma_1))$ of order 27 and of exponent three, by L_2 $(\cong \mathrm{SL}(2, 3))$. If $x \in O_3(C(\sigma_1)) - \langle\sigma_1\rangle$ then $x \sim \sigma_1\sigma_2^{-1}$. A Sylow 3-subgroup of $PSp_4(3)$ is isomorphic to $Z_3 \operatorname{wr} Z_3$.

The subgroup $C(\sigma_1)$ is maximal in $PSp_4(3)$ and $O_{3,2}(C(\sigma_1))$ lies in precisely one maximal subgroup of $PSp_4(3)$, namely $C(\sigma_1)$.

Finally, if B is a Sylow 5-subgroup of $PSp_4(3)$ then $N(B)$ is a Frobenius group of order 20.

Some properties of Aut $PSp_4(3)$. It is well known that Aut $PSp_4(3)$ contains a subgroup of index two isomorphic to $PSp_4(3)$. Put Aut $PSp_4(3) = PSp_4(3)\langle v\rangle$ where v is an involution and $v \in C(t)$. We may choose v so that we have the following relations for $C(t)$:

$$[u, v] = 1, \quad \sigma_i^v = \sigma_i^{-1}, \quad \alpha_i^v = \alpha_i^{-1}, \quad \beta_i^v = \alpha_i\beta_i \qquad (i = 1, 2).$$

The Sylow 2-subgroup $\langle u, v, O_2(C(t))\rangle$ now contains four elementary abelian subgroups of order 16: $\langle v, u, t, \alpha_1\alpha_2\rangle \sim \langle v\alpha_1, u\beta_1\beta_2, t, \alpha_1\alpha_2\rangle$, and $\langle uv, \alpha_1\alpha_2, \alpha_1\beta_1\beta_2, t\rangle$, S which are both normal. In Aut $PSp_4(3)$, $N(S)/S \cong \Sigma_5$. Further, $C(v) \cong Z_2 \times Z_2 \times \Sigma_4$, and, taking $v \sim uvt$, $C(uv) \cong Z_2 \times \Sigma_6$. (In $C(t)$, $vO_2(C(t))$ contains one class of involutions, and $uvO_2(C(t))$ has two classes with representatives uv, uvt.)

From the relations in $C(t)$ we see that σ_1, σ_1^{-1} are conjugate in Aut $PSp_4(3)$ and so $N(\langle\sigma_1\rangle) = C(\sigma_1)\langle v\rangle$, $C(\sigma_1)$ clearly being the same as in $PSp_4(3)$. Also we have $C(\sigma_1\sigma_2^{-1}) = M\langle t, uv\rangle$ and $N(\langle\sigma_1\sigma_2^{-1}\rangle) = M\langle t, u, v\rangle$ while $C(\sigma_1\sigma_2) = M\langle t, u, v\beta\rangle$ and $N(\langle\sigma_1\sigma_2\rangle) = M(\langle vt\rangle \times \langle t, u, \beta\rangle)$. Note that $N(M)/M \cong Z_2 \times \Sigma_4$ and $C(\sigma_1\sigma_2)/M \cong D_8$. (Most of the properties of $PSp_4(3)$ listed above may be found in Janko [**14**].)

For the remainder of Part I we will adopt the following notation: if x was used to denote an element of $PSp_4(3)$ or Aut $PSp_4(3)$ above then x will denote an element of H such that xJ satisfies the same properties in H/J as x did in Aut $PSp_4(3)$. In addition t will belong to $C_H(\langle\sigma_1, \sigma_2\rangle)$ ($\langle\sigma_1, \sigma_2\rangle \cong E_9$), u, v will normalize $\langle\sigma_1, \sigma_2\rangle$, but we do not know if t is an involution in H or if $[u, t] = 1$. (We only have $t^2, [u, t] \in J$.)

2. A nonsimple case in Theorems A, B.

THEOREM 1. *Suppose that G, H, z satisfy the hypotheses of Theorems* A *or* B. *If, in addition, we have $C_J(\sigma_1) = C_J(\sigma_2)$ then $G = H \cdot O(G)$.*

The proof will be given in a series of lemmas which, taken together, will yield $z \not\sim_G h$ for any involution $h \in H - \langle z\rangle$. The theorem then follows immediately from Proposition 2. Much of the proof is independent of the different assumptions about H. However it will be necessary at times to refer to the three possibilities:

Case (i): $H/J \cong PSp_4(3)$, $Z(J) = \langle z\rangle$;

Case (ii): $H/J \cong$ Aut $PSp_4(3)$, $Z(J) = \langle z\rangle$;

Case (iii): $H/J \cong$ Aut $PSp_4(3)$, $Z(J) = \langle z, z_1\rangle \cong E_4$.

As H contains an element y with $y^3 = \sigma_1$, Proposition 5 yields $[\sigma_1, J] \cong Q_8 * Q_8 * Q_8$ and $J_0 = C_J(\sigma_1) = Z(J) * D$ where $D \cong D_8$ or Q_8. We suppose that $J_0 = C_J(\sigma_2)$ also. Since $\langle J, N_H(\langle\sigma_1\rangle), C_H(\sigma_2)\rangle = H$ (see §1) it follows that $J_0 \lhd H$. Note that if U is a Sylow 3-subgroup of $O_{2,3}(C_H(\sigma_1))$ then $U - \langle\sigma_1\rangle$ only contains conjugates of $\sigma_1\sigma_2^{-1}$. It follows therefore that $C_J(\sigma_1\sigma_2^{-1}) \cap [\sigma_1, J] \cong Q_8$. Thus

$C_J(\sigma_1\sigma_2) \cap [\sigma_1, J] \cong Q_8 * Q_8$ and $O_{2,3}(C_H(\sigma_1\sigma_2)) \cap C(J_0) = \langle\sigma_1\sigma_2\rangle \times Z(J) * L_3 * L_4$ where $L_3 \cong L_4 \cong \mathrm{SL}(2, 3)$. This yields that J_1 contains one class of involutions in H in cases (i), (ii) and two classes in case (iii) (with representatives $j_1, j_1z_1, j_1 \in [\sigma_1, J]$). In particular any involution in J is conjugate to an involution in $C_J(B)$ $(C_J(B) \cap [\sigma_1, J] \cong D_8)$.

LEMMA 1.1. *We have* $z \not\sim_G j$ *for any involution* $j \in J - \langle z\rangle$.

PROOF. Since $|H : C_H(Z(J))| \leq 2$, $O_2(C_G(Z(J))) = J$ whence $N_H(Z(J)) = N_G(Z(J))$. It follows that $z^G \cap Z(J) = \{z\}$ and that a Sylow 2-subgroup of H is a Sylow 2-subgroup of G. Let j be any involution in $J - J_1$, with K a Sylow 2-subgroup of $C_H(j)$. By Lemma 6 of **[16]** we have $\Omega_1(Z(K)) \subseteq \langle j, Z(J)\rangle$. Clearly $[H, H'] \cap J \subseteq J_1$ which means $[K, K'] \cap \Omega_1(Z(K)) \subseteq Z(J)$. On the other hand, $K' \cap J_1 \not\subseteq Z(K)$ so $\langle z\rangle \subset [K, K']$. It follows that $\langle z\rangle \lhd N_G(K)$ whence K is a Sylow 2-subgroup of $C_G(j)$; i.e. $z \not\sim_G j$.

Now let $j \in J_1 - Z(J)$ with $z \sim_G j$. From our remarks above we may suppose $j \in C_J(B)$ whence Proposition 4 implies $z \sim j$ in $C_G(B)$. Let L be a Sylow 2-subgroup of $C(j) \cap C_H(B)$. Since $|L : C_J(j) \cap L| \leq 2$, Proposition 7 of **[16]** yields that we must be in cases (ii) or (iii) with $L \cap J \cong Z_2 \times D_8$ or $Z_2 \times Z_2 \times D_8$, respectively. In addition, $L - J$ must contain involutions. However, an involution $x \in C_H(B) - C_J(B)$ must centralize $[B, J_1] \cong D_8 \times Q_8$. As $[x, J_1] \neq 1$, x acts as an outer automorphism on $C(B) \cap [\sigma_1, J]$ $(\cong D_8)$ whence $C(x) \cap C_{J_1}(B) = Z(J)$. It follows that $z \not\sim_G j$ and the lemma is proved.

Let T_i, $i = 1, 2, 3$, denote Sylow 2-subgroups of $C_H(\sigma_1)$, $C_H(\sigma_1\sigma_2^{-1})$, $C_H(\sigma_1\sigma_2)$, respectively. As $T_1 \cap J = J_0$ and $C_H(J_0)$ covers T_1/J_0 $(\cong Q_8)$, we must have $T_1 = J_0 \times T_1^*$, $T_1^* \cong Q_8$ (Q_8 has trivial multiplier **[13**, p. 643**]**); without loss we take $\langle t\rangle = \Omega_1(T_1^*)$. Clearly $[t, \langle\sigma_1, \sigma_2\rangle] = 1$ from which it follows that $[t, C_J(\sigma_1\sigma_2^{-1})] = 1$ and that t interchanges the two quaternion subgroups in $[\sigma_1\sigma_2^{-1}, J]$. Finally an easy calculation yields that any involution in tJ is conjugate (in H) to an involution in $tC_J(\sigma_1\sigma_2^{-1})$.

LEMMA 1.2. *We have* $\langle\sigma_1\rangle \not\sim_G \langle\sigma_1\sigma_2^{-1}\rangle \not\sim_G \langle\sigma_1\sigma_2\rangle$.

PROOF. From Lemma 1.1 and the fact that $\langle z\rangle \subset T_2' \subset T_2 \cap J$ we see that T_2 is a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1})$. As $|T_3| > 2^9$ and $|T_2| < 2^8$, we have $\sigma_1\sigma_2^{-1} \not\sim_G \sigma_1\sigma_2$. If $\langle\sigma_1\rangle \sim_G \langle\sigma_1\sigma_2^{-1}\rangle$ then there exists $g \in G - H$ with $T_1^g \subseteq T_2$, by Sylow's theorem. However $z \in T_1'$ implies $z^g \in T_2' \subseteq T_2 \cap J$. Thus $z^g = z$ by Lemma 1.1, a contradiction.

LEMMA 1.3. *For any involution* $x \in tJ$ *we have* $x \not\sim_G z$.

PROOF. Let x be an involution in tJ. By the remarks preceding Lemma 1.2 we may suppose $x \in tC_J(\sigma_1\sigma_2^{-1})$. Since $[t, C_J(\sigma_1\sigma_2^{-1})] = 1$, $z \in (C(x) \cap T_2)' \subset T_2 \cap J$. It follows from Lemma 1.1 that $C(x) \cap T_2$ is a Sylow 2-subgroup of $C(x) \cap C_G(\sigma_1\sigma_2^{-1})$ and that $z \not\sim x$ in $C_G(\sigma_1\sigma_2^{-1})$. Proposition 4 and Lemma 1.2 now yield $z \not\sim_G x$.

LEMMA 1.4. *No involution in* utJ *is conjugate to* z *in* G.

PROOF. In the notation of §1 we have $ut \in C_H(\sigma_1\sigma_2)$ and that ut inverts $\sigma_1\sigma_2^{-1}$. Thus $C(ut) \cap [\sigma_1, J] \cong Z_2 \times Q_8$ whence $C_J(x)$ contains a subgroup $K \cong Z_4 * Q_8$ for any (involution) $x \in utJ$. If $z \sim_G x$ then $C_G(x)$ contains a 2-subgroup Y with $|Y : X| = 2$, X a Sylow 2-subgroup of $C_H(x)$. Let $y \in Y - X$ and consider K^y ($\subset X$). Since $z^G \cap J = \{z\}$ (Lemma 1.1), $K^y \cap J = 1$ so $K^yJ/J \cong K \cong Z_4 * Q_8$. This is a contradiction, however, as $C_H(ut)J/J$ cannot contain a subgroup isomorphic to K (see §1).

We remark that the proof of Theorem 1 is now complete if we are in case (i). For the remaining two lemmas we assume therefore that we are in cases (ii) or (iii).

LEMMA 1.5. *If x is an involution in uvJ then $z \not\sim_G x$.*

PROOF. As we may assume $uv \in C_H(B)$, $[uv, [B, J]] = 1$ forces $C_J(x)$ to be nonabelian for any (involution) x in uvJ. Hence if X is a Sylow 2-subgroup of $C_H(x)$, we have $z \in X'$. Since $z^G \cap H' = \{z\}$ by the lemmas already proved, X is a Sylow 2-subgroup of $C_G(x)$ and $x \not\sim_G z$.

LEMMA 1.6. *There are no involutions in vJ conjugate to z in G.*

PROOF. Suppose x is conjugate to an involution in vJ. Then $z \sim_G x$ for some involution x in $uvtJ$ (see §1). From the proof of Lemma 1.5 we see that $C_J(x)$ must be elementary abelian. Hence $uvtJ$ has at most two classes of involutions in $\langle uvt, J\rangle$ so we may assume $x \in C_H(\sigma_1\sigma_2^{-1})$. By Lemma 1.2 and Proposition 4 we must have $z \sim x$ in $C_G(\sigma_1\sigma_2^{-1})$. Now $T_2 = \langle x, t, C_J(\sigma_1\sigma_2^{-1})\rangle$ is a Sylow 2-subgroup of $C_H(\sigma_1\sigma_2^{-1})$ and as $[t, C_J(\sigma_1\sigma_2^{-1})] = 1$, $K = C(x) \cap T_2$ must be elementary abelian.

If all involutions in $xC_J(\sigma_1\sigma_2^{-1})$ are conjugate to x (in T_2) then $\langle z^G \cap K\rangle = \langle x, C_J(\sigma_1\sigma_2^{-1})\rangle$, whence $\{y \mid y \in C_J(\sigma_1\sigma_2^{-1}) - \langle z\rangle\} \lhd N(K)$. Thus $\langle z\rangle \lhd N(K)$ which implies K is a Sylow 2-subgroup of $C(x) \cap C_G(\sigma_1\sigma_2^{-1})$ and $x \not\sim_G z$. It follows therefore that we are in case (iii), $K \cong E_{32}$ and z has 5 conjugates in $N(K) \cap C_G(\sigma_1\sigma_2^{-1})$. However $C_J(\sigma_1\sigma_2^{-1})K/K$ ($\cong E_4$) is a Sylow 2-subgroup of $N(K)/K$ (of order 20). This contradicts the structure of GL(5, 2) which completes the proof of the lemma.

3. The proof of Theorem A. In this section we will complete the proof of Theorem A by proving the following two results.

THEOREM 2. *Suppose G satisfies the assumptions of Theorem* A(i) (*i.e.* $H/J \cong PSp_4(3)$). *If, in addition, $G \neq H \cdot O(G)$ and $C_J(\sigma_1) \neq C_J(\sigma_2)$, then $G \cong U_6(2)$.*

THEOREM 3. *Suppose G satisfies the assumptions of Theorem* A(ii) (*i.e.* $H/J \cong$ Aut $PSp_4(3)$). *If, in addition, $G \neq H \cdot O(G)$ and $C_J(\sigma_1) \neq C_J(\sigma_2)$, then G contains a subgroup G_0 of index two, $G_0 \cong U_6(2)$ and $G \subseteq$ Aut G_0.*

We begin with some remarks on the structure of H which apply to both cases.

Since σ_2 acts nontrivially on $C_J(\sigma_1)$, Proposition 5 gives $C_J(\sigma_1) \cong Q_8$ and $J \cong Q_8 * Q_8 * Q_8 * Q_8$ (i.e. J is of type $+$). If $r \in C_J(\sigma_1) - \langle z\rangle$ then r has order four and $C_H(r)$ covers $O_{2,3,2}(C_H(\sigma_1))/C_J(\sigma_1)$, whence $C_H(r) \subset C_H(\sigma_1)J$. Thus r has at least 240 conjugates in H, so all elements of order four in J are conjugate in H. This yields $C_J(B) = \langle z \rangle$.

As any 3-element of $O_{2,3}(C_H(\sigma_1)) - \langle \sigma_1 \rangle$ is conjugate to $\sigma_1\sigma_2^{-1}$, we have (again using Proposition 5) $J = C_J(\sigma_1) * C_J(\sigma_2) * C_J(\sigma_1\sigma_2^{-1})$; i.e. $O_{2,3}(C_H(\sigma_1\sigma_2^{-1})) \cong Z_3 \times \mathrm{SL}(2, 3) * \mathrm{SL}(2, 3)$. Finally we see that $C_J(\sigma_1\sigma_2) = \langle z \rangle$ and that J has only one class of noncentral involutions in H.

Let T_1 be a Sylow 2-subgroup of $C_H(\sigma_1)$. As $T_1 \cap J \cong T_1/T_1 \cap J \cong Q_8$ and $C_H(T_1 \cap J)$ covers $T_1/T_1 \cap J$, we have $T_1 \cong Q_8 \times Q_8$. Without loss take $\Omega_1(T_1) = \langle t, z \rangle$ so that $[t, \sigma_2] = 1$ also. Let T_2, T_3 be Sylow 2-subgroups of $C_H(\sigma_1\sigma_2^{-1})$ and $C_H(\sigma_1\sigma_2)$, respectively, with $t \in T_i$, $i = 2, 3$. Let M be a Sylow 3-subgroup of $C_H(\langle \sigma_1, \sigma_2 \rangle)$ so that $M \cong E_{27}$ and $C_H(\langle \sigma_1, \sigma_2 \rangle) = M \cdot \langle t, z \rangle$. In the notation of §1 take $\langle u, t, z \rangle$ as a Sylow 2-subgroup of $N_H(\langle \sigma_1, \sigma_2 \rangle)$ in case (i) and $\langle v, u, t, z \rangle$ in case (ii). Without loss, assume that $t \sim_{N(M)} u$ ($t\langle z \rangle \sim_{N(M)} u\langle z \rangle$–see §1). Thus $\langle u, t, z \rangle \cong E_8$ or D_8 depending on whether $(ut)^2 = 1$ or z.

Since $\sigma_1^u = \sigma_2$ we have $C_J(\sigma_1)^u = C_J(\sigma_2)$ and $[C_J(\sigma_1\sigma_2^{-1}), u] = 1$. Conversely, $[t, C_J(\sigma_1) * C_J(\sigma_2)] = 1$ while t interchanges the two quaternion groups in $C_J(\sigma_1\sigma_2^{-1})$, so that $C_J(t) \cong E_4 \times Q_8 * Q_8$. Thus tJ contains 3 classes of involutions in $\langle t, \sigma_1, \sigma_2, J \rangle$ with representatives t, tz, tj where t, tz each have four conjugates, tj has 72 and j is any involution in $C_J(t) - Z(C_J(t))$. Let $F = C_J(ut) = C_J(u) \cap C_J(t)$ so that $F \cong E_{32}$. Hence utJ contains only 2 classes of elements with square in $\langle z \rangle$; we take ut, utz as representatives of the two classes.

Let T be a Sylow 2-subgroup of H with $T = J \cdot O_2(C_H(t))\langle u \rangle$ in case (i) and $T = JO_2(C_H(t))\langle u, v \rangle$ in case (ii). Observe that T is also a Sylow 2-subgroup of G because $C_G(J) = Z(J) = \langle z \rangle$. As $\langle u \rangle \cdot \langle \sigma_1, \sigma_2 \rangle$ acts irreducibly on $C_J(t)/Z(C_J(t)) \cong E_{16}$, it follows that $F \lhd T\langle \sigma_1\sigma_2 \rangle$. Clearly $F \lhd C_H(ut)$ so the structure of $PSp_4(3)$ yields that $N_H(F)/J \cong E_{16} \cdot \mathfrak{A}_5$ (case (i)), $E_{16} \cdot \Sigma_5$ (case (ii)). Take $E = C_H(F)$ and observe $u, t \in E$. Thus E covers $O_2(N_H(F))/J$ which yields $|E| = 2^9$ and $N_H(E)/E \cong N_H(F)/J$. Finally we have $E \cong E_{2^9}$ if $[u, t] = 1$ or $E \cong E_{16} \times D_8 * Q_8$ if $[u, t] = z$.

PROOF OF THEOREM 2. This will be carried out in a series of lemmas.

LEMMA 2.1. *The subgroup $\langle \sigma_1\sigma_2^{-1} \rangle$ is not conjugate to either $\langle \sigma_1 \rangle$ or $\langle \sigma_1\sigma_2 \rangle$ in G. Further we have $C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong PSp_4(3)$ or $C_G(\sigma_1\sigma_2^{-1}) = O(C_G(\sigma_1\sigma_2^{-1}))C_H(\sigma_1\sigma_2^{-1})$.*

PROOF. From the remarks above we see that $C_H(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle$ is isomorphic to the centralizer of a (central) involution in $PSp_4(3)$. The second statement now follows from Proposition 6. In particular, T_2 is a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1})$.

Clearly $\langle \sigma_1 \rangle \not\sim_G \langle \sigma_1\sigma_2^{-1} \rangle$ as $T_1 \cong Q_8 \times Q_8 \not\cong T_2$. If $T_3^g \subset T_2$ for some $g \in G$ then $|T_3| = 8$ implies $|C(z^g) \cap T_2| = 8$. This is not possible in T_2 so we have $\langle \sigma_1\sigma_2^{-1} \rangle \not\sim_G \langle \sigma_1\sigma_2 \rangle$, as required.

LEMMA 2.2. *For any involution $j \in J - \langle z \rangle$ we have $z \not\sim_G j$. In addition we may suppose $z \not\sim_G tz$.*

PROOF. Let $j \in T_2 \cap J - \langle z \rangle$ and recall $t \in T_2$. By Proposition 4 and Lemma 2.1 we have that z is conjugate to j, t or tz in G only if it is conjugate in $C_G(\sigma_1\sigma_2^{-1})$. If $C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong PSp_4(3)$ then we may assume $z \sim t \not\sim j \sim tz$ in $C_G(\sigma_1\sigma_2^{-1})$ (see §1). In the other case $\langle z \rangle$ is weakly closed in T_2 with respect to $C_G(\sigma_1\sigma_2^{-1})$ and so z is not conjugate to any of j, t, tz. The proof is completed by recalling that all involutions in $J - \langle z \rangle$ are conjugate in H.

LEMMA 2.3. *The subgroup E is elementary abelian of order 2^9 and is weakly closed in T with respect to G.*

PROOF. If E is not elementary abelian then $[u, t] = z$ whence $z \not\sim_G t$. By Proposition 2 we must have $z \sim_G tj_1$, j_1 an involution in $C_J(t) - Z(C_J(t))$. It follows that $\langle z^G \cap T \rangle = EJ$. Hence if K is a Sylow 2-subgroup of $C_H(tj_1)$ we have $z \in \langle K \cap z^G \rangle' \subset J \cap K$; i.e. $\langle z \rangle \lhd N_G(K)$. Sylow's theorem immediately yields $z \not\sim_G tj_1$ and we have proved that E is elementary abelian.

Let $g \in G$ with $E^g \subset T$. As $|E^g \cap J| \leqslant 32$ and $|E^g \cdot J/J| < 16$, we have $E^g \cap J \cong E_{32}$ and $E^g \subset EJ$. Thus E^g contains all (32) involutions in utJ so $E^g \cap J = F = C_J(ut)$. It follows immediately that $E^g = E$ as $E = C_G(F)$.

LEMMA 2.4. *The group G has precisely three classes of involutions; namely we have $z \sim_G t, j \sim_G tz \sim_G ut$ and $tj \sim_G utz$, where j is an involution in $C_J(t) - Z(C_J(t))$. In addition, $N_G(E)/E \cong L_3(4)$.*

PROOF. As all classes of involutions in H are represented in E, we must have $z \sim_G e$, $e \in E - \langle z \rangle$, by Proposition 2. It follows from Lemma 2.3 and Sylow's theorem that $z \sim_{N(E)} e$; i.e. $N_H(E) \neq N_G(E)$. We will now show $N_G(E)/E \cong L_3(4)$.

If $O(N_G(E)/E) \neq 1$ then there exists a subgroup J_1 of index two in J with J_1E/E centralizing a subgroup of odd order in $O(N_G(E)/E)$. This is not possible however as $\langle z \rangle \subseteq Z(JE) \subset J \cap E$. Hence we have $O(N_G(E)/E) = 1$. As $\sigma_1\sigma_2$ acts fixed-point-free on JE/E, T/E is of type $L_3(4)$ (see [**6**, Lemma 2.6, pp. 79–80]). It follows from a result of Gorenstein and Harada [**4**, Theorem C] that $N_G(E)/E$ is isomorphic to a subgroup of $PGL(3, 4)$. Let B be a Sylow 5-subgroup of $N_H(E)$. Then $C_E(B) = \langle z \rangle$ which implies that $N_G(E)' = N_G(E)$. Thus $N_G(E)/E \cong L_3(4)$ as $N_G(E) \supset N_H(E)$.

We observe that $|N_G(E) : N_H(E)| = 21$ so z has 21 conjugates in E. Hence $z \sim_G t$ and $tz \sim_G j$ by Lemma 2.1. Let $\langle \sigma, \sigma_1\sigma_2 \rangle$ be a Sylow 3-subgroup of $N_G(E)$. Clearly z, t, u are conjugate under the action of σ on $C_E(\sigma_1\sigma_2) = \langle u, t, z \rangle$; it follows therefore that tz, uz, ut are conjugate under σ and $C(\sigma) \cap C_E(\sigma_1\sigma_2) = \langle utz \rangle$. Thus j has at least 210 conjugates in $N_G(E)$. Since utz, tj have 160, 120 conjugates in $N_H(E)$, respectively, the order of $L_3(4)$ and the fact that a Sylow 7-normalizer of $L_3(4)$ is a Frobenius group of order 21 gives $j \not\sim_{N(E)} utz \sim_{N(E)} tj$. The lemma follows from the observation once again that two involutions of E are conjugate in G if and only if they are conjugate in $N_G(E)$.

LEMMA 2.5. *We have $C_G(tj) \subset N_G(E)$ and $C_G(tj)/E \cong U_3(2) \cong E_9 \cdot Q_8$.*

PROOF. As $|C(tj) \cap N(E)/E| = 2^3 \cdot 3^2$ and $N_G(E)/E \cong L_3(4)$, we have $C_G(tj) \cap N(E)/E \cong E_9 \cdot Q_8$ (a Sylow 3-normalizer in $L_3(4)$). Let R be a Sylow 3-subgroup of $C(tj) \cap N_G(E)$ and V a Sylow 2-subgroup of $N(R) \cap C(tj) \cap N_G(E)$. From $V \cap E = \langle tj \rangle$ we conclude $V \cong Z_2 \times Q_8$. Since $C_J(tj) \cong E_8 \times D_8$ there is an involution $j_0 \in C_J(tj) - E$. By Proposition 1 we may assume $j_0 \in V$ (so $\Omega_1(V) = \langle j_0, jt \rangle$) and by Sylow's theorem we may assume $V \subset T$.

Let $Y = EV = T \cap C(tj) \cap N_G(E)$ and let $X = C_G(tj)$. As E char Y, Y must be a Sylow 2-subgroup of X. Further, we compute that j_0E contains two classes of involutions in Y with representatives j_0, j_0tj and that $C_E(V) = \langle tj, z \rangle$. Thus $\{z\} = z^G \cap Z(C_Y(j_0))$ whence Sylow's theorem yields $C_Y(j_0)$ is a Sylow 2-subgroup of $C_X(j_0)$. In particular no involution in $Y - E$ is conjugate to an involution in E in X. An application of Grün's first theorem [3, Theorem 7.4.2] gives $Y \cap X' \subseteq \Omega_1(Y)$. Hence X contains a normal subgroup X_1 of index four with $X_1 \cap Y = \Omega_1(Y) = \langle E, j_0 \rangle$. Proposition 3 applied to X_1 yields that X_1 has a subgroup of index two which does not contain j_0. It follows that X_1 possesses a subgroup X_2 of index two with $N(E) \cap X_2 = E \cdot R$. We conclude that $N_X(V) = VC_X(V)$, which implies $N_X(R) \cap X_2 = C_X(R)$.

If $C_X(R) = R \times \langle tj \rangle$ $(= C(R) \cap N_G(E))$, two applications of Burnside's transfer theorem [3, Theorem 7.4.3] yield $X_2 = O(X_2) \cdot E \cdot R$. The lemma will follow if we can show $C_X(r) \subseteq R \cdot E$, $r \in R^{\#}$, and $O(X) = 1$. To do this, we use the fact that $utz \sim_{N(E)} tj$ and show $C(\sigma_1\sigma_2) \cap C_G(utz) \subset N_G(E)$ and $O(C_G(utz)) = 1$.

As u, t, z are all conjugate in $C_G(utz) \cap N_G(E)$, each of u, t, z acts fixed-point-free on $O(C_G(utz))$. Thus $O(C_G(utz)) = 1$. Recall that

$$\begin{aligned} C(\sigma_1\sigma_2) \cap C_H(utz) &= \langle \sigma_1\sigma_2, \sigma \rangle \cdot \langle u, t, z \rangle \\ &= \langle \sigma_1\sigma_2 \rangle \times \langle utz \rangle \times \langle uz, tz \rangle \langle \sigma \rangle \cong Z_3 \times Z_2 \times \mathfrak{A}_4. \end{aligned}$$

Clearly, $C_G(ut) \cap C_G(utz) = C_H(utz)$, so we conclude that $C(ut) \cap C(\sigma_1\sigma_2) \cap C_G(utz)$ is abelian. A result of Suzuki (see [19]; or see [3, pp. 420–423]) yields $C(\sigma_1\sigma_2) \cap C_G(utz)/\langle \sigma_1\sigma_2, utz \rangle \cong \mathfrak{A}_4$ or $\mathfrak{A}_5$. The second case is not possible as $R \sim_{N(E)} \langle \sigma, \sigma_1\sigma_2 \rangle$ and $N_X(R) = C_X(R) \cdot V$. Thus $C(\sigma_1\sigma_2) \cap C_G(utz) \subset N_G(E)$ as required.

LEMMA 2.6. *If* $W = \langle tz \rangle \times Z(C_J(tz))$ *then*

$$C_G(W) = O_2(C_H(tz)) \cdot \langle \sigma_1\sigma_2^{-1}, u \rangle,$$
$$N_G(W) = C_G(W) \cdot \big(N_G(W) \cap C_G(\sigma_1\sigma_2^{-1})\big)$$

and $N_G(W)/C_G(W) \cong \mathfrak{A}_5$. *Further, we have* $C_G(tz) \subset N_G(W)$ *so that* $C_G(tz) = C_G(W)\langle \sigma_1\sigma_2, l \rangle$ *where* $\langle \sigma_1\sigma_2, l \rangle \cong \Sigma_3$, $z^l = t$ *and* $l \in N_G(E)$.

PROOF. We have $W = \langle tz \rangle \times Z(C_J(tz)) \cong E_{16}$, $W \subseteq E \cap C_H(\sigma_1\sigma_2^{-1})$ and $W \lhd C_H(tz)$. By Lemmas 2.1 and 2.4, $C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong PSp_4(3)$. Since u inverts $\sigma_1\sigma_2^{-1}$ and centralizes a complement to $\langle \sigma_1\sigma_2^{-1} \rangle$ in $C_H(\sigma_1\sigma_2^{-1})$, it follows that $N_G(\langle \sigma_1\sigma_2^{-1} \rangle) = \langle \sigma_1\sigma_2^{-1}, u \rangle \times P$ where $P \cong PSp_4(3)$. Now $N_P(W)/W \cong \mathfrak{A}_5$ (see §1) and as z, tz have precisely 5, 10 conjugates in W, it follows (from the structure of $\mathfrak{A}_8$) that $N_G(W) = N_P(W)C_G(W)$. In particular,

$$C_G(W) = O_2(C_H(tz)) \cdot \langle \sigma_1\sigma_2^{-1}, u \rangle \quad (C_H(tz) = C_G(W)\langle \sigma_1\sigma_2 \rangle).$$

Since tz has 10 conjugates in $N_P(W)$, $C_G(tz) \cap N_P(W) = W\langle \sigma_1\sigma_2, l\rangle$ where $\langle \sigma_1\sigma_2, l\rangle \cong \Sigma_3$. Clearly $l \notin H$ whence $z^l = t$ as $C_W(\sigma_1\sigma_2) = \langle t, z\rangle$. Finally $[l, u] = 1$ so l normalizes $O_2(C_H(tz))\langle u\rangle \supseteq E$ and hence $l \in N_G(E)$.

It remains to show that $C_G(tz) \subset N_G(W)$. Firstly we note that

$$C_G(tz) \cap N_G(E) = O_2(C_H(tz)) \cdot (\langle u\rangle \times \langle \sigma_1\sigma_2, l\rangle) \subset N_G(W),$$

as tz has 210 conjugates in E. Clearly $L = \langle O_2(C_H(tz)), u, l\rangle$ is a Sylow 2-subgroup of $C_G(tz)$, $E \lhd L$ and z has only 2 conjugates in $N(E) \cap C_G(tz)$, namely t, z. As $z^G \cap L \subset E$, Proposition 2 applied to $C_G(tz)/\langle tz\rangle$ yields

$$C_G(tz) = O(C_G(tz)) \cdot N_G \cdot (\langle t, z\rangle).$$

Since $tj \sim_G tjz$ and $C_G(tj) \subset N_G(E)$ (Lemma 2.5), each of z, tj and tjz must act fixed-point-free on $O(C_G(tz))$. Thus $O(C_G(tz)) = 1$ and $C_G(tz) = N_G(\langle t, z\rangle)$ whence $C_G(tz) \subset N_G(W)$. The lemma is proved.

LEMMA 2.7. *The order of G is $2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$.*

PROOF. Thompson's order formula and Lemmas 2.4–2.6 give

$$|G| = 2^{15} \cdot 3^4 \cdot 5\, a(tj) + 2^{13} \cdot 3^2\, a(z) + 2^{13} \cdot 3^4 \cdot 5\, a(tz)$$

where, for any involution $g \in G$,

$$a(g) = \left|\{(x, y) | (xy)^n = g \text{ for some positive integer } n, \text{ with } x \sim_G z, y \sim_G tj\}\right|.$$

In the computations for $a(tj)$ and $a(tz)$ we will use the notation introduced in Lemmas 2.5 and 2.6. Recall that $|z^G \cap N_G(E)| = |z^G \cap E| = 21$ and that $z^G \cap H = \{z\} \cup t^H$. For the rest of the proof we will assume $x \sim_G z, y \sim_G tj$.

$a(tj) = 9$. Since $[R, E] \times \langle tj\rangle = E$, and $[R, E] \lhd C_G(tj)$, we may assume $y \in E$ if $(xy)^n = tj$. In $C_G(tj)$, z has 9 conjugates, while in $C_G(utz)$, z has 12 conjugates. The result follows because $tjz \sim_G tj$ while $ut \sim_G tz$.

$a(z) = 0$. If $(xy)^n = z$ we easily verify that $x \neq z$ and if $x = t$, $y \notin C_H(t)$. Suppose $x = t$ and $y \in H - C_H(t)$. If $[t, y] \in J$, we only have to consider $y = uj_2$, $j_2 \in C_J(u) - C_J(t)$. However $(tuj_2)^2 = (tj_2)^2 \neq z$ as $z \notin [t, J]$. Suppose next that $tJ \nsim_H yJ$, so that $(ty)^m$ lies in a conjugate of tJ or utJ, for some m. Now $F = \langle z\rangle \times [ut, J]$ and $[u, T] \cap tJ = \varnothing$ (see §1) so $(ty)^m J \sim_H tJ$ and $(ty)^q \sim_H \sigma_1\sigma_2^{-1}$ for some integer q. Similarly if $tJ \sim_H yJ$, $(ty)^q \sim_H \sigma_1\sigma_2^{-1}$ for some q (if the product of two central involutions in $PSp_4(3)$ has odd order, the product is conjugate to $\sigma_1\sigma_2^{-1}$). Finally we check that there are no x, y in $N_H(\langle \sigma_1\sigma_2^{-1}\rangle)$ with $(xy)^n = z$ for any n.

$a(tz) = 2^4 \cdot 3^2 \cdot 19$. Recall that

$$N_G(\langle \sigma_1\sigma_2^{-1}\rangle) = \langle \sigma_1\sigma_2^{-1}, u\rangle \times P, \quad P \cong PSp_4(3),$$

and

$$C_G(tz) = O_2(C_H(tz)) \cdot (\langle \sigma_1\sigma_2^{-1}, u\rangle \times \langle \sigma_1\sigma_2, l\rangle).$$

Since P has two classes of involutions with representatives z, tz, it follows that $l \sim_G tz$ and $ul \sim_G utz \sim_G tj$. We put $D = O_2(C_G(tz)) = O_2(C_H(tz))$ and compute that ul has $2^6 \cdot 3^2$ conjugates in $C_G(tz)$ and all involutions in ulD are conjugate to ul. Further, all involutions in lD are conjugate to l while in uD, u has 16 conjugates,

uj_0 ($\sim_G tj$) has 96 ($j_0 \in Z(C_J(t))$), utz has 64 and all other involutions (there are 80) in uD are conjugate to tz in G. (All involutions in uD lie in E.) Finally we have $|z^G \cap D| = 5$ while $|z^G \cap C_G(tz) - D| = |u^{C(tz)}| = 48$.

Suppose $(xy)^n = tz$ and firstly that $x \in D$. If $x = z$ we get no pairs if $y \in D$ but $2^6 \cdot 3^2$ pairs if $y \sim ul$ in $C_G(tz)$. When $x \not\sim z$ in $C_G(tz)$ we take $x = z$ in $C_G(j)$ and easily show there are no pairs in this case. Hence there are $2^7 \cdot 3^2$ pairs if $x \in D$.

Now suppose $x = u$ and observe that $u \cdot utz = tz$ so we get one pair if $y \in C_G(u)$. Since $[\langle u \rangle, D] \subseteq E - tz(E \cap J)$ we get zero pairs if $y \in D$. If $y \in ul_1D$ where $l_1D \sim lD$ then $(u \cdot ul_1d)^2 = (l_1d)^2 = [u, d]$ for those $d \in D$ with ul_1d an involution. There are zero pairs therefore in this case. Next suppose $y \in u_1l_1D$, $uD \neq u_1D \sim uD$, $l_1D \sim lD$ in $C_G(tz)$. Then $(uy)^3 \in l_1D$ and $\langle u, y \rangle D = (\langle u, \sigma_1\sigma_2^{-1} \rangle \times \langle l_1 \rangle) \cdot D$. As $C_G(u) \supseteq C_G(\sigma_1\sigma_2^{-1}) \cap \langle l_1, D \rangle$, $(uy)^3$ is an involution, so we again get zero pairs. Finally if $y \in u_1D$ for $uD \neq u_1D \sim uD$ then $\langle u, y \rangle D = \langle \sigma_1\sigma_2^{-1}, u \rangle D$. Since $utz \sim_G tj$ the structure of $C(\sigma_1\sigma_2^{-1}) \cap C_G(tz)$ shows that we get two pairs for each conjugate of $\langle \sigma_1\sigma_2^{-1} \rangle$ normalized by u. There are 16 such conjugates. We therefore get 48×33 pairs if $x \notin D$. The lemma follows immediately.

LEMMA 2.8. *There exists $k \in G$ such that $G = H \cup HlH \cup HkH$ where $|H^l \cap H| = 2^{13} \cdot 3^2$ and $|H^k \cap H| = 2^6 \cdot 3^4 \cdot 5$. Thus G acts as a rank three permutation group on the cosets of H.*

PROOF. As $H^l = C_G(t)$, $H^l \cap H = C_H(t)$ whence $|H^l \cap H| = 2^{13} \cdot 3^2$. Since $N_G(\langle \sigma_1\sigma_2^{-1} \rangle) = \langle u, \sigma_1\sigma_2^{-1} \rangle \times P$, there exists $u_1 \in N_G(\langle \sigma_1\sigma_2^{-1} \rangle)$, $u_1 \sim_G u$ such that

$$C_G(u_1) \cap C_G(u) \cap N(\langle \sigma_1\sigma_2^{-1} \rangle) = P.$$

Hence there exists $k \in G$ with $C_H(z^k) \cdot J = H$, as $u \sim_G z$. Finally, as $z^k \notin H$ we have $C_H(z^k) \cap J = 1$; i.e. $C_H(z^k) \cong PSp_4(3)$. The lemma follows from Lemma 2.7.

LEMMA 2.9. *The group G is simple and is isomorphic to $U_6(2)$.*

PROOF. From Lemma 2.6, $O(C_G(tz)) = 1$ whence z, t, tz all act fixed-point-free on $O(G)$. It follows that $O(G) = 1$. If $1 \neq N \lhd G$, $z \in N$ because $Z(T) = \langle z \rangle$ and $O(G) = 1$. We observe that $\langle z^G \cap H \rangle = H$ so $H \subseteq N$. Thus $N = G$ as $N_G(T) = T$ ($N_G(T) \subseteq H$), and G is a simple group.

In the notation of Lemma 2.8, we have $(zz^l)^2 = (zt)^2 = 1$ and $(z \cdot z^k)^3 = 1$. (Note that k was chosen in Lemma 2.8 so that

$$N_G(\langle zz^k \rangle) = \langle z, z^k \rangle \times C_H(z^k) \sim_G N_G(\langle \sigma_1\sigma_2^{-1} \rangle) \cong \Sigma_3 \times PSp_4(3).)$$

Thus z^G is a class of 3-transpositions. We conclude that $G \cong U_6(2)$ because of Fischer's result [**1**].

PROOF OF THEOREM 3. We now suppose G, H, z satisfy the assumptions of Theorem 3. To begin the proof we make some remarks on the structure of H in this case.

Recall that $\langle v, t, u, z \rangle$ is a Sylow 2-subgroup of $N_H(\langle \sigma_1, \sigma_2 \rangle)$, with $\sigma_i^v = \sigma_i^{-1}$, $i = 1, 2$. Take $\langle v, t, u, \beta, z \rangle$ to be a Sylow 2-subgroup of $N_G(M)$, where $\langle t, u, \beta, z \rangle / \langle z \rangle \cong D_8$ and note that $N_H(M)/C_H(M) \cong Z_2 \times \Sigma_4$. As $vJ \sim_H uvtJ$

and $C_M(ut) = \langle \sigma_1\sigma_2 \rangle$, we have $C_M(vt) = 1$ whence $vtC_H(M) \in Z(N_H(M)/C_H(M))$, and clearly $[\langle vt \rangle, \langle tu, \beta \rangle] = 1$. Without loss we may assume $v \in N_H(T_1)$ so that $[v, t] = 1$ also (recall $T_1 \cong Q_8 \times Q_8$, $\mho_1(T_1) = \langle t, z \rangle$). It follows immediately that $[\langle vt \rangle, \langle u, t \rangle] = 1$. In addition we will take $T_2 = \langle vu, t \rangle \cdot C_J(\sigma_1\sigma_2^{-1})$ and $T_3 = \langle t, u, z, v\beta \rangle$ (note that $T_3/\langle z \rangle \cong D_8$).

Since vt inverts $\langle \sigma_1, \sigma_2 \rangle$ and $[vt, u] = 1$, we easily compute $C_J(vt) \cong C_J(v) \cong E_{16}$, $C_J(uv) \cong E_{32}$. Further if $h \in H - H'$ with $h^2 \in \langle z \rangle$ then h is conjugate to one of v, uv, uvz in H.

LEMMA 3.1. *The elements σ_1, $\sigma_1\sigma_2^{-1}$, $\sigma_1\sigma_2$ lie in distinct conjugate classes in G.*

PROOF. As in the proof of Lemma 2.1, T_2 is a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1})$. If there exists $g \in G$ with $T_1^g \subset T_2$ then $z^g \in T_2' \subseteq T_2 \cap J$. However $\Omega_1(T_1) = \langle t, z \rangle$ while $C_J(z^g) \cap T_2 \cong Z_2 \times D_8$. We conclude that $\sigma_1 \not\sim \sigma_1\sigma_2^{-1}$. If $T_3^g \subset T_2$ for some $g \in G$ then $|C(z^g) \cap T_2| = 16$, whence there exists $h \in C_H(\sigma_1\sigma_2^{-1})$ with $z^{gh} = vut$. However $C(vut) \cap T_2$ must be abelian in this case as $z \notin \langle t, C_J(\sigma_1\sigma_2^{-1})\rangle'$ and $|C_J(v) \cap T_2| = 4$. Clearly this contradicts $T_3/\langle z \rangle \cong D_8$. It remains to show $\sigma_1 \not\sim_G \sigma_1\sigma_2$. Suppose that T_1 is not a Sylow 2-subgroup of $C_G(\sigma_1)$. By Sylow's theorem there is a subgroup T_1^* of $C_G(\sigma_1)$ with $|T_1^*: T_1| = 2$. It follows that $T_1^* \cong Q_8 \text{ wr } Z_2$ and $\langle z, t \rangle$ char T_1^*. Thus T_1^* is a Sylow 2-subgroup of $C_G(\sigma_1)$. In either case it is immediate that neither T_1^* nor T_1 contains a subgroup isomorphic to T_3. We conclude that $\sigma_1 \not\sim_G \sigma_1\sigma_2$.

LEMMA 3.2. *Either $C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong$ Aut $PSp_4(3)$ or $L_4(3)$, or $C_G(\sigma_1\sigma_2^{-1}) = O(C_G(\sigma_1\sigma_2^{-1})) \cdot C_H(\sigma_1\sigma_2^{-1})$. In addition, $z \not\sim_G j$ for any involution $j \in J - \langle z \rangle$ and we may assume $z \not\sim_G tz$.*

PROOF. By Proposition 4 and Lemma 3.1, $z \sim_G tz$ if and only if $z \sim tz$ in $C_G(\sigma_1)$. Recall $T_1 = T_{11} \times T_{12}$, $T_{11} \cong T_{12} \cong Q_8$ and assume $T_{11}' = \langle z \rangle$, $T_{12}' = \langle t \rangle$. It follows immediately from the last part of the proof of Lemma 3.1 and Burnside's lemma [3, Theorem 7.1.1] that $z \not\sim_G tz$.

Suppose that $C_G(\sigma_1\sigma_2^{-1}) \neq C_H(\sigma_1\sigma_2^{-1})O(C_H(\sigma_1\sigma_2^{-1}))$. By Proposition 6, $C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong L_4(3)$, $U_4(3)$, Aut $PSp_4(3)$ or Aut $G_2(3)$. Since $z \not\sim_G tz$ and $U_4(3)$ has one class of involutions, $U_4(3)$ is not a possibility. We show that $C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \not\cong G_2(3)$ and the lemma follows, as in both of the two remaining cases, $z \not\sim j$ in $C_G(\sigma_1\sigma_2^{-1})$.

If $C_G(\sigma_1\sigma_2^{-1})/\langle \sigma_1\sigma_2^{-1} \rangle \cong$ Aut $G_2(3)$ then $\langle vu, t \rangle \cong D_8$, hence $\langle u, t \rangle \cong D_8$. Let K be a subgroup of index two in $C_G(\sigma_1\sigma_2^{-1})$ with $K/\langle \sigma_1\sigma_2^{-1} \rangle \cong G_2(3)$. Then $K \cap T_2 = \langle vut, C_J(\sigma_1\sigma_2^{-1}) \rangle$ as vut normalizes the two quaternion subgroups of $C_G(\sigma_1\sigma_2^{-1})$ while t, vu interchange them. (Recall that vt inverts M, a Sylow 3-subgroup of $C_H(\sigma_1\sigma_2^{-1})$, and u centralizes a complement to $\langle \sigma_1\sigma_2^{-1} \rangle$ in $O_{2,3}(C_H(\sigma_1\sigma_2^{-1}))\langle vt \rangle$.) The structure of $G_2(3)$ (see [15]) yields that either u or uz centralizes $C_G(\sigma_1\sigma_2^{-1})' \cong G_2(3)$; i.e. $N_G(\langle \sigma_1\sigma_2^{-1} \rangle)'\langle u \rangle \cong \Sigma_3 \times G_2(3)$. Hence $[t, u] = 1$, a contradiction. The lemma is proved.

LEMMA 3.3. *The subgroup E is elementary abelian of order 2^9 and weakly closed in T with respect to G.*

PROOF. If $C_G(\sigma_1\sigma_2^{-1})/\langle\sigma_1\sigma_2^{-1}\rangle \cong L_4(3)$ or Aut $PSp_4(3)$ then $\langle vu, t, z\rangle \cong E_8$ and hence $\langle u, t, z\rangle \cong E_8$; i.e. $E \cong E_{2^9}$. For the case when $\langle z\rangle$ is weakly closed in $C_H(\sigma_1\sigma_2^{-1})$ with respect to $C_G(\sigma_1\sigma_2^{-1})$, the proof of the first part of Lemma 2.3 may be repeated to yield $E \cong E_{2^9}$.

In order to prove E is weakly closed in T, observe that $E^g \subset T$ $(g \in G)$ implies $E^g \cap eJ \neq \varnothing$ for some $e \in E$ with $eJ \sim_H utJ$ (see §1 for a list of subgroups of Aut $PSp_4(3)$ isomorphic to E_{16}). As $C_J(e) = F$ and $E = C_G(F)$, we have $E^g = E$ as required.

LEMMA 3.4. *The subgroup $N_G(E)$ contains a subgroup K of index two with $K/E \cong L_3(4)$ and $K \cap H = N_G(E) \cap H'$.*

PROOF. By Lemma 3.2, $z \sim_G e$ for some $e \in E - \langle z\rangle$. It follows therefore from Lemma 3.3 that $z \sim e$ in $N_G(E)$ whence $N_G(E) \neq N_H(E)$. If $N_G(E)$ contains a subgroup K with $|N_G(E)/E : K/E| = 2$ then $K/E \cap N_H(E)/E \cong E_{16}\mathfrak{A}_5$ and $K/E \cong L_3(4)$ as in the proof of Lemma 2.4.

For $j_1 \in J - E$, $\langle z\rangle \subset [\langle j_1\rangle, E] \subset J \cap E$ whence $C_{N(E)/E}(j_1E) = T^h/E$, $h \in N_H(E)$. If we assume that $N_G(E)/E$ has no subgroup of index two then $N_G(E)/E$ is "fusion-simple" (in the sense of [5]). Further, T/E, a Sylow 2-subgroup of $N_G(E)/E$ is of type $\hat{A}_8$ (see [6, Lemma 2.6, pp. 79–80]). However these conditions contradict a result of Gorenstein and Harada [5, Theorem A]. Thus $N_G(E)/E$ must possess a subgroup of index two and the lemma is proved.

LEMMA 3.5. *There are 3 classes of involutions in E in $N_G(E)$ with representatives z, tz, utz. These classes are not fused in G. Further, $C_G(\sigma_1\sigma_2^{-1})/\langle\sigma_1\sigma_2^{-1}\rangle \cong$ Aut $PSp_4(3)$ and for any involution $h \in H - H'$, we have $z \not\sim_G h$.*

PROOF. We have $z \sim t \not\sim j \sim tz \sim ut \not\sim utz \sim tj$ in $N_G(E)$ as in Lemma 2.4 (j an involution in $C_J(t) - Z(C_J(t))$). Further, these classes are not fused in G because of Lemma 3.3. As $z \sim_G t$, Proposition 4 and Lemmas 3.1, 3.2 yield $C_G(\sigma_1\sigma_2^{-1}) \neq O(C_G(\sigma_1\sigma_2^{-1}))C_H(\sigma_1\sigma_2^{-1})$.

Suppose that $C_G(\sigma_1\sigma_2^{-1})/\langle\sigma_1\sigma_2^{-1}\rangle \cong L_4(3)$. Since $\langle u, z\rangle$ centralizes $C_H(\sigma_1\sigma_2^{-1})/\langle\sigma_1\sigma_2^{-1}\rangle$, the structure of $L_4(3)$ (see [17]) and $u \sim_G z$ yield $N_G(\langle\sigma_1\sigma_2^{-1}\rangle) = \langle\sigma_1\sigma_2^{-1}, uz\rangle \times L$, $L \cong L_4(3)$. However $t \sim_G z$ forces $t \cdot uz \sim z \cdot uz = u$ in $N_G(\langle\sigma_1\sigma_2^{-1}\rangle)$, clearly a contradiction. Thus $C_G(\sigma_1\sigma_2^{-1})/\langle\sigma_1\sigma_2^{-1}\rangle \cong$ Aut $PSp_4(3)$ and, as $z \sim t$ in $C_G(\langle\sigma_1\sigma_2^{-1}\rangle)$, we have z is not conjugate to any involution in $T_2 - \langle t, C_J(\sigma_1\sigma_2^{-1})\rangle$ in $C_G(\sigma_1\sigma_2^{-1})$.

LEMMA 3.6. *The group G contains a subgroup G_0 of index two with $G_0 - U_6(2)$.*

PROOF. Recall that $C_J(uv) \cong E_{32}$ and $C_H(uv)/C_J(uv) \cong Z_2 \times \Sigma_6$ (see §1). Let Y denote a Sylow 2-subgroup of $C_H(uv)$. Clearly $\langle z\rangle \subset Z(Y) \subset \langle J, uv\rangle$ whence $z^G \cap Z(Y) = \{z\}$ by Lemmas 3.5 and 3.2. Thus Y is a Sylow 2-subgroup of $C_G(uv)$, and as $|Y| = 2^{10}$, $uv \not\sim_G tz$ and $uv \not\sim_G tj$; i.e. $uv \not\sim_G x$ for any involution $x \in H'$. It follows immediately from Proposition 3 that G contains a subgroup G_0 of index two with $uv \in G - G_0$. Thus $G_0 \cap H = H'$ whence $G_0 \cong U_6(2)$ by Theorem 2. The lemma is proved.

As $G \subseteq \mathrm{Aut}\, G_0$, the proof of Theorem 3 and hence that of Theorem A has been completed.

4. The proof of Theorem B. Throughout this section we will work under the following assumptions.

Hypothesis 1. *Let G be a finite group, z an involution in G and $H = C_G(z)$. Suppose that $G \neq H \cdot O(G)$ and that H satisfies*:

(I) *$J = O_2(H)$ is the direct product of a group of order two and an extra-special subgroup of order 2^9, $J' = \langle z \rangle$ and $C_H(J) \subseteq J$;*

(II) *$H/J \cong \mathrm{Aut}\, PSp_4(3)$;*

(III) *$C_J(\sigma_1) \neq C_J(\sigma_2)$.*

We will put $Z(J) = \langle z, z_1 \rangle$ and otherwise use the same notation as in the proof of Theorem 3 in §3.

As $T_1 \cap J \cong Z_2 \times Q_8$ and $T_1/J \cap T_1 \cong Q_8$, $T_1/Z(J) \cong E_4 \times Q_8$ and $T_1 = T_1^*(T_1 \cap J)$, $T_1^* \cong Q_8$ (although T_1 is not necessarily a direct product this time). We take $\Omega_1(T_1) = Z(T_1) = \langle t, z, z_1 \rangle$. Further, $T_2 = (\langle z_1 \rangle \times [M, C_J(\sigma_1\sigma_2^{-1})]) \cdot \langle t, uv \rangle$ and $T_3 = \langle z_1, z \rangle \cdot \langle t, u, v\beta \rangle$ with $T_3/Z(J) \cong D_8$. Note that $Z(T_2) \subseteq Z(J)$ and $Z(T_3) \subseteq \langle Z(J), ut \rangle$.

In the same way as in §3 we compute that $J - Z(J)$ has two classes of involutions with representatives j, jz_1–choose j in $[M, C_J(\sigma_1\sigma_2^{-1})]$. (To see $j \not\sim_H jz_1$ use the fact that $j \not\sim jz_1$ in $C_H(\sigma_1\sigma_2^{-1})$.) Further t, tz, tz_1, tz_1z each have 4 conjugates and tj_1, tj_1z_1 each have 72 conjugates in $\langle t, \sigma_1, \sigma_2, J \rangle$. (Here j_1 is an involution in $C_J(t) - Z(C_J(t))$.) Finally, if $x^2 \in \langle z \rangle$ and $xJ \sim_H utJ$, then x is conjugate to (at least) one of ut, utz, utz_1, utz_1z; if $x^2 \in \langle z \rangle$ and $xJ \sim_H vJ$ then $x \sim_H v$ or $x \sim_H vz_1$; or if $x^2 \in \langle z \rangle$ and $xJ \sim_H uvJ$ then x is conjugate to one of uv, uvz, uvz_1, uvz_1z.

We begin the proof of Theorem B by considering the case when $Z(J) = Z(H)$.

THEOREM 4. *Suppose Hypothesis* 1 *holds and, in addition, $Z(J) = Z(H)$. Then $Z(G) = \langle z_1 \rangle$ or $\langle zz_1 \rangle$ and G contains a subgroup G_0 of index two with $G_0/Z(G) \cong U_6(2)$ and $G \subseteq$* Aut G_0.

As usual, the proof will be carried out in a series of lemmas.

LEMMA 4.1. *The involutions z, z_1, zz_1 lie in distinct conjugacy classes in G. Hence a Sylow 2-subgroup of H is a Sylow 2-subgroup of G.*

PROOF. As $H = C_G(\langle z, z_1 \rangle)$, $J = O_2(C_G(\langle z, z_1 \rangle))$, whence $\langle z \rangle = J' \lhd N_G(\langle z, z_1 \rangle)$. If T is a Sylow 2-subgroup of H then $Z(T) = \langle z, z_1 \rangle$, so $\langle z \rangle \lhd N_G(T)$. Thus $T = N_G(T)$ and T is a Sylow 2-subgroup of G (by Sylow's theorem) and z, z_1, zz_1 lie in distinct conjugacy classes by Burnside's lemma [**3**, Theorem 7.1.1].

LEMMA 4.2. *There is an element g in either $C_G(z_1)$ or $C_G(zz_1)$ such that $z^g \in H - \langle z \rangle$.*

PROOF. Suppose that $z^g \in H$, $g \in C_G(z_1)$ or $C_G(zz_1)$, implies $z^g = z$. We will show, in a number of steps below, that $\langle z \rangle$ is weakly closed in H–this will contradict Proposition 2 as we have assumed $G \neq H \cdot O(G)$.

(a) *If $W \supseteq Z(J)$, $W \cong E_8$, then $N_G(W) \subseteq H$.* (For $g \in N_G(W)$, we have $\langle z, z_1\rangle^g \cap \langle z, z_1 \rangle \neq 1$. If $g \notin H$, $z \neq z^g \in W \subset H$ and $g \in C_G(z_1)$ or $C_G(zz_1)$, as $z_1 \nsim_G zz_1$.)

(b) *$z \nsim_G j$ for any $j \in J - \langle z \rangle$.* (If $j \in J - Z(J)$ and K is a Sylow 2-subgroup of $C_H(j)$, then $Z(K) = \langle z, z_1, j\rangle$. Thus $N_G(K) \subseteq H$ by (a) and $z \nsim_G j$. We already have $z_1 \nsim_G z \nsim_G zz_1$–Lemma 4.1.)

(c) *$z \nsim_G x$ for any involution x in tJ.* (Using the structure of Aut $PSp_4(3)$ and the remarks at the beginning of this section we easily compute that if x is an involution in tJ and X is a Sylow 2-subgroup of $C_H(x)$ then $Z(X) \subseteq \langle t, J\rangle$. Suppose there is a 2-subgroup Y of $C_G(x)$ with $|Y : X| = 2$. For $y \in Y - X$, $\langle z, z_1\rangle^y \cap J \neq 1$ so $z \neq z^y \in Z(X)$ with $y \in C_G(z_1)$ or $y \in C_G(zz_1)$ by (b) and Lemma 4.1. We conclude that $z \nsim_G x$.)

(d) *σ_1, $\sigma_1\sigma_2^{-1}$ and $\sigma_1\sigma_2$ lie in distinct conjugacy classes in G.* (As $Z(T_2) = Z(J)$, $Z(T_1) = \langle t, z, z_1\rangle$, $Z(J) \subseteq Z(T_3) \subseteq \langle ut, z, z_1\rangle$, (a) and (b) yield that T_1, T_2, T_3 are Sylow 2-subgroups of $C_G(\sigma_1)$, $C_G(\sigma_1\sigma_2^{-1})$, $C_G(\sigma_1\sigma_2)$, respectively. The result follows as the T_i have different orders.)

(e) *$z \nsim_G x$ for any involution $x \in utJ$.* (If $z \sim_G x$, x would be conjugate to an involution in $Z(T_3) = \langle ut, z, z_1\rangle$. The result follows from Propostion 4, (d), (a) and Burnside's lemma.)

(f) *$z \nsim_G x$ for any involution $x \in uvJ$.* (If $z \sim_G x$ then z is conjugate to an involution in $\langle uv, z, z_1\rangle$. However $\langle uv, z, z_1 \rangle$ is a Sylow 2-subgroup of $C_H(B)$ where B is a Sylow 5-subgroup of H. We get the required contradiction in the same way as in (e).)

(g) *$z \nsim_G v$ and $z \nsim_G vz_1$.* (If z is conjugate to v in G then z must be conjugate to an involution x in $C_G(\sigma_1\sigma_2^{-1})$, $x \in uvtJ \cap C_H(\sigma_1\sigma_2^{-1})$, by (d) and Proposition 4. If X is a Sylow 2-subgroup of $C(x) \cap T_2$ then X is abelian as $[t, J] \cap Z(J) = 1$ and $|C_J(x) \cap T_2| = 8$. We can apply the same argument as given in the second paragraph of the proof of Lemma 1.6 to $\Omega_1(X)$ to show V is a Sylow 2-subgroup of $C(x) \cap C_G(\sigma_1\sigma_2^{-1})$. Thus $v \nsim_G z$. A similar argument yields $z \nsim_G vz_1$ also.)

This completes the proof of the lemma.

Let $F = C_J(\langle u, t\rangle)$ and $E = C_H(F)$. In this case, $F \cong E_{64}$, $|E| = 2^{10}$ and (using the same arguments as in §3) $N_H(E)/E \cong \Sigma_5$.

LEMMA 4.3. *We may assume that $C_G(z_1)$ contains a subgroup U of index two with $C_G(z_1) = U\langle v\rangle$ and $U/\langle z_1\rangle \cong U_6(2)$. Further, E is weakly closed in T with respect to G and $N_U(E)/E \cong L_3(4)$.*

PROOF. By Lemma 4.2 we may assume that there exists $g \in C_G(z_1)$ with $z^g \in H - \langle z \rangle$. Thus $C_G(z_1) \neq O(C_G(z_1)) \cdot H$ by Proposition 2. It follows therefore from Theorem 3 that $C_G(z_1)$ contains a subgroup U with the properties stated. The proof that E is weakly closed in T is the same as that given in Lemma 3.3.

LEMMA 4.4. *We have $N_G(E) \subset C_G(z_1)$ and z_1 is not conjugate (in G) to any involution in $H'\langle z_1\rangle - \langle z_1\rangle$.*

PROOF. If $z \sim_G j$ for some involution $j \in J - Z(J)$ and X is a Sylow 2-subgroup of $C_H(j)$, there exists a 2-group $Y \subseteq C_G(j)$ with $|Y : X| = 2$. For $y \in Y - K$, $z^y \in \langle j, z, z_1 \rangle - \langle z \rangle$ and $y \in C_G(z_1)$ or $C_G(zz_1)$. However this is not possible–see Lemma 3.2. We conclude $z \nsim_G j$ for any $j \in J - \langle z \rangle$. It follows by the argument of Lemma 3.4 that $N_G(E)$ contains a subgroup K of index two with $N_G(E) = K\langle v \rangle$, and $K/E \cong L_3(4)$. Thus $N_G(E) \subseteq C_G(z_1)$ by Lemma 4.3. Finally, recall that each involution in $H'\langle z_1 \rangle$ is conjugate to an involution in E. Hence as E is weakly closed in T and $C_G(z_1) \supseteq N_G(E)$, it follows immediately that $z_1^G \cap H'\langle z_1 \rangle = \{z_1\}$.

LEMMA 4.5. *We have* $G = C_G(z_1)$.

PROOF. We claim that G contains a subgroup G_0 of index two. In order to use the proof of Lemma 3.6 we only have to show $z \nsim_G x$, $x \in uvJ$. (We have already shown $z^G \cap J = \{z\}$ in the proof of Lemma 4.4.) As in the proof of Lemma 4.2(f) let $\langle uv, z, z_1 \rangle$ be a Sylow 2-subgroup of $C_H(B)$, $|B| = 5$. By Proposition 4, z is conjugate to an involution in uvJ if and only if z is conjugate to an involution in $uvZ(J)$ in $C_G(B)$. It follows that z must be conjugate to an involution in $uvZ(J)$ in $C_G(z_1)$ or $C_G(zz_1)$, which is not the case. We conclude, as in Lemma 3.6, that G contains a subgroup G_0 with $G = G_0\langle uv \rangle$.

If $z_1 \notin G_0$ then we conclude from Theorem 3 that G_0 contains a subgroup G_1 of index two with $G_1 \cong U_6(2)$. In this case $G \subseteq C_G(z_1)$ and the lemma is proved. Therefore we will assume $z_1 \in G_0$. It follows that $G_0 \cap H = H'\langle z_1 \rangle$ whence $z_1^G \cap H = \{z_1\}$. Proposition 2 yields $G = C_G(z_1) \cdot O(G)$ and it remains to show that $O(G) = 1$.

Since we may assume $z \sim_G t$, it follows that $[tz, O(G)] = 1$. However $tz \sim_G j_0$ for some $j_0 \in J$ (Lemma 3.5) and $j_0 \sim_H j_0 z$. Thus $[j_0, O(G)] = [j_0 z, O(G)] = 1$ whence $O(G) \subseteq H$ so $O(G) = 1$. The lemma is proved.

It is clear that Theorem 4 follows immediately from Lemmas 4.5 and 4.3. To complete the proof of Theorem B it only remains to consider the case $Z(J) \neq Z(H)$.

THEOREM 5. *Suppose Hypothesis* 1 *holds and, in addition,* $Z(J) \neq Z(H)$. *Then* $G \cong M(22)$.

We proceed (as usual) to give the proof in a series of lemmas, after making the obvious remark that $C_H(Z(J))/J \cong PSp_4(3)$, and if $h \in H - C_H(Z(J))$ then $z_1^h = z_1 z$.

LEMMA 5.1. *The cosets* vJ *and* uvJ *each contain precisely one class of involutions in* H.

PROOF. Let $X = \langle uv, \sigma_1\sigma_2, J \rangle$ so that $X/J \cong \Sigma_3$. By our assumptions above, $\langle uv, Z(J) \rangle$ is a Sylow 2-subgroup of $N_X(\langle \sigma_1\sigma_2 \rangle)$. As $\langle uv, Z(J) \rangle$ is nonabelian, $\langle uv, Z(J) \rangle \cong D_8$ so $N_X(\langle \sigma_1\sigma_2 \rangle)$ contains precisely one class of involutions which are not in J. The result follows for the coset uvJ from Proposition 1, and exactly the same argument applied to $\langle v, \sigma_1\sigma_2, J \rangle$ yields the result for vJ.

LEMMA 5.2. *The involution* z_1 *is not conjugate* (*in* G) *to any involution in* $J - \{z_1, z_1 z\}$.

PROOF. If Y is a Sylow 2-subgroup of $C_H(z_1)$ then clearly $Z(Y) = Z(J) = \langle z, z_1\rangle$. As $J = O_2(C_G(\langle z, z_1\rangle))$ and $J' = \langle z\rangle$, Y must be a Sylow 2-subgroup of $C_G(z_1)$ whence $z_1 \not\sim_G z$. Recall that $J - Z(J)$ has two classes of involutions in H with representatives j, jz_1 (and each has 270 conjugates in H). If K is a Sylow 2-subgroup of $C_H(j)$ then $K \not\subseteq C_H(Z(J))$ so $Z(K) = \langle z, j\rangle$. If j is conjugate to z_1 in G, then there exists $g \in G$ with $\langle z, j\rangle^g = \langle z, z_1\rangle$. Thus $z^g = z$ and $g \in H$ which is a contradiction. The same argument shows $z_1 \not\sim_G jz_1$ and the lemma follows.

LEMMA 5.3. *We have* $\langle \sigma_1\rangle \not\sim_G \langle \sigma_1\sigma_2^{-1}\rangle \not\sim_G \langle \sigma_1\sigma_2\rangle$.

PROOF. Note that $|T_2| = 2^8$ and $Z(T_2) = \langle z\rangle$, so that T_2 is a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1})$. Now $|T_1| = 2^7$ and $Z(T_1) \cong E_8$ whence if $T_1^g \subset T_2$ for some $g \in G$, $|Z(T_2)| \geqslant 4$. We conclude that $\langle \sigma_1\rangle \not\sim_G \langle \sigma_1\sigma_2^{-1}\rangle$. If $T_3^g \subset T_2$ for some $g \in G$ then $z \in T_3'$ implies $z^g \in T_2' \subset T_2 \cap J$. However $|T_3| = 2^5$ while $|C(z^g) \cap T_2| > 2^7$, clearly a contradiction. The lemma is proved.

LEMMA 5.4. *Either* $C_G(\sigma_1\sigma_2^{-1}) = O(C_G(\sigma_1\sigma_2^{-1})) \cdot C_H(\sigma_1\sigma_2^{-1})$ *or* $C_G(\sigma_1\sigma_2^{-1})$ *contains a subgroup* U *of index two with* $U\langle z_1\rangle = C_G(\sigma_1\sigma_2^{-1})$ *and* $U/\langle \sigma_1\sigma_2^{-1}\rangle \cong U_4(3)$. *Further, if* z *is conjugate to some involution in* $T_2 - \langle z\rangle$ *in* G *then, with appropriate choice of elements, we have the following fusion in* $C_G(\sigma_1\sigma_2^{-1})$:

$$z \sim j \sim t \sim tz \sim uv \not\sim uvtz_1 \sim jz_1 \sim tz_1z \not\sim z_1 \sim tz_1.$$

PROOF. If z is not conjugate to any involution in $T_2 - \langle z\rangle$ then $C_G(\sigma_1\sigma_2^{-1}) = O(C_G(\sigma_1\sigma_2^{-1})) \cdot C_H(\sigma_1\sigma_2^{-1})$ by Proposition 2. Hence we will suppose $z \sim_G x$ for some $x \in T_2 - \langle z\rangle$. By Proposition 4 and Lemma 5.3, $z \sim x$ in $C_G(\sigma_1\sigma_2^{-1})$. Let $X = C_G(\sigma_1\sigma_2^{-1})$ and $Y = C_H(\sigma_1\sigma_2^{-1})$. We take uv to be an involution (in $N_Y(\langle \sigma_1\sigma_2\rangle)$) and $C_Y(uv) \cap \langle t, z, z_1\rangle = \langle z, tz_1\rangle$ so that $t^{uv} = tz$ ($\langle t, z, z_1\rangle = C_Y(\sigma_1\sigma_2) \cap T_2$). As t, uv interchange the two quaternion subgroups in $O_{2,3}(Y)'$ ($= [M, Y \cap J]$), we have $C_J(t) \cap Y \cong E_{16}$, $C_J(uv) \cap Y \cong E_8$ and $C_J(uvtz_1) \cap Y \cong Z_4 \times Z_2$. Thus Y contains 9 classes of involutions with representatives $z, z_1, j, jz_1, t, tz_1, tz_1z, uv, uvtz_1$. Further, T_2 contains precisely one elementary abelian subgroup of order 32, namely $W = \langle t\rangle \times (C_J(t) \cap T_2)$.

Suppose at first that z is not conjugate (in X) to any involution in $W - \langle z\rangle$. Thus $z \sim_X uv$ or $z \sim_X uvtz_1$. If K is a Sylow 2-subgroup of $C_Y(uv)$ then $Z(K) = \langle uv, z, j'\rangle$ (for some $j' \in J \cap Y - Z(J)$). However $\{j', j'z\} \lhd N_X(K)$ (by Lemma 5.1) whence $\langle z\rangle \lhd N_X(K)$ so K is a Sylow 2-subgroup of $C_X(uv)$ and $z \not\sim_X uv$. If L is a Sylow 2-subgroup of $C_Y(uvtz_1)$ then $\langle z\rangle \subseteq \mho^1(L) \subseteq L \cap J$, so $\langle z\rangle \lhd N_X(L)$ and $z \not\sim_X uvtz_1$ either. We have proved that $z \sim_X w$ for some $w \in W - \langle z\rangle$ and thus $z \sim w$ in $N_X(W)$ (as W is normal in any Sylow 2-subgroup which contains it). Thus $N_X(W) \neq N_Y(W)$.

We have $C_X(W) = W \times \langle \sigma_1\sigma_2^{-1}\rangle$, $N_Y(W)/C(W) \cong \Sigma_4$, $T_2/W \cong D_8$ and T_2 is a Sylow 2-subgroup of $N_X(W)$. If $xC(W)$ centralizes $j_2C(W)$ in $N_X(W)/C_X(W)$, where $\langle j_2\rangle C(W)/C(W) = Z(T_2C(W)/C(W))$ then x normalizes $C_W(j_2) = \langle z, z_1, j'\rangle$ for some $j' \in W \cap J$, and therefore x normalizes $\{z_1z, z_1\}$ by Lemma 5.2; i.e. $x \in N_Y(W)$. It now follows from a result of Gorenstein and Walter [7, Theorem 1] that $N_X(W)/C_X(W) \cong \Sigma_5$, $L_2(7)$ or $\mathfrak{A}_6$.

In $N_Y(W)$, z has 1 conjugate, z_1 has 2 conjugates, j, jz_1 each have 6, t has 8, and tz_1, tz_1z each have 4 conjugates. If $N_X(W)/C_X(W) \cong \Sigma_5$, as z_1 must have 10 conjugates we may assume (without loss) that j still has only 6 conjugates in $N_X(W)$. This forces $z \in \langle j^{N(W)} \rangle \subset J \cap W$ which is not possible as z has 5 conjugates in $N_X(W)$ (recall $j \sim jz$ in $T_2 \cap J$). If $N_X(W)/C_X(W) \cong L_2(7)$, z, z_1 must have 7, 14 conjugates, respectively, and we again may assume that j has only 6 conjugates in $N_X(W)$. This forces $J \cap W = \langle z^{N(W)}, j^{N(W)} \rangle \lhd N_X(W)$, a contradiction. We have proved therefore that $N_X(W)/C_X(W) \cong \mathfrak{A}_6$ and (as above) determine that $z \sim j \sim t$ ($z \sim t \sim tz$ forces $z \sim j$ as $[\langle \sigma_1\sigma_2 \rangle, W] \subset [M, J]$), $z_1 \sim tz_1$ and $jz_1 \sim tz_1z$ (as we may interchange tz_1, tz_1z if necessary). Since $\mathfrak{A}_6$ has one class of involutions, it follows that all involutions in $N_X(W) - W$ are conjugate either to z or jz_1 (as j_2W contains two classes in Y with representatives j_2, j_2z_1, for some $j_2 \in (J \cap T_2) - W$). It follows from the uniqueness of W (in T_2) that z_1 is not conjugate to any involution in $\langle t, uv \rangle \cdot [M, Y \cap J]$ and so X has a subgroup X_1 of index two with $z_1 \in X - X_1$. Clearly $z \in X_1 \cap W$ whence $X_1 \cap W = \langle z^{N(W)} \rangle \cong E_{16}$.

We see that X_1 has no subgroup of index two and that $X_1 \cap Y$ satisfies the assumptions for the centralizer of an involution in Proposition 6. Thus we conclude that $X_1/\langle \sigma_1\sigma_2^{-1} \rangle \cong U_4(3)$ as $\langle t, uv \rangle \cong D_8$. This completes the proof of the lemma.

As above we take $F = C_J(\langle u, t \rangle) \cong E_{64}$ and $E = C_H(F)$. Thus $|E| = 2^{10}$ and $N_H(E)/E \cong \Sigma_5$. Let T be a Sylow 2-subgroup of $N_H(E)$ (whence T is a Sylow 2-subgroup of G).

Lemma 5.5. *The subgroup E is elementary abelian* (*of order* 2^{10}) *and weakly closed in T with respect to G.*

Proof. As we chose $T_3 = \langle u, t, v\beta \rangle Z(J)$, u normalizes $\langle t, Z(J) \rangle$ whence $[u, t] = 1$ or $(ut)^2 = z$ which implies $tz_1^u = tz_1z$. Thus if $(ut)^2 = z$, z cannot be fused to any involution in $T_2 - \langle z \rangle$ by Lemma 5.4 and so $z^G \cap H \subset \{z\} \cup tj_1^H \cup tj_1z_1^H$. This leads to a contradiction as in the proof of Lemma 2.3. Hence $[t, u] = 1$ and E is elementary abelian. The proof that E is weakly closed in T follows as in the proof of Lemma 3.3.

Lemma 5.6. *We have $N_G(E)/E \cong M_{22}$* (*the Mathieu group on* 22 *letters*) *and $C_G(\sigma_1\sigma_2^{-1}) = U\langle z_1 \rangle$ where $U/\langle \sigma_1\sigma_2^{-1} \rangle \cong U_4(3)$.*

Proof. By Lemma 5.4 we have that $z \sim_G e$ for some $e \in E - \langle z \rangle$. Hence Lemma 5.5 forces $z \sim e$ in $N_G(E)$ and $N_G(E) \neq N_H(E)$. We first show that for any $j_0 \in J - E$, $O(C_{N(E)/E}(j_0E)) = 1$. Suppose $gE \in O(C(j_0E))$. Then as $JE/E \subseteq C(j_0E)$ it follows that $[g, K] \subseteq E$ for K/E a subgroup of index two in JE/E. As $\langle z, z_1 \rangle \subset Z(K) \subset J \cap E$ and g normalizes $Z(K)$ it follows by Lemma 5.2 that g normalizes $\{z_1, z_1z\}$. Hence $g \in H$ so $g \in E$ and $O(C(j_0E)) = 1$. This forces, in addition, that $O(N_G(E)/E) = 1$ also.

If $N_G(E)/E$ possesses a subgroup L/E of index two then $L/E \cong L_3(4)$ as in the proof of Lemma 2.4. However this means z must have 21 conjugates in $N_G(E)$ so $z \sim_G tz_1$ or tz_1z, against Lemma 5.4. Thus $N_G(E)/E$ does not contain a subgroup of index two. If $j_0 \in J - E$ then $|C_{N(E)/E}(j_0E)| < 2^7 \cdot 15$ as $z \in [j_0, E]$ of order at most 16, and $N_G(JE) \subset H$. Thus as T/E is of type $\hat{A}_8$ (see Lemma 3.4), the combined work of Gorenstein and Harada **[5]** and Phan **[18]** yields $N_G(E)/E \cong M_{22}$ or $U_4(3)$. Since $N_G(E)/E \cong U_4(3)$ implies $|N_G(E){:}\ N_H(E)| > |E|$, we have $N_G(E)/E \cong M_{22}$, and z has 231 conjugates in $N_G(E)$. It follows that $z \sim j \sim t \sim ut$ say, where $[ut, v\beta] = 1$ in T_3, $z_1 \sim tz_1$, and all other involutions in E (there are 770) are conjugate to jz_1 in $N_G(E)$. (This last fact follows from some simple computations and the structure of M_{22}–however it is not needed in this work.)

LEMMA 5.7. *We have* $C_G(z_1)/\langle z_1\rangle \cong U_6(2)$ *and* $G \cong M(22)$.

PROOF. As $\langle z\rangle$ is not weakly closed in $C_H(z_1)$ with respect to $C_G(z_1)$ (use either Lemma 5.4 or 5.6), we have $C_G(z_1)/\langle z_1\rangle \cong U_6(2)$ by Theorem 2. We next show that $z_1 \in C_G(z_1)'$ so that $C_G(z_1) \cong \hat{U}_6(2)$–a two-fold covering group of $U_6(2)$.

Recall that $Z(T_1) = \langle t, z, z_1\rangle$ and the Sylow 2-subgroup $\langle v, T_1\rangle$ of $N_G(\langle\sigma_1\rangle)$ has centre $\langle tz_1, z\rangle$. As $z^G \cap Z(T_1) = \{z, t, tz\}$, $z_1^G \cap Z(T_1) = \{z_1, z_1z, tz_1\}$, it follows immediately from Sylow's theorem that T_1 is a Sylow 2-subgroup of $C_G(\sigma_1)$. Now T_3 contains $\langle u, t, z, z_1\rangle \cong E_{16}$ so $\sigma_1 \nsim_G \sigma_1\sigma_2$ whence Lemma 5.3 and Proposition 4 imply $z \sim t \sim tz$ in $C_G(\sigma_1)$. Thus $z \sim t \sim tz$ in $N(T_1) \cap C_G(\sigma_1)$ by Burnside's lemma. If $z_1 \notin T_1'$ then $T_1 \cong Z_2 \times Q_8 \times Q_8$ (see §3, introductory remarks) with $\mho^1(T_1) = \langle t, z\rangle$. However it is now impossible for z, t, tz to be conjugate in $N(T_1) \cap C_G(\sigma_1)$ so we have proved $z_1 \in T_1'$ whence $z_1 \in C_G(z_1)'$.

It follows from a result of Griess **[8]** that $C_G(z_1)$ is uniquely determined. Thus D. Hunt's result **[11]** yields $G \cong M(22)$ once we have observed that G is simple. As $z \sim_G t \sim_G tz$, we have $O(G) = 1$ immediately, so if $1 \neq N \lhd G$ then $z \in N$. Hence, as $\langle z^G \cap H\rangle = H$ we have $H \subset N$, so $N = G$ as $Z(T) = \langle z\rangle$. This completes the proofs of the lemma and Theorem 5.

5. The proof of Theorem C. Throughout this section, G will denote a finite group which satisfies the assumptions of Theorem C. In addition we will assume that $G \neq H \cdot O(G)$ so that $z \sim_G h$, for some $h \in H - \langle z\rangle$ by Proposition 2. From our assumptions on the structure of H we have $C_H(c) = \langle c\rangle \times P$ where $P/\langle z\rangle \cong PSp_4(3)$. Thus $P \cong Z_2 \times PSp_4(3)$ or $P\hat{S}p_4(3)$ (the covering group for $PSp_4(3)$). Further, $N_H(\langle c\rangle) = (\langle c\rangle \times P)\langle v\rangle$, and we will take all the elements used to describe the structure of Aut $PSp_4(3)$ in §1 to lie in $P\langle v\rangle$. In particular we see that $t^2 = 1$.

As $|J'| = 2$, we have $J' = \langle z\rangle$, and if $\langle z_1, z_2\rangle = [\langle c\rangle, Z(J)]$, then $Z(J) = \langle z, z_1, z_2\rangle$ and $J = \langle z_1, z_2\rangle \times J_0$, $J_0 \cong Q_8 * Q_8 * Q_8 * Q_8$. Clearly $H' = C_H(c) \cdot J$ and $H'' = P \cdot J = C_H(Z(J))$. If $T = \langle J, t, \alpha_1, \beta_1, \alpha_2, \beta_2, u, v\rangle$ then T is a Sylow 2-subgroup of H and we take $z_2^v = z_1z_2$ so $Z(T) = \langle z, z_1\rangle$. As usual, $T_2 = \langle C_J(\sigma_1\sigma_2^{-1}), uv, t\rangle$ is a Sylow 2-subgroup of $C_H(\sigma_1\sigma_2^{-1})$ and $Z(T_2) = \langle z, z_1\rangle$.

Repeating the arguments of previous sections we see that J has 5 classes of involutions with representatives $z(1)$, $z_1(3)$, $z_1z(3)$, $j(270)$, $jz_1(810)$ where $j \in [C_J(\sigma_1\sigma_2^{-1}), M]$. (The numbers in brackets of course denote the number of conjugates of each of the involutions.) In tJ, $t(4)$, $tz(4)$, $tz_1(12)$, $tz_1z(12)$, $tj_1(72)$, $tj_1z_1(216)$ ($j_1 \in C_J(t) - Z(C_J(t))$) are representatives of the classes of involutions, where $t \sim_H tz$ and $tz_1 \sim_H tz_1z$ only if $P \cong P\hat{S}p_4(3)$.

In utJ, $ut(16)$, $utz(16)$, $utz_1(48)$, $utz_1z(48)$ represent the classes of elements with square in $\langle z\rangle$, while if $x^2 \in \langle z\rangle$, $x \in vJ$ implies $x \sim_H v$, and $x \in uvJ$ implies $x \sim_H uv$ or $x \sim_H uvz$. In addition, $C_J(v) \cong E_{32}$ while $C_J(uv) \cong E_{64}$, $C_J(ut) \cong E_{128}$ and $C_J(t) \cong E_{16} \times Q_8 * Q_8$.

Finally we consider the elements of order three. If $y^3 = 1$, $y \in H$, then y is conjugate to an element in $\langle c, \sigma_1, \sigma_2\rangle^{\#}$, $C_H(\langle c, \sigma_1, \sigma_2\rangle) = (M \times \langle c\rangle)\langle t, z\rangle$ and $N_H(\langle c, \sigma_1, \sigma_2\rangle) = C_H(\langle c, \sigma_1, \sigma_2\rangle) \cdot \langle u, v\rangle$. We see that there are 9 conjugacy classes of elements of order 3 in H. In the table below Y denotes a Sylow 2-subgroup of $C_H(y)$.

Conjugacy classes of elements of order three in H

y	$Y \cap J = C_J(y)$	$Y/Y \cap J$	$Z(Y)$
σ_1	$\cong E_4 \times Q_8$	$\cong Q_8$	$= \langle t, Z(J)\rangle$
$\sigma_1\sigma_2^{-1}$	$\cong E_4 \times Q_8 * Q_8$	$\cong E_4$	$= \langle z_1, z\rangle$
$\sigma_1\sigma_2$	$= Z(J)$	$\cong D_8$	$\subseteq \langle ut, z_1, z\rangle$
c	$= \langle z\rangle$	type $PSp_4(3)$	$\subseteq \langle z, t\rangle$
$c\sigma_1^{-1}$	$\cong Q_8 * Q_8 * Q_8$	$\cong Q_8$	$= \langle z\rangle$
$c\sigma_1$	$= \langle z\rangle$	$\cong Q_8$	$= \langle t, z\rangle$
$c\sigma_1\sigma_2^{-1}$	$\cong Q_8$	$\cong Z_2$	$= \langle t, z\rangle$
$c\sigma_1\sigma_2$	$\cong Q_8 * Q_8$	$\cong E_4$	$\subseteq \langle u, z\rangle$
$c(\sigma_1\sigma_2)^{-1}$	$\cong Q_8 * Q_8$	$\cong E_4$	$\subseteq \langle t, z\rangle$

LEMMA 6.1. *The involutions z, z_1, z_1z lie in distinct conjugacy classes in G. Further, a Sylow 2-subgroup T of H is a Sylow 2-subgroup of G.*

PROOF. As $J = O_2(C_G(Z(T)))$ and $\langle z\rangle = J'$, $N_G(Z(T)) \subseteq H$ whence $N_G(T) = T$ and the result follows from the theorems of Sylow and Burnside.

LEMMA 6.2. *If $\langle\sigma_1\sigma_2^{-1}\rangle^g \subset H$ for some $g \in G$ then there exists $h \in H$ with $\langle\sigma_1\sigma_2^{-1}\rangle^h = \langle\sigma_1\sigma_2^{-1}\rangle^g$.*

PROOF. Let $y^3 = 1$, $y \in H$ but $y \nsim_H \sigma_1\sigma_2^{-1}$. We have to show $y \nsim_G \sigma_1\sigma_2^{-1}$. As $Z(T_2) = \langle z, z_1\rangle$ we see that T_2 is a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1})$ by Lemma 6.1. As $|T_2| = 2^9$ it is immediate that $\sigma_1\sigma_2^{-1} \nsim_G c\sigma_1$. Further, if $\sigma_1\sigma_2^{-1} \sim_G y$, then if $|Y| < 2^9$, we must have $|Z(Y)| \geqslant 8$. Hence it only remains to consider $y = \sigma_1$ or $\sigma_1\sigma_2$. We have $\sigma_1 \nsim_G \sigma_1\sigma_2^{-1}$ by the argument of Lemma 3.1. If $T_3^g \subset T_2$ for some $g \in G$ (T_3 a Sylow 2-subgroup of $C_H(\sigma_1\sigma_2)$, as usual), then since $|T_3| = 2^6$, $z^g \in uv(T_2 \cap J)$ ($|C(uvt) \cap T_2| = 2^5$). However in this case $Z(T_3) = \langle ut, z, z_1\rangle \cong E_8$ while $C(z^g) \cap T_2$ has centre of order 16. We conclude that $\sigma_1\sigma_2^{-1} \nsim_G \sigma_1\sigma_2$ and the lemma is proved.

As in the proof of Lemma 5.4, let $W = \langle t\rangle \times (C_J(t) \cap T_2)$, the unique elementary abelian subgroup of order 64 in T_2.

LEMMA 6.3. *If $X = C_G(\sigma_1\sigma_2^{-1})$ and $Y = C_H(\sigma_1\sigma_2^{-1})$ then one of the following holds:*
(i) *$X = O(X) \cdot Y$ and $z^G \cap Y = \{z\}$;*

(ii) $N_X(W)/C_X(W) \cong \mathcal{A}_7$ *and with appropriate choice of* t, uv *we have the following fusion of involutions in* X:

$$z \sim t \sim tz_1z \sim jz_1 \sim uv \sim uvt \not\sim z_1 \sim j \sim tz_1 \sim uvz \not\sim z_1z \sim tz;$$

(iii) $N_X(W)/C_X(W) \cong Z_3 \cdot \Sigma_5$, $\langle z_1, z_2\rangle \lhd N_X(W)$ *and* $|z^G \cap W| = 5$ *with* $z \sim_G t$.

PROOF. As usual, Proposition 4 and Lemma 6.2 imply that for any involution $y \in Y$, $z \sim_G y$ if and only if $z \sim_X y$. Assume $X \neq O(X)Y$ so that $z \sim_X y$ for some $y \in Y - \langle z\rangle$ by Proposition 2. We note that $Y = Y_0 \cdot Y_1$ where $Y_0 \cap Y_1 = 1$, $Y_0 = \langle z_1, z_2, c\rangle \cong \mathcal{A}_4$, $Y_1 = \langle uv, t, M\rangle[M, Y \cap J]$ with $Y_1/\langle \sigma_1\sigma_2^{-1}\rangle$ satisfying the assumptions of the centralizer of an involution in Proposition 6.

Suppose at first that $z \not\sim_X w$ for any $w \in W - \langle z\rangle$. Then it follows, as in the proof of Lemma 1.6, that $z \not\sim_X uv$ and if $z \sim_X uvt$, then $C_Y(uvt) = \langle \sigma_1\sigma_2^{-1}\rangle \times L$ where $L \cong E_{32}$. In this case we have $N_Y(L)/C_Y(L) \cong E_8$ whence z has 9 conjugates in $N_X(L)$. Thus $N_X(L)$ contains a normal subgroup L_1 with $|L_1 : L| = 2$. However $L_1 \subset T_2$ and $\{z\} = Z^G \cap Z(L_1)$ which yields $\langle z\rangle \lhd N_X(L)$, a contradiction. We conclude that $z \sim_X w$ for some $w \in W - \langle z\rangle$ and hence $z \sim w$ in $N_X(W)$.

For the rest of the proof of this lemma, we will use the "bar-convention" for $N_X(W)/C_X(W)$. (Note that $C_X(W) = W \times \langle \sigma_1\sigma_2^{-1}\rangle$.) We have $\overline{N_Y(W)} \cong Z_3 \cdot \Sigma_4$ and $\overline{\langle c\sigma_1\sigma_2\rangle} = O_3(\overline{N_Y(W)})$. If $\overline{\langle c\sigma_1\sigma_2\rangle} \not\subset O(\overline{N_X(W)})$ then $\overline{N_X(W)}/O(\overline{N_X(W)}) \cong \mathcal{A}_7$ by Gorenstein's and Walter's result [7]. It follows immediately that $\overline{N_X(W)} \cong \mathcal{A}_7$ and that z has 35 conjugates in $N_X(W)$. Since $\overline{c\sigma_1\sigma_2} \sim \overline{\sigma_1\sigma_2}$ in $\overline{N_X(W)}$, $j \sim z_1$ and z_1, z_1z have 21, 7 conjugates, respectively, in $N_X(W)$. The rest of the fusion in (ii) follows from an appropriate choice of t, uv and the fact that $\mathcal{A}_7$ has one class of involutions.

Suppose now that $\langle \overline{c\sigma_1\sigma_2}\rangle \subset O(\overline{N_X(W)})$. If $j_0 \in T_2 \cap J - W$ then $[j_0, W] = \langle z, j\rangle$ for some $j \in J \cap W$. Thus $|C_{\overline{N(W)}}(\overline{j_0})| = 2^3 \cdot 3$ or $2^3 \cdot 3^2$ whence $O(\overline{N(W)}) = \overline{\langle c\sigma_1\sigma_2\rangle}$ or $|O(\overline{N(W)})| = 3^4$ (recall that $\overline{j_0} \in O_2(\overline{N_Y(W)})$). In the latter case, as $|C_W(c\sigma_1\sigma_2)| = 16$, $\langle \overline{c\sigma_1\sigma_2}\rangle \not\subset Z(O(\overline{N_X(W)}))$. Thus $\overline{N_Y(W)}$ acts faithfully on $Z(O(\overline{N_X(W)}))$ which forces $|Z(O(\overline{N_X(W)}))| \geqslant 27$. Hence $O(\overline{N_X(W)})$ must be abelian, which is a contradiction. It follows that $O(\overline{N(W)}) = \langle \overline{c\sigma_1\sigma_2}\rangle$ and, therefore, $\langle z_1, z_2\rangle = [c\sigma_1\sigma_2, W] \lhd \overline{N_X(W)}$ and $\langle z, t, j_1, j_2\rangle = C_W(c\sigma_1\sigma_2) \lhd N_X(W)$, where $\langle j_1, j_2\rangle \times Z(J) = J \cap W$. It follows from Gorenstein's and Walter's result [7] and the fact that $\overline{N_X(W)}$ has a subgroup of index two that $\overline{N_X(W)} \cong Z_3 \cdot \Sigma_5$. Finally we see that z has only 5 conjugates in $N_X(W)$, whence $|z^G \cap X| = 5$ also, and take $z \sim_G t$ (replacing t by tz if necessary). The lemma is proved.

As in previous sections let $E = C_H(F)$, where $F = C_J(\langle u, t\rangle)$ $(\cong E_{2^7})$. In this case $|E| = 2^{11}$ and $N_H(E)/E \cong Z_3 \cdot \Sigma_5$.

LEMMA 6.4. *The subgroup* E *is elementary abelian of order* 2^{11} *and is weakly closed in* T *with respect to* G. *In addition*, $N_G(E)/E \cong M_{23}$ (*the Mathieu group on* 23 *letters*) *or* $N_G(E)/E$ *contains a subgroup of index two which is isomorphic to* $PGL(3, 4)$.

PROOF. If E is nonabelian then $(ut)^2 = z$ and $t \sim_H tz$. Hence we must be in case (i) of Lemma 6.3, so $z^G \cap H \subseteq \{z\} \cup tj_1^H \cup tj_1z_1^H$. This leads to a contradiction, as in the proof of Lemma 2.3. The proof that E is weakly closed in T follows as in Lemma 3.3.

Lemma 6.3 implies that there exists $e \in E - \langle z \rangle$ with $z \sim e$ in $N_G(E)$; i.e. $N_G(E) \neq N_H(E)$. The argument in Lemma 5.6 combined with $z^G \cap J = \{z\}$ in cases (i), (iii) of Lemma 6.3 and $\langle zz_1^G \cap J \rangle = Z(J)$ in case (ii) yield that $O(C_{N(E)/E}(jE)) = 1$ (where $j \in J - E$). In particular we have $O(N_G(E)/E) = 1$.

If $N_G(E)/E$ has a subgroup of index two then this subgroup is isomorphic to $PGL(3, 4)$ by Gorenstein's and Harada's result **[4]**. If $N_G(E)/E$ has no subgroup of index two then $N_G(E)/E \cong M_{22}$, M_{23}, or $U_4(3)$ (by **[5]**, **[18]**) as j_0E has centralizer of order $< 2^7 \cdot 45$ in $N_G(E)/E$ and $N_G(JE) \subset H$. It now follows that $N_G(E)/E \cong M_{23}$, as neither M_{22} nor $U_4(3)$ contains a subgroup isomorphic to $E_{16}(Z_3 \cdot \Sigma_5) \cong N_H(E)/E$.

LEMMA 6.5. *If* $N_G(E)/E \cong PGL(3, 4) \cdot Z_2$ *then* $\langle z_1, z_2 \rangle \lhd G$ *with* $G/\langle z_1, z_2 \rangle \cong$ Aut $U_6(2)$.

PROOF. In this case, z has 21 conjugates in $N_G(E)$ so we are in case (iii) of Lemma 6.3. As $N_X(W)/W \cong Z_3 \cdot \Sigma_5$ and $\langle z_1, z_2 \rangle \lhd N_X(W)$, it follows from Theorem 2 that $C_G(\langle z_1, z_2 \rangle)/\langle z_1, z_2 \rangle \cong U_6(2)$. In addition, $C_G(z_1) \subset N_G(\langle z_1, z_2 \rangle)$ by Theorem 3. Further, $N_G(\langle z_1, z_2 \rangle)/\langle z_1, z_2 \rangle \cong$ Aut $U_6(2)$, and the structure of Aut $U_6(2)$ yields $N_G(E) \subset N_G(\langle z_1, z_2 \rangle)$. Thus $z^G \cap H' = \{z\} \cup t^H$ and, in particular, $z^G \cap J = \{z\}$ (by Lemmas 6.3 and 6.4).

As $t \nsim_G tz$, $[uv, t] = 1$, and as $z \sim_X t$, we have $(uv)^2 = 1$. If $z \sim_G uv$ then $z \sim_X uv$ by Proposition 4 and Lemma 6.2. If K is a Sylow 2-subgroup of $C_Y(uv)$ then $Z(K) = \langle z, z_1, j, uv \rangle$ for some $j \in J \cap T_2$, and we can repeat the argument in Lemma 1.6 to yield $\langle z \rangle \lhd N_X(K)$ whence $z \nsim_G uv$. Following the proof of Lemma 3.6 we have that G contains a subgroup G_0 of index two with $uv \in G - G_0$. If $N = N_G(\langle z_1, z_2 \rangle) \cap G_0$ then $N/\langle z_1, z_2 \rangle \cong U_6(2) \cdot Z_3$ and $N \supset C_G(\langle z_1, z_2 \rangle)$.

We will prove that $C(x) \cap G_0 \subset N$ for any involution $x \in N$. Note that N has 7 classes of involutions with representatives z_1, z, z_1z, tz, utz, tzz_1, $utzz_1$ (which are not fused in G). Firstly, we observe that $O(C_G(x)) = 1$ for any involution $x \in N$ as $\langle z_1, z_2 \rangle \subset C_G(x)$ and $C_G(z_1) \subset N_G(\langle z_1, z_2 \rangle)$. If $C_N(x) \subset C_G(\langle z_1, z_2 \rangle)$ then $C(x) \cap G_0 \subset N$ by Proposition 2. Thus as $H \cap G_0 \subset N$ and the argument of Lemma 2.6 may be repeated to yield $C_G(tz) \subset N_G(\langle t, z \rangle) \subset N \cdot \langle uv \rangle$, it only remains to prove that $C(utz) \cap G_0 \subset N$.

Let $C = C(utz) \cap G_0$ and recall $C \cap N \subset N_N(E)$ with $C \cap N/E \cong E_9 \cdot$ SL(2, 3) (see Lemma 2.5). Take $\langle \sigma_1\sigma_2, d, c \rangle$ to be a Sylow 3-subgroup of $C \cap N$ with $R = \langle \sigma_1\sigma_2, d \rangle \subset C_G(\langle z_1, z_2 \rangle)$. From the structure of $PGL(3, 4)$, we see there are 3 classes of subgroups of order three in $N_G(E)$ with representatives $\langle \sigma_1\sigma_2 \rangle$, $\langle c \rangle$, $\langle cd \rangle$. Further, we have

$$C_E(\sigma_1\sigma_2) = \langle z_1, z_2, z, u, t \rangle \cong E_{32} \quad \text{and} \quad C(cd) \cap N_G(E) \cong Z_3 \times E_8 \cdot F_{21}$$

(F_{21}, a Frobenius group of order 21). It follows that $C_E(\sigma_1\sigma_2)$ is a Sylow 2-subgroup of $C(\sigma_1\sigma_2) \cap G_0$ and hence $\langle \sigma_1\sigma_2 \rangle \nsim_G \langle c \rangle$. If $\langle cd \rangle \sim_G \langle \sigma_1\sigma_2 \rangle$ then $C(cd) \cap G_0$

would contain a Sylow 2-subgroup of order 32 which would be normalized by a nonabelian group of order 21. Since $7 \nmid |N_G(C_E(\sigma_1\sigma_2))|$ we see that $\langle cd\rangle \not\sim_G \langle\sigma_1\sigma_2\rangle$.

We have proved that $\langle\sigma_1\sigma_2^G \cap \langle\sigma_1\sigma_2, d, c\rangle\rangle = R$ and we will therefore show that $\langle\sigma_1\sigma_2, d, c\rangle$ is a Sylow 3-subgroup of C by proving that $N_C(R) \subset C \cap N$. Since $C_C(R) \cap N$ has Sylow 2-subgroup $\langle z_1, z_2, utz\rangle$ it follows that $N_C(R) = C_C(R) \cdot (C \cap N)$. Further,

$$N(\langle z_1, z_2, utz\rangle) \cap C_C(R) = R \times \langle z_1, z_2, utz\rangle$$

whence $O(C_C(R)) = R$, and Burnside's transfer theorem [**3**, Theorem 7.4.3] yields $C_C(R) \subset N$.

We have shown therefore that $\langle\sigma_1\sigma_2, d, c\rangle$ is a Sylow 3-subgroup of C. As $N_C(\langle\sigma_1\sigma_2, d, c\rangle)/\langle R, utz\rangle \cong Z_6$, Grün's transfer theorem [**3**, Theorem 7.4.2] yields that C has a normal subgroup C_0 of index three with $C_0 \cap N = C_G(\langle z_1, z_2\rangle)$. It follows from Proposition 2 that $C \subset N$ as required. This completes the proof that N contains the centralizer (in G_0) of each of its involutions. By [**3**, Theorem 9.2.1], it follows that $G_0 = N$ or that N contains one class of involutions. Thus $N = G_0$ and the lemma is proved.

LEMMA 6.6. *If $N_G(E)/E \cong M_{23}$ then $G \cong M(23)$.*

PROOF. If $N_G(E)/E \cong M_{23}$, Lemma 6.3(ii) holds and z, z_1, zz_1 have 1771, 253, 23 conjugates, respectively, in $N_G(E)$. Further, as M_{23} has one class of involutions, G has precisely 3 classes of involutions. In addition (with appropriate choice of ut), we have $z_1 \sim j \sim tz_1z \sim ut$, $zz_1 \sim tz$ and if $e \in E^{\#}$ is not conjugate to one of these involutions in $N_H(E)$ then $e \sim z$ in $N_G(E)$.

Since $C_H(zz_1)/\langle zz_1\rangle$ satisfies the hypotheses of Theorem 5, we have $C_G(zz_1)/\langle zz_1\rangle \cong M(22)$. As $z_1 = [z_2, v]$, $zz_1 \in C_G(zz_1)'$ so $C_G(zz_1)$ is isomorphic to the double cover of $M(22)$ (see [**9**]). It follows that $G \cong M(23)$ (see [**12**]) once we have proved that G is simple.

As $z \sim_G jz_1 \sim jz_1z$, $O(G) = 1$. Thus if $N \lhd G$, N contains one of z, z_1, zz_1. However, $\langle z^G \cap H\rangle = \langle z_1^G \cap H\rangle = H$, while $\langle zz_1^G \cap H\rangle = H'$, so we conclude $H \subset N$, whence $N = G$ as $N_G(T) = T$. This completes the proof of Lemma 6.6 and Theorem C.

6. The proof of Theorem D. In this section we will use the notation in the statement of Theorem D, and in addition we put $J = O_2(H) \cong E_4 \times Q_8 * Q_8$; $T = J\langle u, t\rangle$, a Sylow 2-subgroup of H; $W = \langle t\rangle \times C_J(t)$, the unique elementary abelian subgroup of order 64 in T; and let $\langle\sigma_i\rangle$ be a Sylow 3-subgroup of K_i, $i = 1, 2, 3$. Let $[u, z_2] = z_1$ so that $Z(T) = \langle z, z_1\rangle$ (recall $O_2(K_3) = \langle z_1, z_2\rangle \lhd H$). We suppose $G \neq H \cdot O(G)$ and give the proof in a series of lemmas.

LEMMA 7.1. *The elements z, z_1, zz_1 lie in distinct conjugacy classes in G and T is a Sylow 2-subgroup of G.*

PROOF. As $Z(T) = \langle z, z_1\rangle$ and $\mho^1(T') = \langle z\rangle$, the lemma follows from Sylow's theorem and Burnside's lemma.

LEMMA 7.2. *Either $N_G(W)/W \cong \mathcal{Q}_7$ and G has precisely 3 conjugacy classes of involutions with*

$$z \sim t \sim tz_1z \sim jz_1 \sim u \sim ut \nsim z_1 \sim j \sim tz_1 \sim uz \nsim z_1z \sim tz;$$

or $N_G(W)/W \cong Z_3 \cdot \Sigma_5$ with $\langle z_1, z_2\rangle \lhd N_G(W)$, $|z^G \cap W| = 5$ and

$$z \sim t \nsim z_1 \nsim zz_1 \sim tz_1 \nsim j \sim tz \nsim jz_1 \sim tzz_1.$$

PROOF. Lemma 6.3.

LEMMA 7.3. *If $N_G(W)/W \cong Z_3 \cdot \Sigma_5$ then $\langle z_1, z_2\rangle \lhd G$ with $G/C_G(\langle z_1, z_2\rangle) \cong \Sigma_3$ and $C_G(\langle z_1, z_2\rangle)/\langle z_1, z_2\rangle \cong PSp_4(3)$.*

PROOF. By Proposition 6, $C_G(\langle z_1, z_2\rangle)/\langle z_1, z_2\rangle \cong PSp_4(3)$. As in the proof of Lemma 6.5, $C_T(u)$ is a Sylow 2-subgroup of $C_G(u)$. Thus G contains a subgroup G_0 of index two (Proposition 3) with $G_0 \cap H = K_1 \cdot K_2 \cdot K_3\langle t\rangle$. Since $\langle z_1, z_2\rangle$ is strongly closed in $T \cap G_0$, Proposition 2 yields

$$C_G(z_1) = O(C_G(z_1)) \cdot (N_G(\langle z_1, z_2\rangle) \cap C_G(z_1)).$$

As $z \sim t$ and $tz \sim j \sim jz$ in $C_G(z_1)$, we have $[tz, O(C_G(z_1))] = 1$ whence

$$[j, O(C_G(z_1))] = [jz, O(C_G(z_1))] = 1.$$

It follows that $[z, O(C_G(z_1))] = 1$ so $O(C_G(z_1)) = 1$. We can show that $C(x) \cap G_0 \subset N_G(\langle z_1, z_2\rangle)$ for all involutions in $N_G(\langle z_1, z_2\rangle)$, in the same way as was done in Lemma 6.5. It follows that $\langle z_1, z_2\rangle \lhd G$ by [**3**, Theorem 9.2.1].

For the rest of this section we will assume $N_G(W)/W \cong \mathcal{Q}_7$.

LEMMA 7.4. *We have $C_G(zz_1) = U\langle z_2\rangle$ where $U/\langle zz_1\rangle \cong U_4(3)$ (U is a covering group of $U_4(3)$) and $\langle z_2, zz_1\rangle \lhd C_G(z_2)$ with $C_G(z_2)/\langle z_2, zz_1\rangle \cong$ Aut $PSp_4(3)$ (and $C_G(\langle z_2, zz_1\rangle) \cong E_4 \times PSp_4(3)$).*

PROOF. The structure of $C_G(zz_1)$ follows from Lemma 5.4. As $\langle z, jz_1\rangle$ normalizes $O(C_G(z_2))$ we have $O(C_G(z_2)) = 1$. In $C(z_2) \cap N_G(W)$ ($\cong E_{64} \cdot \Sigma_5$), z_2z has 5 conjugates while z_1z has only two, namely z_1z, z_2z_1z. It follows therefore from Proposition 2 that $\langle z_1z, z_2\rangle \lhd C_G(z_2)$. Thus $C_G(z_2)/\langle z_1z, z_2\rangle \cong$ Aut $PSp_4(3)$ by Proposition 6 (as $C_G(z_2)$ contains a subgroup of index two, namely $C_G\langle z_1z, z_2\rangle$).

LEMMA 7.5. *The order of G is* $2^9 \cdot 3^9 \cdot 5 \cdot 7 \cdot 13$.

PROOF. Let $a(g) = |\{(x, y)|(xy)^n = g$ for some positive integer n where $x \sim_G z$, $y \sim_G zz_1\}|$ for any involution $g \in G$. We compute that $a(z) = 2^2 \cdot 3^3$, $a(z_1) = 3^6 \cdot 5$ and $a(z_1z) = 0$. The lemma follows from Thompson's order formula (see Lemma 2.7).

LEMMA 7.6. *We have $K = O_3(C_G(\sigma_1\sigma_2^{-1})) \cong E_{3^5}$, $C_G(\sigma_1\sigma_2^{-1})/K \cong \Sigma_6$ and*

$$C_G(\sigma_1\sigma_2^{-1}) = \langle\sigma_1\sigma_2^{-1}\rangle \times (C_G(tz) \cap C_G(\sigma_1\sigma_2^{-1}))$$

with $\langle tz, \sigma_1\sigma_2^{-1}\rangle \cong \Sigma_3$.

PROOF. Put $S = \langle z_1, z_2, z, u\rangle$, a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1}) \cap C_G(z_1)$ (recall $z_1 \sim_G z_2$). As $S' = \langle z_1\rangle, S$ is a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2^{-1})$. Further, $\langle z_1, z_2, uz\rangle \cong D_8$ with $z_1 \sim z_2 \sim z_1z_2$ in $C_H(\sigma_1\sigma_2^{-1})$ and $z_1 \sim uz \sim uzz_1$ in $C_G(\sigma_1\sigma_2^{-1}) \cap C_G(zz_1)$.

Since $S - \langle z_1, z_2, uz\rangle$ contains 3 conjugates each of z and zz_1, Proposition 3 yields that $C_G(\sigma_1\sigma_2^{-1})$ has a subgroup of index two with Sylow 2-subgroup isomorphic to D_8. It now follows from Gorenstein and Walter [7] that $C_G(\sigma_1\sigma_2^{-1})/K \cong \Sigma_6$ (we know that $C_H(\sigma_1\sigma_2^{-1}) \cong E_9 \cdot (Z_2 \times \Sigma_4)$, $C_G(\sigma_1\sigma_2^{-1}) \cap C_G(zz_1) \cong E_{3^4} \cdot (Z_2 \times \Sigma_4)$). We get $|K| = 3^5$ by applying the Brauer-Wielandt formula [20] to $K\langle z_1, z_2\rangle$.

Recall that tz inverts $\sigma_1\sigma_2^{-1}$ and that $[t, S] = 1$. It follows that $C_G(tz)$ covers $N_G(\langle\sigma_1\sigma_2^{-1}\rangle)/K$ because tS contains precisely one conjugate of zz_1, namely tz. Finally, $C_K(z) = \langle\sigma_1, \sigma_2\rangle$ so $C_K(tz) \neq 1$. It follows that $C_K(tz) \cong E_{81}$ and the lemma is proved.

LEMMA 7.7. *The group G is a rank three extension of $C_G(zz_1)$, and zz_1^G is a class of 3-transpositions.*

PROOF. We have $|C_G(z_2) \cap C_G(zz_1)| = 2^8 \cdot 3^4 \cdot 5$ by Lemma 7.4, and by Lemma 7.6 there exists $g \in G$ with $|C_G(zz_1) \cap C_G(zz_1^g)| = 2^4 \cdot 3^6 \cdot 5$ (since $zz_1 \sim_G tz$). The lemma follows from Lemma 7.5 and $\langle tz, \sigma_1\sigma_2^{-1}\rangle \cong \Sigma_3$.

LEMMA 7.8. *We have $G \cong P\Omega(7, 3) = B_3(3)$.*

PROOF. Suppose $1 \neq N \lhd G$. Then $N \cap Z(T) \neq 1$ as each of z_1, z_1z_2, z_2 acts fixed-point free on $O(G)$. This implies that $W \subseteq N$ and hence $N_G(W) \subseteq N$. Thus $T \subset N$ and so $N = G$ as $N_G(T) = T$. It follows now from Fischer's result [1] that $G \cong P\Omega(7, 3)$.

PART II

The following result is proved.

THEOREM. *Let G be a finite group, z an involution in G and suppose $H = C_G(z)$ satisfies*:

(i) *$J = O_2(H)$ is extra-special of order 2^{13} with $C_H(J) \subseteq J$;*

(ii) *H contains a normal subgroup K with $K/O_{2,3}(H) \cong U_4(3)$ and $O_{2,3}(H)/J \cong Z_3$.*

Then one of the following holds:

(a) *$G = H \cdot O(G)$;*

(b) *$H/K \cong Z_2$ and G is a simple group with $|G| = |M(24)'|$;*

(c) *$H/K \cong Z_2 \times Z_2$ and $G \cong M(24)$.*

The notation used will be as in Part I, and, in addition, P_{27} will denote a nonabelian group of order 27 and exponent 3.

1. Some properties of $U_4(3)$. Much of the information given below comes from Phan [18]. We will, in general, follow his notation. The group $U_4(3)$ is a simple group of order $2^7 \cdot 3^6 \cdot 5 \cdot 7$ and has only one class of involutions. For an involution $\bar{t}$ in $U_4(3)$ we have $C(\bar{t}) = (\bar{L}_1 * \bar{L}_2)\langle\bar{u}, \bar{v}\rangle$ where $\bar{L}_i = \langle\bar{a}_i, \bar{b}_i\rangle\langle\bar{\sigma}_i\rangle \cong SL(2, 3)$, $\langle\bar{a}_i, \bar{b}_i\rangle \cong Q_8$, $\langle\bar{\sigma}_i\rangle \cong Z_3$ for $i = 1, 2$, and the relations involving $\bar{u}$, $\bar{v}$ are as follows:

$$\bar{u}^2 = (\bar{u}\bar{v})^2 = 1, \quad \bar{v}^2 = \bar{t} \qquad (\text{i.e. } \langle\bar{u}, \bar{v}\rangle \cong D_8);$$

$$\bar{a}_1^{\bar{u}} = \bar{a}_2, \quad \bar{b}_1 = \bar{b}_2^{\bar{u}}, \qquad \bar{\sigma}_1^{\bar{u}} = \bar{\sigma}_2;$$

$$\bar{a}_i^{\bar{v}} = \bar{a}_i^{-1}, \quad \bar{b}_i^{\bar{v}} = \bar{a}_i\bar{b}_i, \quad \bar{\sigma}_i^{\bar{v}} = \bar{\sigma}_i^{-1}, \qquad i = 1, 2.$$

The Sylow 2-subgroup $O_2(C(\bar{t}))\langle \bar{u}, \bar{v}\rangle$ contains precisely two elementary abelian subgroups of order 16:

$$\bar{F}_1 = \langle \bar{t}, \bar{a}_1\bar{a}_2, \bar{b}_1\bar{b}_2, \bar{u}\rangle, \quad \bar{F}_2 = \langle \bar{t}, \bar{a}_1\bar{a}_2, \bar{b}_2\bar{a}_2\bar{b}_1, \bar{u}\bar{v}\rangle$$

with $N(\bar{F}_i)/\bar{F}_i \cong \mathfrak{A}_6$, $i = 1, 2$.

There are four classes of elements of order 3 in $U_4(3)$ with representatives $\bar{\sigma}_1, \bar{\sigma}_1\bar{\sigma}_2^{-1}, \bar{\sigma}_1\bar{\sigma}_2, \bar{\sigma}_3$. For the Sylow 3-subgroup $\langle \bar{\sigma}_1, \bar{\sigma}_2\rangle$ of $C(\bar{t})$ we have $N(\langle \bar{\sigma}_1, \bar{\sigma}_2\rangle) = \bar{M}\langle \bar{u}, \bar{v}\rangle$ where $\bar{M} \cong E_{81}$ and $N(\bar{M})/\bar{M} \cong \mathfrak{A}_6$. Further, $C(\bar{\sigma}_1\bar{\sigma}_2^{-1})/\bar{M} \cong C(\bar{\sigma}_1\bar{\sigma}_2)/\bar{M} \cong \mathfrak{A}_4$ and $N(\langle \bar{\sigma}_1\bar{\sigma}_2^{-1}\rangle)/\bar{M} \cong N(\langle \bar{\sigma}_1\bar{\sigma}_2\rangle)/\bar{M} \cong \Sigma_4$. For $\bar{\sigma}_1$ we have $C(\bar{\sigma}_1) = \bar{U}_1 \cdot \bar{L}_2$ where $\bar{U}_1 \cong P_{27} * P_{27}$, $N\langle \bar{\sigma}_1\rangle = C(\bar{\sigma}_1)\langle \bar{v}\rangle$ and $\bar{L}_2\langle \bar{v}\rangle$ acts irreducibly on $\bar{U}_1/\langle \bar{\sigma}_1\rangle$. Clearly $\bar{U}_1\langle \bar{\sigma}_2\rangle$ is a Sylow 3-subgroup of $U_4(3)$ with centre $\langle \bar{\sigma}_1\rangle$. Finally, we may assume $C(\bar{\sigma}_3) \subset \bar{U}_1$, $N(\langle \bar{\sigma}_3\rangle) = C(\bar{\sigma}_3)\langle \bar{t}\rangle$ with $C(\bar{\sigma}_3) \cong Z_3 \times P_{27}$, and we note that $\bar{U}_1(\bar{M})$ contains 2 (20) conjugates of $\bar{\sigma}_1$, 48 (30) conjugates of each of $\bar{\sigma}_1\bar{\sigma}_2^{-1}, \bar{\sigma}_1\bar{\sigma}_2$ and 144 (0) conjugates of $\bar{\sigma}_3$.

The Sylow 5- and Sylow 7-normalizers in $U_4(3)$ are Frobenius groups of orders 20, 21, respectively.

We now consider $U_4(3)\langle \bar{\pi}_1, \bar{\pi}_2\rangle$ where $\bar{\pi}_1, \bar{\pi}_2$ are involutory (outer) automorphisms of $U_4(3)$ which satisfy the following relations:

$$C(\langle \bar{\pi}_1, \bar{\pi}_2\rangle) \cap C(\bar{t}) = \bar{L}_1 * \bar{L}_2 = O_{2,3}(C(\bar{t})),$$

$(\bar{\pi}_3)^2 = \bar{t}$ where $\bar{\pi}_3 = \bar{\pi}_1\bar{\pi}_2$ and $[\bar{\pi}_1, \bar{u}] = [\bar{\pi}_3, \bar{v}] = 1$, $[\bar{\pi}_1, \bar{v}] = [\bar{\pi}_2, \bar{u}] = [\bar{\pi}_2, \bar{v}] = [\bar{\pi}_3, \bar{u}] = \bar{t}$. For $i = 1, 2$, $\langle \bar{\pi}_i, \bar{F}_i\rangle \cong E_{32}$, while if $j \neq i$ and $j \in \{1, 2, 3\}$, $\langle \bar{\pi}_j, N(\bar{F}_i)\rangle/\bar{F}_i \cong \Sigma_6$.

There are 6 classes of involutions outside $U_4(3)$ with representatives $\bar{\pi}_i, \bar{\pi}_i\bar{a}_1\bar{a}_2$ $(i = 1, 2)$, $\bar{\pi}_3\bar{u}, \bar{\pi}_3\bar{a}_1$ which have centralizers in $U_4(3)$ isomorphic to $PSp_4(3)$, $E_{16} \cdot E_9 \cdot Z_4$, Σ_6 and $U_3(3)$, respectively. In addition, we note that we may assume

$$\bar{\pi}_1 \sim \bar{\pi}_1\bar{u}\bar{t}, \quad \bar{\pi}_1\bar{a}_1\bar{a}_2 \sim \bar{\pi}_1\bar{u} \sim \bar{\pi}_1\bar{v}, \quad \bar{\pi}_2 \sim \bar{\pi}_2\bar{u}\bar{v}\bar{t},$$
$$\bar{\pi}_2\bar{a}_1\bar{a}_2 \sim \bar{\pi}_2\bar{u}\bar{v} \sim \bar{\pi}_2\bar{v} \quad \text{and} \quad \bar{\pi}_3\bar{u} \sim \bar{\pi}_3\bar{v} \sim \bar{\pi}_3\bar{u}\bar{v}.$$

With regard to the elements of order 3, $\bar{\sigma}_1$ centralizes $\bar{\pi}_1, \bar{\pi}_2$ and $\bar{\pi}_3\bar{a}_1$, $\bar{\sigma}_1\bar{\sigma}_2^{-1}$ centralizes $\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_2\bar{u}\bar{v}$ and $\bar{\pi}_3\bar{u}\bar{v}$, $\bar{\sigma}_1\bar{\sigma}_2$ centralizes $\bar{\pi}_1, \bar{\pi}_1\bar{u}, \bar{\pi}_2, \bar{\pi}_3\bar{u}$ while $\bar{\sigma}_3$ centralizes conjugates of $\bar{\pi}_i\bar{a}_1\bar{a}_2$ $(i = 1, 2)$ and $\bar{\pi}_3\bar{a}_1$.

Finally we remark that $U_4(3)\langle \bar{\pi}_1, \bar{\pi}_2\rangle$ is the maximal subgroup of Aut $U_4(3)$ ($|\text{Aut } U_4(3)\colon U_4(3)\langle \bar{\pi}_1, \bar{\pi}_2\rangle| = 2$) in which $\bar{\sigma}_1\bar{\sigma}_2^{-1} \nsim \bar{\sigma}_1\bar{\sigma}_2$.

2. Some properties of H. We will use the notation of the theorem and note that if T is a Sylow 2-subgroup of H then $Z(T) = \langle z\rangle$ so T is a Sylow 2-subgroup of G. Let $\langle c\rangle$ denote a Sylow 3-subgroup of $O_{2,3}(H)$. Thus $C_K(c)/\langle c, C_J(c)\rangle \cong U_4(3)$. If $C_J(c) \neq \langle z\rangle$ then $C_J(c)$ and $[\langle c\rangle, J]$ are both extra-special groups of order $< 2^{11}$ (see Part I, Proposition 5). It follows from assumption (i) and **[13**, pp. 356, 357] that at least one of the orthogonal groups $O^+(10, 2)$ or $O^-(10, 2)$ must contain a subgroup isomorphic to $U_4(3)$. This is easily seen to be impossible by considering the Sylow 3-subgroups of these three groups. Therefore we have proved $C_J(c) = \langle z\rangle$. It follows immediately that J is the central product of 6 quaternion groups (or 6 dihedral groups) of order 8.

The "bar convention" will be used for $C_K(c)/\langle c, z\rangle$ and this group will be identified with $U_4(3)$ (as described in §1). As each element of order three in $U_4(3)$ is conjugate to its inverse we may take σ_i ($i = 1, 2, 3$), $\sigma_1\sigma_2^{-1}$, $\sigma_1\sigma_2$ to be elements of order three in $C_K(c)$, and also assume each of these elements is conjugate to its inverse in $C_K(c)$. As $C(\sigma_1)$ acts irreducibly on $O_3(C(\bar{\sigma}_1))/\langle\bar{\sigma}_1\rangle$ and there exists $\bar{d} \in C(\bar{\sigma}_1) - O_3(C(\bar{\sigma}_1))$ with $(\bar{d})^3 = \bar{\sigma}_1$, it follows that σ_1 lies in the centre of a Sylow 3-subgroup of $C_K(c)$. Let U_1 be a Sylow 3-subgroup of $O_3(C(\sigma_1) \cap C_K(c))$ so that $O_3(C(\bar{\sigma}_1)) = \bar{U}_1 = U_1 \times \langle z\rangle/\langle c, z\rangle$. If M is a Sylow 3-subgroup of $O_3(C(\sigma_1\sigma_2) \cap C_K(c))$ then $M \cong E_{3^5}$ and $O_3(C(\overline{\sigma_1\sigma_2})) = \bar{M} = M \times \langle z\rangle/\langle c, z\rangle$. Finally we take t, u, uv to be involutions in $C_K(c)$, so that $\langle c, \sigma_1, \sigma_2\rangle$ is a Sylow 3-subgroup of $C_K(t)$ and $[\langle u, v\rangle, \langle c, \sigma_1, \sigma_2\rangle] = \langle\sigma_1, \sigma_2\rangle$.

Since we have assumed $\sigma_1^v = \sigma_1^{-1}$, $(\sigma_1 c)^v = \sigma_1^{-1}c$, it follows that $C_J(\sigma_1 c) \neq \langle z\rangle$ and $[\sigma_1, J]$ is the central product of 2, 4 or 6 quaternion groups. As $\sigma_1 \in U_1'$ and $\bar{L}_2\langle\bar{v}\rangle$ acts irreducibly on $\bar{U}_1/\langle\bar{\sigma}_1\rangle$, it follows that $[\sigma_1, J]$ admits a group of automorphisms of order $\geqslant 3^6$. Hence $[\sigma_1, J] = J(3^6 \nmid |O_8^+(2)|)$, $C_J(\sigma_1) = \langle z\rangle$ and $C_J(\sigma_1 c) \cong C_J(\sigma_1^{-1}c) \cong Q_8 * Q_8 * Q_8$. Let $J_2 = C_J(\sigma_1 c)$, $J_3 = C_J(\sigma_1^{-1}c)$ so $J = J_2 * J_3$ and put $U_i = C_K(J_i) \cap U_1$, $i = 2, 3$.

As $U_2 \cap U_3 \cap Z(U_i) = 1$, it follows that $U_1 = U_2 \times U_3$ and $U_i \cong P_{27}$, $i = 2, 3$. In particular, $c \in U_1'$ so $c \in C_K(c)'$. Further, $C(J_i) \cap C_K(\sigma_1) = U_i \times \langle z\rangle$ and $C_K(\sigma_1)/\langle U_i, z\rangle$ is isomorphic to a Sylow 3-normalizer in $PSp_4(3)$ ($\cong P_{27}\mathrm{SL}(2, 3)$), $i = 2, 3$. By the Frattini argument, $N_H(\langle\sigma_1, c\rangle)$ covers H/K. If $x \in N_H(\langle\sigma_1, c\rangle) - K$, replacing x by xv if necessary, we may assume x normalizes both $\langle\sigma_1 c\rangle$ and $\langle\sigma_1^{-1}c\rangle$ and also, therefore, J_i, U_i, $i = 2, 3$. Either $[x, J_2] \neq 1$ or $[x, J_3] \neq 1$ so for $i = 2, 3$ (or perhaps both), $\langle x, C_K(\sigma_1)\rangle/\langle U_i, z\rangle$ is isomorphic to a subgroup of Aut $PSp_4(3) \cong O_6^-(2)$. It follows immediately that $x^2 \in C_K(\sigma_1)$ and $H = K$ or $H/K \cong Z_2$ or $Z_2 \times Z_2$. We collect all these results in the following lemma.

LEMMA 1. *The subgroup $J = O_2(H)$ is the central product of six quaternion (dihedral) groups and $C_J(c) = C_J(\sigma_1) = \langle z\rangle$. We have $K/J \cong C_K(c)/\langle z\rangle$ is isomorphic to a nonsplit extension of a group of order three by $U_4(3)$ and either $H = K$ or $H/K \cong Z_2$ or $Z_2 \times Z_2$. Finally if $J_2 = C_J(\sigma_1 c)$, $J_3 = C_J(\sigma_1^{-1}c)$ and $U_1 = O_3(C_K(\sigma_1))$, then $J_i \cong Q_8 * Q_8 * Q_8$, $U_i = C_K(J_i) \cap U_1 \cong P_{27}$, $i = 2, 3$ and $U_1 = U_2 \times U_3$.*

If μ, ν are elements of order 5, 7, respectively, in $C_K(c)$, then from Lemma 1 it follows that $\langle\mu\rangle$, $\langle\nu\rangle$ are Sylow 5-, 7-subgroups, respectively, in H and $C_J(\mu) \cong Q_8 * Q_8$, while $C_J(\nu) = \langle z\rangle$. The 48 elements of order three in $U_i - Z(U_i)$, $i = 2, 3$, are all conjugate in $N_K(\langle\sigma_1\rangle)$ and are therefore conjugate to $\sigma_1\sigma_2^{-1}$ or $\sigma_1\sigma_2$ in K. We will assume that $U_2 - \langle\sigma_1 c\rangle$ (only) contains conjugates of $\sigma_1\sigma_2^{-1}$. It follows immediately therefore that

$$C_J(\sigma_1\sigma_2^{-1}) \cong Q_8 * Q_8 * Q_8 * Q_8$$

and we take

$$C_K(\sigma_1\sigma_2^{-1}) = C_J(\sigma_1\sigma_2^{-1}) \cdot M \cdot \langle uv, t\rangle \cdot \langle\sigma_3\rangle.$$

Also, $\sigma_1\sigma_2^{-1} \not\sim_H \sigma_1\sigma_2^{-1}c^i$ and $C_J(\sigma_1\sigma_2^{-1}c^i) \cong Q_8$, $i = \pm 1$.

With this choice of elements we also have $\sigma_1\sigma_2 \sim_K \sigma_1\sigma_2 c^i$, $i = \pm 1$,

$$C_J(\sigma_1\sigma_2) = \left[\langle \sigma_1\sigma_2^{-1}\rangle, J\right] \cong Q_8 * Q_8$$

and

$$C_K(\sigma_1\sigma_2) = C_J(\sigma_1\sigma_2) \cdot M \cdot \langle u, t\rangle.$$

In the same way, $\sigma_3 \sim_K \sigma_3 c^i$, $i = \pm 1$, $C_J(\sigma_3) \cong Q_8 * Q_8$ and $C_K(\sigma_3) = C_J(\sigma_3) \cdot (C_K(\sigma_3) \cap U_1)$. We have proved

LEMMA 2. *The group H has 7 classes of subgroups of order three with representatives $\langle c\rangle$, $\langle \sigma_1\rangle$, $\langle \sigma_1 c\rangle$, $\langle \sigma_1\sigma_2^{-1}\rangle$, $\langle \sigma_1\sigma_2^{-1}c\rangle$, $\langle \sigma_1\sigma_2\rangle$, $\langle \sigma_3\rangle$, where*

$$C_J\left(\sigma_1\sigma_2^{-1}\right) \cong Q_8 * Q_8 * Q_8 * Q_8, \quad C_J\left(\sigma_1\sigma_2^{-1}c\right) \cong Q_8$$

and

$$C_J(\sigma_1\sigma_2) \cong C_J(\sigma_3) \cong Q_8 * Q_8.$$

Further, we have $C_J(\mu) \cong Q_8 * Q_8$, $C_J(\nu) = \langle z\rangle$, where $\langle \mu\rangle$, $\langle \nu\rangle$ are, respectively, Sylow 5-, 7-subgroups of H.

As t centralizes the subgroup $\langle c, \sigma_1\sigma_2^{-1}\rangle$ which acts faithfully on $C_J(\sigma_1\sigma_2)$ we have $[t, C_J(\sigma_1\sigma_2)] = 1$. If $M_1 = [\langle t, uv\rangle, M] \times \langle c\rangle \cong E_{81}$ then $M = M_1 \times \langle \sigma_1\sigma_2^{-1}\rangle$ and $C_H(t) \cap M_1 = \langle c, \sigma_1\sigma_2\rangle$. Let $\tau_1 \in M_1$ with

$$C_J(\tau_1) \cap C_J\left(\sigma_1\sigma_2^{-1}\right) \cong Q_8 * Q_8 * Q_8$$

(as $C_H(\sigma_1\sigma_2^{-1})/O_{2,3}(C_H(\sigma_1\sigma_2^{-1})) \cong \mathcal{Q}_4$, M_1 acts faithfully on $C_J(\sigma_1\sigma_2^{-1})$). Now τ_1 has 4 conjugates in $\langle uv, t\rangle \cdot M_1$ whence $C_J(\sigma_1\sigma_2^{-1}) \cdot M_1$ is the central product of four subgroups each isomorphic to SL(2, 3). These four subgroups are permuted by $C_H(\sigma_1\sigma_2^{-1})/C_J(\sigma_1\sigma_2^{-1}) \cdot M$ ($\cong \mathcal{Q}_4$) acting in the natural way. Hence $C(t) \cap C_J(\sigma_1\sigma_2^{-1}) \cong E_{32}$ and therefore $C_J(t) \cong E_{16} \times Q_8 * Q_8$. It follows that tJ contains 3 classes of involutions in K with representatives t, tz, tj (j an involution in $C_J(\sigma_1\sigma_2) - \langle z\rangle$), having 16, 16, 288 conjugates, respectively, in $N_K(tJ)$ ($t \nsim_K tz$ as $(ut)^2 = 1$ and $v^2 \in t\langle z\rangle$).

In the same way as above we see that $[uv, C_J(\sigma_1\sigma_2)] = 1$ and $C_J(uv) \cap C_J(\sigma_1\sigma_2^{-1}) \cong E_{32}$. Further, $Z = Z(C_J(\langle uv, t\rangle)) \cong E_8$, all involutions in $Z - \langle z\rangle$ are conjugate in $\langle J, c\rangle$ and therefore $C_H(Z)$ covers F_2J/J where $\bar{F}_2 = \langle F_2, c, z\rangle/\langle c, z\rangle \cong E_{16}$. Similarly $C_H(Z)$ covers $N_K(F_2)J/J$ so if $j_2 \in Z - \langle z\rangle$ then $C_K(j_2)/C_J(j_2) \cong E_{16} \cdot \mathcal{Q}_6$ (as this subgroup is maximal in $U_4(3)$).

Let j_1 be an involution in $C_J(\sigma_1 c) - \langle z\rangle$. Without loss we may take $U_2\langle \sigma_2 c\rangle$ to be a Sylow 3-subgroup of $C(j_1) \cap C_K(\sigma_1 c)$ whence $U_2\langle \sigma_2 c\rangle$ is a Sylow 3-subgroup of $C_H(j_1)$. It now follows by counting the number of involutions in $J - \langle z\rangle$ that $|C_K(j_1)/C_J(j_1)| = 2^5 \cdot 3^4 \cdot 5$ or $2^6 \cdot 3^4 \cdot 5$. The structure of $U_4(3)$ immediately yields $C_K(j_1)/C_J(j_1) \cong PSp_4(3)$. We have therefore determined all conjugacy classes of involutions in K.

LEMMA 3. *There are precisely two conjugacy classes of involutions in $J - \langle z\rangle$ in H with representatives j_1, j_2 where $C_K(j_1)/C_J(j_1) \cong PSp_4(3)$ and $C_K(j_2)/C_J(j_2) \cong E_{16} \cdot \mathcal{Q}_6$. In $K - J$ there are precisely three classes of involutions with representatives t, tz, tj where $j \in C_J(\sigma_1\sigma_2) - \langle z\rangle$.*

If $H \neq K$, it follows from our assumptions and the lemmas above that $H = K\langle \pi_i \rangle$ or $H = K\langle \pi_1, \pi_2 \rangle$ where $\pi_i \in N_H(\langle c \rangle)$ and $\pi_i\langle c, z\rangle/\langle c, z\rangle = \bar{\pi}_i$, $i = 1, 2, 3$, correspond to the elements introduced in the previous section. In addition we may assume $\pi_i^2 \in \langle z \rangle$, $i = 1, 2$, and $\pi_3^2 = t$.

Suppose $[\pi_i, c] = 1$ for $i = 1, 2$ or 3. As we may assume $[\pi_i, \langle \sigma_1, \sigma_2 \rangle] = 1$ and as $C_K(\pi_i)$ covers $C_K(\sigma_1)/O_{2,3}(C_K(\sigma_1))$, it follows we may assume $C(\pi_i) \cap U_1 = \langle U_2, c \rangle$, $C(\pi_i t) \cap U_1 = \langle U_3, c \rangle$. Thus $\pi_i^v = \pi_i t$ so $i = 1$ or 2; i.e. π_3 inverts c. As U_2 acts irreducibly on $J_3/\langle z \rangle$, $[\pi_i, J_3] = 1$. Further π_i inverts an element λ in $U_3 - \langle \sigma_1^{-1}c \rangle$, $\lambda \sim_K \sigma_1\sigma_2^{-1}$. As $\langle \lambda \rangle \sim \langle \lambda\sigma_1^{-1}c \rangle \sim \langle \lambda\sigma_1 c^{-1} \rangle$ in U_3 and $C(\lambda) \cap J_2 \cong Q_8$, it follows that $[\pi_i, C(\lambda) \cap J_2] = 1$ while π_i swaps the two quaternion groups $C(\lambda\sigma_1^{-1}c) \cap J_2$, $C(\lambda\sigma_1 c^{-1}) \cap J_2$. Thus

$$C_J(\pi_i) \cong E_4 \times Q_8 * Q_8 * Q_8 * Q_8,$$

and we suppose $j_1 \in Z(C_J(\pi_i))$. Then all involutions in $Z(C_J(\pi_i)) - \langle z \rangle$ are conjugate to j_1 (in $\langle c, J \rangle$) and $C_K(j_1)$ covers $C_K(\pi_i)/C_J(\pi_i) \cdot \langle c \rangle \cong PSp_4(3)$.

If $\langle \mu \rangle$ is a Sylow 5-subgroup of $C_K(\pi_i)$ then $C_J(\mu) \cap C_J(\pi_i) = Z(C_J(\pi_i))$. Now if $\langle \mu' \rangle$ is a Sylow 5-subgroup of $C_K(j_2)$, as $\langle \pi_2, F_2, J \rangle / J \cong E_{32}$ we may suppose $j_2 \in C_J(\pi_2) \cap C_J(\mu')$. Since $Z(C_J(\pi_i))$ only contains conjugates of j_1 we have proved that π_2 inverts c also.

Finally, we will show $[\pi_1, c] = 1$. From §1, $\pi_1 u$ centralizes a Sylow 3-subgroup of $N(\bar{F}_1)$ in $C_K(c)/\langle c, z \rangle$. However a Sylow 3-subgroup of $N(\bar{F}_1)$ contains conjugates of $\bar{\sigma}_1\bar{\sigma}_2$, $\bar{\sigma}_3$ and $C_K(\sigma_1\sigma_2) = C_J(\sigma_1\sigma_2) \cdot M\langle u, t \rangle$. Therefore $c \in C(\pi_1) \cap C_K(c)$ because M does not contain conjugates of σ_3; i.e. a Sylow 3-subgroup of $N_K(F_1 J)$ cannot split over $\langle c \rangle$.

If $h \in H - K$ with $h^2 \in \langle z \rangle$ and $c^h = c^{-1}$, then $C_J(h)$ is elementary abelian of order 64 or 128 depending whether $h \sim_H hz$ or not. Since $3^3 \nmid |\mathrm{GL}(5, 2)|$, we must have $C_J(\pi_2) \cong C_J(\pi_3 a_1) \cong E_{128}$. Let $\langle \sigma_1\sigma_2^{-1}, \sigma_3' \rangle$ be a Sylow 3-subgroup of $C_H(\pi_2 uv)$, $\sigma_3 \sim_K \sigma_3'$, and note that $C_J(\sigma_3') \subset C_J(\sigma_1\sigma_2^{-1})$. Thus $\pi_2 uv$ acts fixed-point-free on $O_{3,2,3}(C_H(\sigma_3'))/O_{3,2}(C_H(\sigma_3')) \cong E_9$ whence $\pi_2 uv$ normalizes the two quaternion subgroups in $C_J(\sigma_3')$ and so $\pi_2 uv \sim_H \pi_2 uvz$. Similarly, $\pi_3 u$ acts fixed-point-free on $\langle \sigma_1\sigma_2^{-1}, c \rangle$ and so normalizes the two quaternion subgroups in $C_J(\sigma_1\sigma_2)$. Thus $C_J(\pi_2 uv) \cong C_J(\pi_3 u) \cong E_{64}$. These results are collected in the following lemma.

LEMMA 4. *If* $\pi_1 \in H$ *then* $[\pi_1, c] = 1$ *and* $C_J(\pi_1) \cong E_4 \times Q_8 * Q_8 * Q_8 * Q_8$. *If* $\pi_2 \in H$ *then* π_2 *inverts* c *and if* $h \in \pi_2 K$ *with* $h^2 \in \langle z \rangle$, *h is conjugate to one of* π_2, $\pi_2 z$, $\pi_2 uv$ *with* $C_J(\pi_2) \cong E_{128}$, $C_J(\pi_2 uv) \cong E_{64}$. *Finally, if* $\pi_3 \in H$ *then* π_3 *inverts c also and any element in* $\pi_3 K$ *with square in* $\langle z \rangle$ *is conjugate to one of* $\pi_3 a_1$, $\pi_3 a_1 z$, $\pi_3 u$ *where* $C_J(\pi_3 a_1) \cong E_{128}$ *and* $C_J(\pi_3 u) \cong E_{64}$.

3. The proof of case (b).

LEMMA 5. *Let G be a finite group which satisfies the assumptions of the theorem. Then* $z \not\sim_G j_1$, *and if* $z \sim_G j_2$, *then either* $H = K\langle \pi_2 \rangle$ *or* $|H : K| = 4$.

PROOF. If x is an involution in $J - \langle z \rangle$, $z \sim_G x$ implies that $O_2(C_H(x))/C_J(x)$ contains a normal elementary abelian subgroup of order 32 by [**16**, Proposition 7].

It follows from Lemma 3 and the structure of $N(\bar{F}_2)$ given in §1 that $z \nsim_G j_1$ and $z \sim_G j_2$ implies $\pi_2 \in H$. The lemma is proved.

The following simple result will be needed a number of times in this section.

PROPOSITION. *Let G be a finite group, z an involution in G and $H = C_G(z)$. Suppose that $P \neq 1$ is a p-subgroup of H (p an odd prime) which satisfies*

(*) if $g^{-1}Pg \subset H$, $g \in G$, then there exists $h \in H$ so that $g^{-1}Pg = h^{-1}Ph$.

Then for any involution $x \in C_H(P)$, $x \sim_G z$ if and only if $x \sim_{N(P)} z$. If, in addition, $N_G(P) = N_H(P)C_G(P)$, then $x \sim_G z$ if and only if $x \sim_{C(P)} z$.

For the rest of this section, all results will be proved under the following assumptions.

Hypothesis 1. *Let G be a finite group which satisfies the assumptions of the theorem. In addition, suppose $G \neq H \cdot O(G)$ and $H = K$, $H = K\langle \pi_2 \rangle$ or $H = K\langle \pi_3 \rangle$.*

LEMMA 6. *We have $z \nsim_G \pi_3 u$ and $z \sim_G j_2$ if and only if $z \sim_G \pi_2$ and $j_1 \sim_G \pi_2 z$.*

PROOF. If $H = K\langle \pi_3 \rangle$ then $z \nsim_G j_2$ by Lemma 5. If $\langle \mu \rangle$ is a Sylow 5-subgroup of $C_H(\pi_3 u)$ then by the proposition, $z \sim_G \pi_3 u$ if and only if $z \sim \pi_3 u$ in $C_G(\mu)$. For some $j' \in C_J(\mu) - \langle z \rangle$, $\langle \pi_3 u, z, j' \rangle$ is a Sylow 2-subgroup of $C_H(\mu) \cap C_H(\pi_3 u)$. If $z \sim_G \pi_3 u$ then $\{j', j'z\} \lhd N_G(\langle \pi_3 u, z, j' \rangle)$ whence $\langle \pi_3 u, z, j' \rangle$ is a Sylow 2-subgroup of $C_G(\mu)$. Thus $z \nsim_G \pi_3 u$.

Now suppose $H = K\langle \pi_2 \rangle$. Without loss we take $\pi_2 \in C_H(\mu)$ and observe that if $\pi_2^2 = 1$, $C_H(\mu)/\langle \mu \rangle$ is isomorphic to the centralizer of a central involution in $\mathcal{A}_8$ $(C_J(\pi_2) \cap C_J(\mu) \cong E_8)$. Since $C_H(j_2)/O_2(C_H(j_2)) \cong \mathcal{A}_6$, if we assume $j_2 \in C_J(\mu)$ also we have $C_H(j_2)$ does not cover $N_H(\langle \mu \rangle)/C_H(\mu)$ whence j_2 has 12 conjugates in $C_H(\mu)$ (the other 6 are conjugate to j_1). If $C_G(\mu)$ has a subgroup K of index two then $K \cap C_H(\mu) = \langle \mu \rangle \times C_J(\mu) \cdot \langle c \rangle$ and $\langle z \rangle$ is weakly closed in $C_H(\mu)$ with respect to $C_G(\mu)$. On the other hand if $C_G(\mu)$ has no subgroup of index two we see easily that $\pi_2^2 = 1$. A result of D. Held **[24]** yields $C_G(\mu)/\langle \mu \rangle \cong \mathcal{A}_8$, $\mathcal{A}_9$ or $\mathrm{Hol}(E_8) = E_8 \cdot L_2(7)$. The last case cannot occur as $z \nsim_G j_1$.

Thus we have proved that in $C_G(\mu)$, either z is not conjugate to any involution in $C_H(\mu) - \langle z \rangle$ or $z \sim j_2 \sim \pi_2$, say, and $j_1 \sim \pi_2 z$. The corresponding result for G follows from the proposition.

LEMMA 7. *Either $\langle \sigma_1 \sigma_2 \rangle$ satisfies condition (*) of the proposition (i.e. $\langle \sigma_1\sigma_2 \rangle^H = \langle \sigma_1 \sigma_2 \rangle^G \cap H$) or $\sigma_1 \sigma_2 \sim_G c$, $H = K\langle \pi_2 \rangle$ and*

$$z \sim_G j_2 \sim_G t \sim_G tj \sim_G \pi_2, \quad j_1 \sim_G tz \sim_G \pi_2 z.$$

PROOF. We begin by listing some properties of Sylow 2-subgroups of the centralizers of elements of order three in H. In the table, y is an element of order 3, Y is a Sylow 2-subgroup of $C_H(y)$, $Y/Y \cap J$ is given for $H = K\langle \pi_2 \rangle$ (and $H = K\langle \pi_3 \rangle$). Recall that $Y \cap J = \langle z \rangle$ or $Y \cap J$ is the central product of quaternion groups.

y	Y	$\lvert Y\rvert$	$\lvert Y \cap J\rvert$	$Y/Y \cap J$
c	T_0	2^8	2	type $U_4(3)$
σ_1	T_1	2^5	2	$Q_8 \times Z_2$ $(Q_8 * Z_4)$
$\sigma_1\sigma_2^{-1}$	T_2	2^{12}	2^9	E_8 (D_8)
$\sigma_1\sigma_2$	T_3	2^8	2^5	D_8
$\sigma_1 c$	T_4	2^{10}	2^7	Q_8
$\sigma_1\sigma_2^{-1}c$	T_5	2^5	2^3	$Z_2 \times Z_2$
σ_3	T_6	2^6	2^5	Z_2

In addition we note that when $H = K$, T_i $(i = 1, \ldots, 6)$ has order 2^8, 2^4, 2^{11}, 2^7, 2^{10}, 2^5, 2^5, respectively.

If $H = K$ then $\langle z\rangle = T_3' \cap Z(T_3)$, while if $H \neq K$, $Z(T_3) \subset T_3' \subseteq \langle t, T_3 \cap J\rangle$ whence $\langle z\rangle = Z(T_3) \cap [T_3, T_3']$. Thus T_3 is a Sylow 2-subgroup of $C_G(\sigma_1\sigma_2)$ so $\sigma_1\sigma_2^{-1} \not\sim_G \sigma_1\sigma_2 \not\sim_G \sigma_1 c$, and, in addition, $c \not\sim_G \sigma_1\sigma_2$ if $H = K$. Clearly $\Omega_1(T_5') = \langle z\rangle$ and $Z(T_6) = \langle z\rangle$ so T_5, T_6 are Sylow 2-subgroups of $C_G(\sigma_1\sigma_2^{-1}c)$, $C_G(\sigma_3)$, respectively. In particular, $\sigma_1\sigma_2^{-1}c \not\sim_G \sigma_1\sigma_2 \not\sim_G \sigma_3$. Since $Z(T_3) = \langle t, z\rangle$, if x is an involution in T_3 then $C_G(x) \cap T_3$ contains an elementary abelian subgroup of order 16. As T_1 clearly contains no such subgroup, $\sigma_1 \not\sim_G \sigma_1\sigma_2$.

It remains to consider the case $\sigma_1\sigma_2 \sim_G c$. We may assume here that $t \sim_G z$ $(Z(T_0) = Z(T_3) = \langle t, z\rangle)$, whence

$$C_G(t) \cap C_G(\sigma_1\sigma_2)/\langle t, \sigma_1\sigma_2\rangle \cong U_4(3).$$

As $U_4(3)$ has one class of involutions, $z \sim \pi_3 u$ or $\pi_3 ut$ in $C_G(\sigma_1\sigma_2)$. This is impossible as $\pi_3 u \sim_H \pi_3 ut$ and $z \not\sim_G \pi_3 u$ by Lemma 6. Thus we have $H = K\langle \pi_2\rangle$.

We note that if $j \in T_3 \cap J$ then $j \sim_H j_1$ as $C_H(j_2)$ only contains conjugates of $\sigma_1\sigma_2^{-1}$, σ_3. It follows therefore from $z\langle t\rangle \sim j\langle t\rangle$ that $z \sim tj$ and $j_1 \sim tz$. Also from $z\langle t\rangle \sim \pi_2\langle t\rangle$ we may assume $z \sim \pi_2$ and $\pi_2 t \sim j_1$ (thus $\pi_2 t \sim_H \pi_2 z$). This completes the proof of the lemma.

LEMMA 8. *If $z \not\sim_G j_2$ then $z^G \cap K = \{z\}$. In particular, $H \neq K$.*

PROOF. If $z \not\sim_G j_2$ then $\langle \sigma_1\sigma_2\rangle$ satisfies (*) of the proposition. It follows therefore from $Z(T_3) = \langle t, z\rangle$ and Burnside's lemma [**13**, IV, 2.5] that $t \not\sim_G z \not\sim_G tz$. If $z \sim_G tj$, Lemma 6 implies that $z^G \cap T_3 \subset J\langle t\rangle \cap T_3$. However

$$\langle z^G \cap C(tj) \cap T_3\rangle = \langle tj\rangle \times C_J(tj) \cap T_3 \cong Z_2 \times Z_2 \times D_8$$

and so $C(tj) \cap T_3$ is a Sylow 2-subgroup of $C(tj) \cap C_G(\sigma_1\sigma_2)$. Thus $z^G \cap K = \{z\}$. If $H = K$, Glauberman's theorem [**2**] yields $G = H \cdot O(G)$ against Hypothesis 1. The lemma is proved.

LEMMA 9. *We have $H = K\langle \pi_2\rangle$.*

PROOF. Suppose $H = K\langle \pi_3\rangle$ whence $z \not\sim_G j_2$. We will show $z \not\sim_G \pi_3 a_1$, and hence $z^G \cap H = \{z\}$ by Lemmas 6 and 8. However, this will contradict Glauberman's result [**2**]; i.e. we will have shown $H \neq K\langle \pi_3\rangle$.

We may assume $\pi_3 a_1 \in T_1$, where $T_1/\langle z\rangle \cong Q_8 * Z_4$ in this case. Since $\langle\sigma_2\rangle$ normalizes T_1 but does not centralize $T_1 \cap K \cong Z_2 \times Q_8$, $Z(T_1) = \langle\pi_3, z\rangle$ say with $\pi_3^2 = t$. By Lemma 8, $tz \nsim_G z \nsim_G t$ so T_1 is a Sylow 2-subgroup of $C_G(\sigma_1)$. It follows immediately that $\langle\sigma_1\rangle$ satisfies $(*)$ of the proposition as $\mho^1(T_5') = \langle z\rangle$ whereas T_1' contains t or tz. In order to complete the proof of the lemma we only need to show $z \nsim \pi_3 a_1$ in $C_G(\sigma_1)$.

Now $C(\pi_3 a_1) \cap T_1 = \langle\pi_3 a_1, \pi_3, z\rangle$ so $\mho^1(C(\pi_3 a_1) \cap T_1) = \langle t\rangle$. Thus if $\pi_3 a_1 \sim z$ in $C_G(\sigma_1)$ there exists $g \in C_G(t) \cap C_G(\sigma_1)$ with $z^g \in T_1 - \langle z\rangle$. However $T_1/\langle t\rangle \cong E_{16}$ (because $\pi_3 a_1$ is an involution, $a_1^2 = t$ also) whence $z\langle t\rangle$, $z^g\langle t\rangle$ are conjugate in $N(T_1)/\langle t\rangle$ by Burnside's lemma. This is clearly impossible so $H \neq K\langle\pi_3\rangle$. It follows therefore from Lemma 8 that $H = K\langle\pi_2\rangle$.

LEMMA 10. *We have $z \sim_G j_2$ if and only if $z \sim_G \pi_2 uv$. Therefore we have $z \sim_G j_2$. Further, $c \sim_G \sigma_1\sigma_2$ and the fusion of involutions has been completely determined.*

PROOF. If $\sigma_3 \sim_G \sigma_1$ then there exists $g \in G$ with $T_1^g \subset T_6$. However $T_1 \cong Z_2 \times Z_2 \times Q_8$ so $T_1^g \cap T_6 \cap J$ is abelian of order 16, or isomorphic to $Z_2 \times Q_8$. Both of these cases are impossible ($T_6 \cap J \cong Q_8 * Q_8$). It follows that $\langle\sigma_3\rangle$ satisfies condition $(*)$ of the proposition–see the proof of Lemma 7. Since T_6 is of type $G_2(3)$, we easily show $z \sim j_2$ in $C_G(\sigma_3)$ if and only if $z \sim \pi_2 uv$ (T_6 has at most three classes of involution in $C_H(\sigma_3)$ with representatives $z, j_2', \pi_2 uv, j_2' \sim_H j_2$). The first part of the lemma now follows from the proposition.

If $z \nsim_G j_2$, it follows from Lemmas 6 and 8 that $z^G \cap H = \{z\}$. We have the same contradiction as in the previous two lemmas. It remains to show therefore that $z \sim_G j_2$ implies $\sigma_1\sigma_2 \sim_G c$.

Suppose to the contrary that $\sigma_1\sigma_2 \nsim_G c$. By Lemma 7, $\langle\sigma_1\sigma_2\rangle$ satisfies $(*)$ of the proposition, whence by Lemma 6 we have $z \sim \pi_2$ in $C_G(\sigma_1\sigma_2)$. Let $X = C_G(\sigma_1\sigma_2)$, $Y = C_H(\sigma_1\sigma_2)$ and $W = \langle\pi_2, t\rangle \times (C_J(\pi_2) \cap T_3) \cong E_{32}$. We easily see that T_3 contains only two elementary abelian subgroups of order 32, both of which are normal in T_3. Thus $z \sim \pi_2$ in $N_X(W)$. As above, $Z(T_3) = \langle t, z\rangle$ so z, t, tz lie in distinct conjugacy classes in X. We have $C_X(W) = W \times \langle\sigma_1\sigma_2\rangle$, $N_Y(W)/C_X(W) \cong \Sigma_4$, and in $N_Y(W)$, $z(1)$, $t(1)$, $tz(1)$, $j(6)$, $tj(6)$, $\pi_2(8)$, $\pi_2 z(8)$ ($j \in T_3 \cap J$) represent the classes of involutions in W, with the number of conjugates given in brackets. Since $27 \nmid |\mathrm{GL}(5, 2)|$, z cannot have 9 conjugates in $N_X(W)$ so z has 15 conjugates in $N_X(W)$: $z \sim \pi_2 \sim tj$. Replacing t by tz if necessary we have $tz \sim j \sim \pi_2 z$ (recall $j \sim_H j_1 \sim_G \pi_2 z$ by Lemma 6) and $\langle t\rangle \lhd N_X(W)$. It now follows from Part I, Proposition 6, $z\langle t\rangle \sim j\langle t\rangle$ in $N_X(W)/\langle t\rangle$ and the fact that $\pi_2 u(J \cap T_3)$ does not contain involutions, that $C_X(t)/\langle t, \sigma_1\sigma_2\rangle \cong U_4(3)$. However, this yields $z\langle t\rangle \sim_X u\langle t\rangle$ so $z \sim_X t$ or $z \sim_X tz$, a contradiction. This completes the proof of the lemma.

LEMMA 11. *Let $F = C_J(t) \cap C_J(u)$ and $E = C_H(F)$. Then $E \cong E_{2^{11}}$ and $N_G(E)/E \cong M_{24}$, the Mathieu group on 24 letters. Further we have $C_G(j_1) \cap N_G(E)/E \cong$ Aut M_{22}.*

PROOF. As $u \sim t \sim ut$ in $N_H(\langle \sigma_1\sigma_2, c\rangle)$ and $C_J(\langle u, t\rangle) \cap C_J(\sigma_1\sigma_2) \cong E_8$, we have $F = C_J(t) \cap C_J(u) \cong E_{2^7}$. In the same way as in Part I, §3, we have $|C_H(t)\colon N_H(F) \cap C_H(t)| = |C_H(u)\colon N_H(F) \cap C_H(u)| = 3$, from which it follows that $N_H(F)/J \cong E_{16} \cdot Z_3 \cdot \Sigma_6$. Hence $E = C_H(F)$ covers $O_2(N_H(F))/J$ whence $E \cong E_{2^{11}}$.

Let T be a Sylow 2-subgroup of H so that $C_H(t)$ covers T/J. Suppose $E^g \subset T$ for some $g \in G$. We will show $E^g = E$. Observe first that $2^6 < |E^g \cap J| < 2^7$, $2^4 < |E^g \cdot J/J| \leqslant 2^5$ and $E^g \cap tJ \neq \varnothing$. If $E^g \subset K \cap T$ then we have $E^g \cap uJ \neq \varnothing$ or $E^g \cap uvJ \neq \varnothing$. Since $C_J(t_1) \subset C_J(t)$ for any $t_1 \in tJ$, and as $C_J(uv) \cap C_J(t) \cong E_4 \times Q_8 * Q_8$, we have $E^g \cap J = F$ and so $E^g = E$. If $E^g \not\subset K \cap T$ then E^g contains conjugates of $\pi_2 uv$, whence $|E^g \cap J| = 64$. It follows that $E^g \cap uJ \neq \varnothing$ or $E^g \cap uvJ \neq \varnothing$ whence $E^g \cap J \subset F$. However if $e \in E^g$ and $eJ \sim_H \pi_2 uvJ$, then as e inverts c, $|C_F(e)| = 2^4$. This contradiction means we have proved that E is weakly closed in T with respect to G.

There are 1771 conjugates of z in E (the remaining 276 are conjugate to j_1) and they are all conjugate to z in $N_G(E)$ by the above remarks. Thus $|N_G(E)/E| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = |M_{24}|$. If R is a Sylow 23-subgroup of $N_G(E)$ then $C_E(R) = 1$ from which it follows that $N_G(E)/E$ is a simple group. A result of R. Stanton **[28]** yields $N_G(E)/E \cong M_{24}$. The last fact, that $C_G(j_1) \cap N_G(E)/E \cong \operatorname{Aut} M_{22}$, follows from the structure of M_{24} (see **[23]**, for example).

LEMMA 12. *We have* $C_G(j_1)/\langle j_1\rangle \cong \operatorname{Aut} M(22)$ *with* $j_1 \in C_G(j_1)'$.

PROOF. Set $C_J(\sigma_1\sigma_2) = \langle r_1, s_1\rangle * \langle r_2, s_2\rangle$ where $\langle r_i, s_i\rangle \cong Q_8$, $i = 1, 2$, are chosen so that we have

$$r_1^u = r_2, \quad s_1^u = s_2, \quad r_1^{\pi_2} = r_2^{-1}, \quad s_1^{\pi_2} = r_2 s_2.$$

Let $j = r_1 r_2$, $z_1 = s_1 s_2$ and $j_0 = s_1 r_2 s_2$. It follows that we have

$$\langle u, \pi_2\rangle \subseteq C_H(j), \quad z_1^u = z_1, \quad z_1^{\pi_2} = z_1 j, \quad j_0^u = j_0 jz, \quad j_0^{\pi_2} = j_0.$$

Recall that $j_1 \sim_H j \sim_H z_1 \sim_H j_0$ and that $|O_3(C_H(\sigma_1\sigma_2))| = 3^3$. If $D = \langle j, z_1, z\rangle$ then $D \subseteq C_J(u) \cap C_J(t) = F$ so $[D, E] = 1$. Also $[D, O_3(C_J(\sigma_1\sigma_2))] = 1$, so from the structure of $PSp_4(3)$ we have $D \lhd C_H(j)$. Clearly $C_H(D)/C_J(D) \cong PSp_4(3)$ and $C_H(D)\langle j_0, \pi_2\rangle = C_H(j)$.

Let l be an involution in $C_J(j) - \langle z, j\rangle$ and L a Sylow 2-subgroup of $C_H(l) \cap C_H(j)$. Since $Z(L \cap J) = \langle l, j, z\rangle$, it follows that $Z(L) = \langle l, j, z\rangle$. If S is a Sylow 2-subgroup of $C_H(j_0) \cap C_H(j)$ then as $j_0 \notin S'$ we have $\langle z\rangle \subseteq S' \cap Z(S) \subseteq \langle j, z\rangle$. Thus S is a Sylow 2-subgroup of $C_G(j_0) \cap C_G(j)$ (as $j \sim_H jz \sim_H j_1$), and if $j_0 \sim l$ in $C_G(j)$ for $l \in C_J(j)$, then $j_0 \sim l$ in $C_H(j)$.

In $N_G(E) \cap C_G(j)$ we see that z (and hence jz) has 231 conjugates and z_1 has 44. Thus if $e \in E$ with $e \sim_G j_1$ then e is conjugate to j, jz or z_1 in $N_G(E) \cap C_G(j)$. If $x \in (N_G(E) \cap C_G(j))' - E$ with $x \sim_G j_1$, then $xE \sim j'E$ where we may take $j' \in C_J(u) \cap C_J(j) - E$ (M_{22} has only one class of involutions). As all involutions in uJ which are conjugate to j_1 in G lie in E, it follows that x is conjugate to an involution in $C_J(j)$.

Since $N_H(E) \cap C_H(j)/E \cong E_{32} \cdot \Sigma_5$ and $C_K(\sigma_1\sigma_2) \cap N_H(E) \cap C_H(j) = C_E(\sigma_1\sigma_2)\langle\sigma_1\sigma_2\rangle\langle j_0\rangle$, the structure of M_{22} yields $j_0 \notin (N_G(E) \cap C_G(j))'$. Combining this with our work in the two paragraphs above we see that j_0 is not conjugate to any involution in $(N_G(E) \cap C_G(j))'$. Thus Thompson's transfer theorem [**29**, Lemma 5.38] yields that $C_G(j)$ contains a subgroup X of index two with $j_0 \notin X$.

It follows that $C_H(j) \cap X/\langle j \rangle$ satisfies the assumptions of Part I, Theorem B. Thus either $\langle j, z_1 \rangle \lhd X$ with $X/\langle j, z_1\rangle \cong U_6(2)$, or $X/\langle j \rangle \cong M(22)$. The first case is impossible as $\langle j, z_1 \rangle \ntriangleleft C_H(j)$, so we conclude $X/\langle j \rangle \cong M(22)$. Finally, $j \in C_H(j)'$ and j_0 acts on $C_H(j) \cap X$ as an outer automorphism whence $C_G(j)/\langle j \rangle \cong \operatorname{Aut} M(22)$.

LEMMA 13. *The group G has order* $2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29 = |M(24)'|$.

PROOF. Thompson's order formula yields

$$|G| = 2^{21} \cdot 3^7 \cdot 5 \cdot 7\, a(j_1) + 2^{19} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13\, a(z)$$

where, for any involution $g \in G$,

$$a(g) = |\{(x, y)|\ (xy)^n = g \text{ for some positive integer } n, \text{ with } x \sim_G z, y \sim_G j_1\}|.$$

$a(z) = 2^2 \cdot 3^3 \cdot 7 \times 2{,}820{,}636$. In the computation of $a(z)$, x, y will be involutions in $H - \langle z \rangle$ with $(xy)^n = z$ for some n and $x \sim_G z, y \sim_G j_1$. Thus x is conjugate in H to one of

$$j_2(2 \cdot 3^5 \cdot 7),\quad t(2^4 \cdot 3^4 \cdot 5 \cdot 7),\quad tj(2^5 \cdot 3^6 \cdot 5 \cdot 7),$$
$$\pi_2(2^7 \cdot 3^3 \cdot 7),\quad \pi_2uv(2^8 \cdot 3^5 \cdot 5 \cdot 7);$$

while y is conjugate to either $j_1(2^2 \cdot 3^3 \cdot 7)$, $tz(2^4 \cdot 3^4 \cdot 5 \cdot 7)$ or $\pi_2z(2^7 \cdot 3^3 \cdot 7)$, where the number in brackets is the number of conjugates of the involution in H. Note that if either $x \nsim_G xz$ or $y \nsim_G yz$ then $(xy)^n = z$ implies n is odd.

Suppose at first that $y \sim_H j_1$. A computation yields that there are 1728 conjugates of j_2 in $J - C_J(j_1)$ whence there are $2^2 \cdot 3^3 \cdot 7 \times 1728$ pairs (x, y) with $x \sim_H j_2$. By the remark above we see there are no such pairs when $x \sim_H t$ or $x \sim_H \pi_1$. We have $(tjy)^2 = z$ if and only if $[j, y] = z$ and $\langle j, y \rangle \subset C_J(t)$. There are 32 such y's and so $2^{10} \cdot 3^6 \cdot 5 \cdot 7$ pairs (x, y) with $x \sim_H tj$ ($[tj, J]$ is elementary abelian so $|tjy| \leqslant 4$). For π_2uv, $(\pi_2uvy)^n = z$ implies $n = 4$; i.e. $(\pi_2uvy)^2$ is of order four in J. Now there is precisely one class (in H) of elements of order four in J with representative r_1 say, where $C_H(r_1)/C_J(r_1) \cong E_{81} \cdot \Sigma_5$. We compute that π_2uv inverts 64 elements of order four (in J) which are all conjugate under the action of $C_H(\pi_2uv)$. There are 6 conjugates of y with $(\pi_2uvy)^2$ of order four so there are $2^{15} \cdot 3^6 \cdot 5 \cdot 7$ pairs (x, y) with $x \sim_H \pi_2uv$.

From the remarks above, if $(xtz)^n = z$ then either $n = 1$ and $x = t$ or $xtz = \sigma z$, σ of odd order ($\neq 1$). (In this latter case, $xz \cdot tz = \sigma$, $\langle xz, tz \rangle$ is a dihedral group of order $2|\sigma| = 2n$.) If $y_1, \ldots, y_m$ are representatives of the $N_H(\langle\sigma\rangle)$-classes of $(tz)^H$ which invert σ, then

$$I = \sum_{i=1}^{m} (|\sigma| - 1)|C_H(tz)|/|C_H(y_i) \cap N_H(\langle\sigma\rangle)|$$

gives the number of involutions $x \in H$ with $xtz \sim_H \sigma z$. If $\sigma \sim_H \mu$ then $I = 2^{11} \cdot 3^2$; if $\sigma \sim_H \sigma_1\sigma_2$, $I = 2^{10}$; if $\sigma \sim_H \sigma_1\sigma_2^{-1}$, $I = 2^8$; if $\sigma \sim_H \sigma_3$, $I = 2^{11} \cdot 3$; otherwise

$I = 0$. Hence there are

$$2^4 \cdot 3^4 \cdot 5 \cdot 7(1 + 2^{11} \cdot 3^2 + 2^{10} + 2^8 + 2^{11} \cdot 3)$$

pairs (x, y) with $y \sim_H tz$.

Similarly if $y \sim_H \pi_2 z$ and $\sigma \sim_H c$ then $I = 2^7$; while if $\sigma \sim_H \sigma_1\sigma_2$, $I = 2^8 \cdot 3 \cdot 5$; otherwise $I = 0$. Hence there are $2^7 \cdot 3^3 \cdot 7(1 + 2^7 \cdot 2^8 \cdot 3 \cdot 5)$ pairs (x, y) with $y \sim_H \pi_2 z$. Thus

$$a(z) = 2^2 \cdot 3^3 \cdot 7 \times 2{,}820{,}636.$$

$a(j_1) = 3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \times 3{,}608{,}577$. Again we will take $x \sim_G z$, $y \sim_G j_1$ but this time with $(xy)^n = j$ for some n. Clearly $x, y \in C_G(j)$ and we will use the same notation as in Lemma 12. There are precisely three classes of involutions in $M(22)$ and another three classes in Aut $M(22) - M(22)$. In $C_G(j)' - \langle j \rangle$ we have four classes of involutions with representatives z, jz, z_1, j_2, where $j_2 \in C_J(j) - C_J(\sigma_1\sigma_2)$ and $j_2 \sim j_2 j$, $z_1 \sim z_1 j$ in $C_G(j)$. In $C_G(j) - C_G(j)'$ there are two classes of involutions with representatives j_0, j_0' where $\langle j_0', z_1 \rangle \times \langle j \rangle = C_J(\mu) \cap C_J(j)$, $\langle \mu \rangle$ a Sylow 5-subgroup of $C_H(j)$. Further we have $j_0 \sim j_0 j$, $j_0' \sim j_0' j$ in $C_G(j)$. In addition, there is one class of elements of order four in $C_G(j) - C_G(j)'$ with square j; namely $(j_0 u z_1)^2 = j$. Since $C_G(j)'$ is isomorphic to the centralizer of a 3-transposition in $M(23)$, no element of $C_G(j)'$ squares to j (see **[1]**).

If $(xy)^n = j$, n odd, then by the above remarks, $x \sim z$, $y \sim jz$ in $C_G(j)$. It is enough therefore to determine the number of conjugates of $z\langle j \rangle$ in $C_G(j)$ whose product with $z\langle j \rangle$ has odd order. With the use of the character table for $M(22)$ **[25]** and the properties of $M(22)$ given in Part I, we compute that there are 1, $2^{14} \cdot 3^3$, 2^{10}, $2^{13} \cdot 5$ conjugates of $z\langle j \rangle$ whose product with $z\langle j \rangle$ is conjugate to 1, μ, $\sigma_1\sigma_2^{-1}$, $\sigma_1\sigma_2$, respectively. Hence if n is odd we get $3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \times 484{,}353$ pairs.

Now suppose n is even. By the remarks above we need only consider $x, y \in N_G(\langle j_0 u z_1 \rangle) \cap C_G(j) - C_G(j_0 u z_1) \cap C_G(j)$. Let $w = j_0 u z_1$ and we will use the relations given for $C_H(\sigma_1\sigma_2)$ in the proof of Lemma 12. In addition, as there exists $g \in G$ with $\sigma_1\sigma_2^g = c$ and $z^g = t$, we have $[u, v] = [u, \pi_2] = [v, \pi_2] = tz$. A computation yields that $\langle t, j_0\pi_2 uv \rangle$ $(\cong E_4) \subseteq C_H(w)$ and that uz_1 inverts w. Also, $C(u) \cap O_3(C_H(\sigma_1\sigma_2)) = \langle \sigma_1\sigma_2, \sigma \rangle \cong E_9$ is a Sylow 3-subgroup of $C_H(w)$. Without loss we assume $[u, C_J(\sigma_1\sigma_2 c)] = 1$ so that $C_J(u) \cap C_J(\sigma_1\sigma_2 c^{-1}) \cong E_8$. Thus

$$J_0 = C_J(u) \cap [\langle \sigma_1\sigma_2 \rangle, J] \cong E_4 \times Q_8 * Q_8$$

and $C_J(w) = J_0 \times \langle j \rangle$.

From the structure of Aut M_{22}, we have $C(wE)$ in $N_X(E)/E$ is isomorphic to $E_8 \cdot \mathrm{GL}(3, 2)$. Further, $C_E(j_0) = C_E(w) \cong E_{2^7}$ and w has 128 conjugates in wE, j_0 has 16, while the remaining 112 involutions are conjugate to j_0' in $C_G(j)$ (with $j_0' \sim_G z$). It follows that

$$C_H(w) = \langle w \rangle \times J_0\langle \sigma_1\sigma_2, \sigma \rangle \langle t, j_0\pi_2 uv \rangle.$$

Let $V = J_0\langle t, j_0\pi_2 uv \rangle$ and $W = V \cap E$. We easily see that $V \times \langle w \rangle$ is a Sylow 2-subgroup of $C_G(w)$ and that V is of type $\Omega_7(3)$ (see Part I, §6). Further, we can show $N_G(W) \cap C_G(w)/W \times \langle w\rangle \cong L_2(7)$ (the argument is similar to Part I, Lemma 6.3). It follows that $C_G(w)/\langle w\rangle$ is simple and, hence, $C_G(w)/\langle w\rangle \cong PSp_6(2)$ by a result of Solomon [27]. Since uz_1 centralizes $V\langle\sigma_1\sigma_2, \sigma\rangle$, assuming $uz_1 \sim_G j_1$ (rather than uz_1z) we have $N_G(\langle w\rangle) = \langle uz_1, w\rangle \times P$, $P \cong PSp_6(2)$.

The group P has 4 classes of involutions with representatives z, z_2, z_2z, j_3 where $z_2 \in Z(J_0) - \langle z\rangle$, $j_3 \in (C_J(\sigma_1\sigma_2c) \cap W) - \langle z\rangle$ (so all four involutions lie in W). It follows that uz_1P has 5 classes of involutions with only uz_1, uz_1z_2 conjugate to j_1 (we assume z_2 has only 7 conjugates in W and $C_P(z_2)/W \cong \Sigma_5$). The coset $w \cdot uz_1P = j_0P$ also has 5 classes of involutions with only j_0, j_0z conjugate to j_1 (recall j_0E contains only 8 conjugates of j, and $j_0 \sim_J j_0z$).

Since $xy = w$ implies $x \sim yw$, the remarks above yield that if $xy = w$ then $x \sim uz_1z$ and $y \sim j_0z$ or $x \sim j_0z_2$ and $y \sim uz_1z_2$ in $N_G(\langle w\rangle)$. From the structure of $P \cong PSp_6(2)$ we compute that $|\{z^g | g \in P \text{ and } z \cdot z^g \text{ has odd order}\}| = 1 + 2^7$ while $|\{z_2^g | g \in P \text{ and } z_2 \cdot z_2^g \text{ has odd order}\}| = 1 + 2^5$. Also, uz_1z has $2 \cdot 3^2 \cdot 5 \cdot 7$ conjugates in $N_G(\langle w\rangle)$ while uz_1z_2 has $2 \cdot 3^2 \cdot 7$ conjugates. Hence if n is odd we get

$$2^8 \cdot 3^5 \cdot 5 \cdot 11 \cdot 13 \times (2 \cdot 3^2 \cdot 5 \cdot 7 \times 129 + 2 \cdot 3^2 \cdot 7 \times 33)$$
$$= 3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \times 3{,}124{,}224 \text{ pairs.}$$

It follows that

$$a(j_1) = 3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \times 3{,}608{,}577$$

and therefore that $|G| = |M(24)'|$.

LEMMA 14. *The group G is simple.*

PROOF. We have $O(G) = 1$ because $z \sim_G j_2 \sim_G j_2z$ and $O(H) = 1$. If $1 \neq N \lhd G$ then $z \in N$ since if T is a Sylow 2-subgroup of H (and hence of G), $Z(T) = \langle z\rangle$. Thus $H = \langle z^G \cap H\rangle \subseteq N$ and so $N = G$ (using $Z(T) = \langle z\rangle$ and the Frattini argument).

This completes the proof of the theorem in case (b).

4. The proof of case (c). Throughout this section we will assume:

Hypothesis 2. Let G be a finite group which satisfies the assumptions of the theorem. In addition, suppose that $G \neq H \cdot O(G)$ and $H = K\langle\pi_1\rangle$ or $H = K\langle\pi_1, \pi_2\rangle$.

LEMMA 15. *The group G has a subgroup G_0 of index two with $G_0 \cap H = K\langle\pi_2\rangle$. Further G_0 is a simple group and $|G_0| = |M(24)'|$.*

PROOF. Recall that $C_J(\pi_1) \cong E_4 \times Q_8 * Q_8 * Q_8 * Q_8$ and that $C_K(\pi_1)/C_J(\pi_1) \cong Z_3 \times PSp_4(3)$. Let $Z = Z(C_J(\pi_1)) \cong E_8$ so that $C_H(Z)$ covers $C_K(\pi_1)/C_J(\pi_1) \cdot \langle c\rangle$. Thus $Z - \langle z\rangle$ contains 6 conjugates of j_1. Let S denote a Sylow 2-subgroup of $C_H(\pi_1)$ and note that $Z(S) \subseteq \langle Z, \pi_1\rangle$. Suppose at first that $\pi_1^2 = 1$.

Since $z \in S' \cap Z(S) \subseteq Z$ we see that S is a Sylow 2-subgroup of $C_G(\pi_1)$. As $|S| < 2^{19}$ it follows that π_1 is not conjugate (in G) to any involution in J. Now let $T_3\langle v\rangle$ be a Sylow 2-subgroup of $N_H(\langle\sigma_1\sigma_2\rangle)$ with $u, \pi_1 \in T_3 \subseteq C_H(\sigma_1\sigma_2)$. As $\langle t\rangle \lhd T_3$ and $[v, \pi_1] \in t\langle z\rangle$, $\langle t, z\rangle \subseteq (T_3 \cap C_H(t))'$. Let $j \in C_J(u) \cap T_3$ as usual so that $\langle u, C_J(j) \cap T_3\rangle' = \langle j, z\rangle$. Since $C_H(tj) \cap N_H(\langle\sigma_1\sigma_2\rangle)$ covers $\langle v\rangle T_3/T_3 \cap J$ we have $(C(tj) \cap T_3\langle v\rangle)' \supseteq \langle t, z, j\rangle$. Thus if l is any involution in tJ and L is a Sylow 2-subgroup of $C_H(l)$ we have $l \in L'$. It follows immediately that $l \nsim_G \pi_1$.

If $\pi_1^2 \neq 1$ then $\pi_1^2 = z$, so $(\pi_1 r)^2 = 1$ where r is an element of order 4 in $C_J(\pi_1)$. We compute that $C_H(\pi_1 r)$ has Sylow 2-subgroup S_0 of order 2^{14} if $H = K\langle\pi_1\rangle$, 2^{15} if $|H : K| = 4$. In either case $Z(S_0) \subseteq \langle\pi_1, r, Z\rangle$ whence $z \in S_0' \cap \Omega_1(Z(S_0)) \subset Z$. It follows that S_0 is a Sylow 2-subgroup of $C_G(\pi_1 r)$ and that $\pi_1 r$ is not conjugate (in G) to any involution in K.

We will now show that π_1 (or $\pi_1 r$) is not conjugate to any involution in $K\langle\pi_2\rangle - K$. Let l be an involution in $K\langle\pi_2\rangle - K$ and L a Sylow 2-subgroup of $C_H(l)$. If $z \sim_G j_2$ then there exists $g \in N_G(\langle z, j_2\rangle)$ with $C_J(j_2)^g$ covering $O_2(C_H(j_2))/C_J(j_2) \cong \langle\bar{F}_2, \bar{\pi}_2\rangle \cong E_{32}$. It follows that l is conjugate to some involution in J. Suppose now $z \nsim_G j_2$. Clearly $Z(L) \subseteq \langle l, C_J(l)\rangle$, whence if $Z(L) = \langle l, z\rangle$, $L' \cap Z(L) = \langle z\rangle$. In particular, L is a Sylow 2-subgroup of $C_G(l)$. If $Z(L) \supset \langle l, z\rangle$ and there exists $L_1 \subseteq C_G(l)$ with $|L_1 : L| = 2$, as $\langle z\rangle \ntrianglelefteq L_1$ and $(Z(L) \cap J - \langle z\rangle) \ntrianglelefteq L_1$, we have without loss $z \sim_G l$ and lz conjugate (in G) to (all) those involutions in $Z(L) \cap J - \langle z\rangle$. Combining these results we see that if L is not a Sylow 2-subgroup of $C_G(l)$ then l is conjugate to an involution in J. Thus $l \nsim_G \pi_1$ and if $l \sim_G \pi_1 r$ then $l \sim_H \pi_2$ or $\pi_2 z$ (as $C_H(\pi_2 uv)$ has Sylow 2-subgroup of order 2^{14}). However, as $z^G \cap Z(L) = \{z\}$ in this case we must have $l \sim_H \pi_1 r$, which is impossible.

By Thompson's transfer theorem **[29]** we have that G contains a subgroup G_0 of index two with $G_0 \cap H = K$, $K\langle\pi_2\rangle$, or $K\langle\pi_3\rangle$. The lemma now follows from our results in §3.

LEMMA 16. *We have* $\pi_1^2 = 1$ *and* $C_G(\pi_1) \cong Z_2 \times M(23)$.

PROOF. From the proof of Lemma 6, $C_G(\mu) \cap G_0/\langle\mu\rangle \cong \mathfrak{A}_8$ or $\mathfrak{A}_9$. Since π_1 acts as an outer automorphism on $C_G(\mu) \cap G_0$, we have $C_G(\mu)/\langle\mu\rangle \cong \Sigma_8$ or Σ_9. As $c \sim_G \sigma_1\sigma_2$, $C_G(c)/O_3(C_G(c)) \cong Z_2 \cdot U_4(3) \cdot Z_2$ whence $C_G(c) \cap C_G(\mu)$ contains a four group, i.e. $C_G(\mu)/\langle\mu\rangle \cong \Sigma_9$. Thus $\pi_1^2 = 1$ and $C_H(\pi_1)$ covers H/K. It follows that $C_H(\pi_1) \cap G_0$ satisfies the asumptions of Part I, Theorem C. Replacing π_1 by $\pi_1 z$ if necessary we have $C_G(\pi_1) \cap C_G(\mu) \cong Z_2 \times Z_5 \times \Sigma_7$. Hence $\langle z\rangle$ is not weakly closed in $C_G(\pi_1)$ and $C_{G_0}(\pi_1)$ does not contain a normal four group. It follows immediately from Theorem C of Part I that $C_G(\pi_1) \cong Z_2 \times M(23)$.

LEMMA 17. *The conjugacy class* π_1^G *is a class of* 3-*transpositions and* $G \cong M(24)$.

PROOF. From the character table for $M(23)$ given in **[26]** we see that the only proper subgroups of $M(23)$ of index less than $|G\colon C_G(\pi_1)| = 306{,}936$ have index 31,671 $(= 3^4 \cdot 17 \cdot 23)$, 137,632 $(= 2^5 \cdot 11 \cdot 17 \cdot 23)$, 275,264 $(= 2^6 \cdot 11 \cdot 17 \cdot 23)$. Hence by Lemma 16 we see that there exists $g \in G$ (in fact we may take $g \in J - C_J(\pi_1)$) with $C_G(\pi_1) \cap C_G(\pi_1^g)$ of order $2 \cdot |M(22)|$ and index 31,671 in $C_G(\pi_1)$. From the remarks before Lemma 4, there is an element $\lambda \sim_H \sigma_1\sigma_2^{-1}$ which is inverted by π_1 and such that $C_J(\pi_1) \supset C_J(\lambda)$. Thus $C_H(\pi_1) \cap C_H(\lambda) = \Lambda$ where $C_H(\lambda) = \langle\lambda\rangle \times \Lambda$ (because $\pi_1 \nsim_H \pi_1 z$). As any Sylow 2-subgroup of Λ has centre equal to $\langle z\rangle$, it follows that $|C_G(\pi_1) \cap C_G(\pi_1^\lambda)| = 2^{13} \cdot n$, n odd.

We have shown therefore that G is a rank three extension of $C_G(\pi_1)$, whence π_1^G is a class of 3-transpositions. Since $G'' = G_0' = G_0$ by Lemma 14, Fischer's result **[1]** yields $G \cong M(24)$.

The proof of the theorem is now complete.

PART III

The following result is proved.

THEOREM. *Let G be a finite group, z an involution in G and suppose $H = C_G(z)$ satisfies*:

(i) *$J = O_2(H)$ is extra-special of order 2^{13} with $C_H(J) \subseteq J$;*

(ii) *H contains a normal subgroup K with $K/O_{2,3}(H) \cong U_4(3)$ and $O_{2,3}(H)/J \cong Z_3$;*

(iii) *$G \neq H \cdot O(G)$ and $|H : K| = 2$.*

Then $G \cong M(24)'$.

The following corollary is an immediate consequence of Part II.

COROLLARY. *Suppose G, H, z satisfy only* (i) *and* (ii) *of the theorem. Then one of the following holds*:

(a) $G = H \cdot O(G)$;

(b) $H/K \cong Z_2$ *and* $G \cong M(24)'$;

(c) $H/K \cong Z_2 \times Z_2$ *and* $G \cong M(24)$.

Throughout this part we will assume conditions (i), (ii), (iii) hold. In Part II we have shown that under these hypotheses G is a simple group of the same order as $M(24)'$ and that if j is an involution in G with $j \nsim_G z$, then $C_G(j)$ is a nonsplit extension of a group of order two by Aut $M(22)$. We will begin by listing properties of G which were determined in Part II, and as far as possible we will use the same notation.

Let T be a Sylow 2-subgroup of H with $T \cap C_H(j)$ a Sylow 2-subgroup of $C_G(j)$. Let E be the unique elementary abelian subgroup of order 2^{11} in T so that $C_G(E) = E$ and $N_G(E)/E \cong M_{24}$ (the Mathieu group). Put $C = C_G(j)'$ so that $C/\langle j\rangle \cong M(22)$. In $N_G(E)$, j has 276 conjugates in E which break up into 4 classes in $N_C(E)$ with representatives j (1 conjugate), z_1 (22), $z_1 j$ (22) and jz (231). We have $z_1 \sim z_1 j$ in $C_G(j) \cap N_G(E)$, and $N_C(E)/E \cong M_{22}$.

For the element $\sigma_1\sigma_2^{-1}$ of order three we have that

$$C_H(\sigma_1\sigma_2^{-1}) = C_J(\sigma_1\sigma_2^{-1}) \cdot M \cdot \langle t, uv, \pi_2\rangle\langle\sigma_3\rangle$$

where

$$C_J(\sigma_1\sigma_2^{-1}) \cong Q_8 * Q_8 * Q_8 * Q_8, \quad M \cong E_{3^5}$$

and

$$C_H(\sigma_1\sigma_2^{-1})/O_{2,3}(C_H(\sigma_1\sigma_2^{-1})) \cong Z_2 \times \mathfrak{A}_4.$$

In addition,

$$N_H(\langle\sigma_1\sigma_2^{-1}\rangle) = C_H(\sigma_1\sigma_2^{-1}) \cdot \langle uz\rangle, \quad uz \sim_G j,$$

$$N_H(\langle\sigma_1\sigma_2^{-1}\rangle)/O_{2,3}(C_H(\sigma_1\sigma_2^{-1})) \cong Z_2 \times \Sigma_4,$$

and $N_H(\langle\sigma_1\sigma_2^{-1}\rangle) - C_H(\sigma_1\sigma_2^{-1})$ contains precisely one class of involutions (with representative uz) which are conjugate to j in G. Finally,

$$C_H(uz) \cap C_H(\sigma_1\sigma_2^{-1}) = C_J(\langle uz, \sigma_1\sigma_2^{-1}\rangle) \cdot M_0 \cdot \langle t, \pi_2, uv\rangle$$

where

$$C_J(\langle uz, \sigma_1\sigma_2^{-1}\rangle) \cong Z_2 \times Z_2 \times Q_8 * Q_8, \quad M_0 = C_M(uz) \cong E_{27}$$

and in fact $C_H(uz) \cap C_H(\sigma_1\sigma_2^{-1})$ is isomorphic to the centralizer of an involution in $\Omega_7(3)$.

LEMMA 1. *We have* $\Omega_1 = C_G(uz) \cap C_G(\sigma_1\sigma_2^{-1}) \cong \Omega_7(3)$ *and* $C_G(\Omega_1) = \langle uz, \sigma_1\sigma_2^{-1}\rangle \cong \Sigma_3$.

PROOF. As $C_G(\sigma_1\sigma_2^{-1}) \supset C_G(\sigma_3)$ and $z \sim \pi_2 uv$ in $C_G(\sigma_3)$ (see Part II, Lemma 10), Theorem D of Part I yields that $\Omega_1 \cong \Omega_7(3)$. To complete the proof it is enough to show that $C_H(\Omega_1) = \langle z, uz, \sigma_1\sigma_2^{-1}\rangle$. Now $\Omega_1 \cap H \supseteq \langle c, \sigma_1\sigma_2\rangle$ and we may assume $j_2 \in \Omega_1 \cap J$ ($j_2 \sim_G z$). The structures of $C_H(\langle c, \sigma_1\sigma_2\rangle)$ and $C_H(j_2)$ now give the required result.

1. The construction of the subgroup $X \cong M(23)$. Let $N_C(E) \subset L \subset N_G(E)$ with $L/E \cong M_{23}$ and $j \sim_L z_1$. (L exists because of the structure of M_{24}–see **[23]**.) As j has 23 conjugates in $L, j \nsim_L z_1 j$. Put $C_1 = C_G(z_1)'$ and note that z_1^C, j^{C_1} are classes of 3-transpositions in C, C_1, respectively, with $z_1 z_1^x \sim_G \sigma_1\sigma_2^{-1}$ or $[z_1, z_1^x] = 1$ for all $x \in C$.

Take $k \in C_1 - C$ with $k \sim_G j$ (thus $kj \sim_G \sigma_1\sigma_2^{-1}$). We now define $\mathfrak{J} = \{j\} \cup z_1^C \cup k^C$ and let $X = N_G(\mathfrak{J})$. By Lemma 1, $\Omega = C_G(k) \cap C_G(j) = C_C(k) \cong \Omega_7(3)$; and we also have $C_C(z_1) = C_G(z_1) \cap C_G(j)$. It follows therefore that

$$\mathfrak{J}^s \cap \mathfrak{J} = \{j\} \quad \text{for all } s \in C_G(j) - C. \tag{1}$$

Since $N_H(\langle\sigma_1\sigma_2^{-1}\rangle)$ contains a Sylow 2-subgroup of $N_G(\langle\sigma_1\sigma_2^{-1}\rangle)$ and $N_H(\langle\sigma_1\sigma_2^{-1}\rangle) - C_H(\sigma_1\sigma_2^{-1})$ contains precisely one class of involutions conjugate to j in G,

$$\mathfrak{J}^s \cup \mathfrak{J} \supset \{g \in G \mid g \sim_G j, gj \sim_G \sigma_1\sigma_2^{-1}\}. \tag{2}$$

(Note that $\mathfrak{J}^s = \mathfrak{J}^{s_1}$ for $s, s_1 \in C_G(j) - C$.)

We prove next that

$$xz_1 \sim_G \sigma_1\sigma_2^{-1} \quad \text{for all } x \in \mathfrak{J} - C_1. \tag{3}$$

If $x \in z_1^C$ this is clear as z_1^C is a class of 3-transpositions in C. Suppose that $x \in k^C$. From the character table of $M(22)$ (see **[25]**) we have

$$C = \Omega(C_1 \cap C) \cup \Omega w(C_1 \cap C)$$

where $w \in N_C(E) - C_1$, $w^2 \in C_1$ and $z_1^w = z_2 \notin C_G(k)$. (Since $k \notin C_G(E) = E$ and $N_C(E)/E \cong M_{22}$ is 2-transitive on the 22 conjugates of z_1 in $N_C(E)$, we can find such a w.) Hence $k^C = k^{(C_1 \cap C)} \cup k^{w(C_1 \cap C)}$ and it is enough to show that $k^w z_1 \sim_G \sigma_1\sigma_2^{-1}$.

As $k \in C_1$, $k^w \in C_2 = C_G(z_2)'$ and $[k^w, z_1] \neq 1$. Now $k^w j \sim_G \sigma_1\sigma_2^{-1}$ by definition of $\mathfrak{J}$ so $k^w z_1 \sim_G \sigma_1\sigma_2^{-1}$ also as j^{C_2} is a class of 3-transpositions in C_2 and $j \sim z_1$ in $L \cap C_2$. Thus (3) is proved.

Choose $g \in C_1 \cap N_G(E) - C$ with $g^2 \in C$ and let

$$z_3 = j^g(|C_1 \cap N_G(E): C_1 \cap N_C(E)| = 2 \cdot 11).$$

We will show that $g \in X = N_G(\mathcal{J})$. From the structure of L we can find $y \in N_C(E)$ with $z_1^y = z_3$, whence $[x, z_3] = 1$ or $xz_3 \sim_G \sigma_1\sigma_2^{-1}$ for all $x \in \mathcal{J}$ by (3).

As $z_3^{C \cap C_1} \subseteq z_3^C = z_1^C$ and $j^{C_1} = \{j\} \cup z_3^{C \cap C_1} \cup k^{C \cap C_1}$, it follows that $j^{C_1} = \mathcal{J} \cap C_1 - \{z_1\}$, whence $(\mathcal{J} \cap C_1)^g = \mathcal{J} \cap C_1$.

Let $x \in \mathcal{J} - C_1$. Thus $xz_1 \sim_G \sigma_1\sigma_2^{-1}$ by (3), so $x^g z_1 \sim_G \sigma_1\sigma_2^{-1}$ also. If $[x, z_3] = 1$, $x^g \in C$ whence $x^g \in \mathcal{J}$ as required. Therefore we will assume $xz_3 \sim_G \sigma_1\sigma_2^{-1}$ so $x^g j \sim_G \sigma_1\sigma_2^{-1} \sim_G x^g z_1$. It follows that $\langle x^g, j, z_1\rangle \cong \Sigma_4$ with $x^g j z_1$ of order four.

Suppose $x^g \notin \mathcal{J}$. By (2) we have $x^g \in \mathcal{J}^s$, $s \in C_G(j) - C$. Choose s so that $z_1^s = z_1 j$ (see the remarks above). Let $x^g = x_1^s$ for some $x_1 \in \mathcal{J}$. Then $[z_1, x_1] = 1$ or $x_1 z_1 \sim_G \sigma_1\sigma_2^{-1}$ by (3) so $(x_1z_1)^s$ has order two or three. However $(x_1z_1)^s = x^g z_1 j$ has order four, a contradiction. Thus $x^g \in \mathcal{J}$ and therefore $g \in N_G(\mathcal{J}) = X$.

We have proved that $C \subset X \subset G$. From above we have $C = C_X(j)$. Since $j \sim_G z_1 \sim_G jz_1$, $O(X) = 1$. Let N be a minimal normal subgroup of X. From $C/\langle j\rangle \cong M(22)$ and $j \in C'$ it follows that $j \in N$. Hence $C \subset N$ as $j \sim_X z_1$. As N is minimal normal, the structure of C yields that N is simple whence $N \cong M(23)$ by a result of D. Hunt **[12]**. Now $T \cap C$ is a Sylow 2-subgroup of N with $Z(T \cap C) = \langle z, j\rangle$. Finally, as z, j, jz have nonisomorphic centralizers in N (see **[12]** or Part I) the Frattini argument yields $N = X$. We have proved

LEMMA 2. *The group G contains a subgroup X isomorphic to $M(23)$.*

2. Some properties of X.

LEMMA 3. *We have $G = X \cup XsX \cup XrX$ where $X \cap X^s = C$ and $|X : C| =$ 31,671 while $D = X \cap X^r$ satisfies $|X : D| = 275{,}264$.*

PROOF. From the character table for $M(23)$ in **[26]**, we see that X has only four permutation characters of degree $\leqslant 306{,}936 = |G : X|$:

$$\varphi_0(1) = 1; \quad \varphi_1(1) = 31{,}671; \quad \varphi_2(1) = 137{,}632; \quad \varphi_3(1) = 275{,}264.$$

If φ is the permutation character for the representation of G on the cosets of X, then by considering values of φ, φ_i $(i = 1, 2, 3)$ on involutions we get $\varphi = \varphi_0 + \varphi_1 + \varphi_3$. The lemma follows.

LEMMA 4. *Let D be a subgroup of X of index 275,264. Then D contains precisely two classes of involutions with representatives z $(|C_D(z)| = 2^{12} \cdot 3^5)$ and jz $(|C_D(jz)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 7)$.*

PROOF. We first note that $|D| = 2^{12} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 13$ and φ_3 is the permutation character associated with D. From the character table, $\varphi_3(z) = 320$, $\varphi_3(j) = 0$, $\varphi_3(jz) = 2816$ which yields $z^D = z^X \cap D$ and $|C_D(z)| = 2^{12} \cdot 3^5$, $j^X \cap D = \varnothing$ and $|C_D(jz)| = 2^{10} \cdot 3^5 \cdot 5 \cdot 7 \cdot n$, $1 \leqslant n < 4$. It remains to prove that $n = 1$.

Let ν be an element of order 7 in $C_D(jz)$. Then $C_X(\nu) \cong Z_7 \times \Sigma_5$ and $|N_X(\langle\nu\rangle): C_X(\nu)| = 6$. From $\varphi_3(\nu) = 10$ it follows that $|C_D(\nu)| = 7 \times 12$ whence, as φ_3 vanishes on elements of order 42, $C_D(\nu) \cong Z_7 \times \mathfrak{A}_4$. Hence all involutions which centralize an element of order 7 in D are conjugate in D. This yields $n = 1$.

LEMMA 5. *There exists an involution $d \sim_X j$ with $d \in N_X(D)$ and $C_D(d) \cong \Omega_7(3)$.*

PROOF. Without loss we may assume $z \in D$. If M^* is a Sylow 3-subgroup of $C_D(z)$ then M^* is a Sylow 3-subgroup of $C_X(z)$. Since $j^X \cap D = \varnothing$,

$$[M^*, J] = J \cap D \cong Q_8 * Q_8 * Q_8 * Q_8.$$

Hence $J \cap D - \langle z \rangle$ contains an involution $j_2 \sim_X z$, whence a Sylow 2-subgroup of $C_D(z)/C_D(z) \cap J$ must be elementary abelian of order 8 by Proposition 7 of [16]. The structure of Aut $PSp_4(3)$ yields $C_D(z)/C_D(z) \cap J \cong E_{81} \cdot (Z_2 \times \mathcal{A}_4)$–see Part I. It follows from the structure of $C_X(z)$ that there exists $d \in N_X(C_D(z))$ with $tz \sim d$ in $H \cap X$ ($tz \sim_G j$ and $N_X(M^*) \cap C_X(z)/\langle z, M^* \rangle \cong Z_2 \times \Sigma_4$–see Part I). A computation yields that $C_X(d) \cap C_D(z)$ is isomorphic to the centralizer of an involution in $\Omega_7(3)$. Clearly $O(C_D(d)) = 1$, and as $|C_D(d)| > |C_X(d)|\,|D|/|X|$, Part I, Theorem D yields $C_D(d) \cong \Omega_7(3)$.

Let $\langle \nu \rangle$, $\langle \rho \rangle$ be Sylow 7-, 13-subgroups, respectively, of $C_D(d)$. Then $C_X(\rho) \cong Z_{13} \times \Sigma_3$, $C_D(\rho) \cong Z_{13} \times Z_3$, $C_D(\rho) \cap C_D(d) = \langle \rho \rangle$, and in each case $N(\langle \rho \rangle)/C(\rho) \cong Z_6$ (see [25] and [26]). Thus d normalizes $N_D(\langle \rho \rangle)$. Similarly it follows from $C_D(d) \cap C_D(\nu) \cong Z_7 \times Z_2$, $N_D(\langle \nu \rangle) \cap C_D(d)/C_D(\nu) \cap C_D(d) \cong Z_6$ (see [25]) and the structure of $N_X(\langle \nu \rangle)$, $N_D(\langle \nu \rangle)$ that d also normalizes $N_D(\langle \nu \rangle)$. Hence $d \in N_X(D)$ by Sylow's theorem.

LEMMA 6. *We have $D' \cong D_4(3)$, $D/D' \cong Z_3$, and if $D_0 = \langle D, d \rangle$, then $D_0/D' \cong \Sigma_3$ and $D_0 \subseteq$ Aut D'. Further $D_0 - D$ contains precisely one class of involutions conjugate to j in X i.e. $j^X \cap D_0 = d^{D_0} = d^D$.*

PROOF. The last statement follows from the fact that the representation of X on the cosets of D_0 must have permutation character φ_2. As j^X is a class of 3-transpositions, d^{D_0} is a class of 3-transpositions in D_0. Clearly $D_0 = \langle d^{D_0} \rangle$, $Z(D_0) = O_2(D_0) = O_3(D_0) = 1$, so Fischer's result [1] yields $D_0' \neq D_0''$.

If T_0 is a Sylow 2-subgroup of $C(z) \cap D_0$ then $Z(T_0) = \langle z \rangle$ and $\langle z^D \cap C_D(z) \rangle = O_{2,3,2}(C_D(z))$. Thus the Frattini argument yields $D' = D''$ and $|D : D'| < 3$. Hence $|D : D'| = 3$ and $\langle D', d \rangle \cong D_4(3) \cdot Z_2 \cong O_8^+(3)$ by Fischer [1]. The other assertions in the lemma follow from the structure of $C_X(z) \cap D_0$.

LEMMA 7. *The subgroup X contains exactly one conjugacy class of subgroups isomorphic to D_0 ($\cong D_4(3) \cdot \Sigma_3$).*

PROOF. Suppose $D_0^* \subset X$, $D_0^* \cong D_0$ but D_0^* is not conjugate to D_0. We may assume $d \in D_0 \cap D_0^*$. Now $C_D(d) \cong \Omega_7(3)$ and this subgroup contains no conjugates of d. By [30] $(17 \cdot 2 \cdot 4)$, $M(22)$ contains precisely two classes of subgroups isomorphic to $\Omega_7(3)$. If $[d^x, d] \neq 1$ for $x \in X$ then $C_X(\langle d^x, d \rangle) \cong \Omega_7(3)$ and this subgroup contains conjugates of d. Thus we may assume that $D_0 \cap D_0^* \supseteq C_D(d) \times \langle d \rangle$.

Let $\langle \rho \rangle$ be a Sylow 13-subgroup of $C_D(d)$. Then

$$\langle \rho \rangle \times \langle d \rangle \subseteq C_D(d) \times \langle d \rangle \subseteq D_0 \cap D_0^*,$$
$$C_{D_0}(\rho) = C_X(\rho) \cong Z_{13} \times \Sigma_3,$$

whence $D_0 \cap D_0^* \supset C_{D_0}(d)$. It follows immediately that $D_0 = D_0^*$ and the lemma is proved.

3. The graph for G.[2] In this section we will consider various graphs, all of which will be undirected without loops or multiple edges. If Γ is a graph and x a vertex in Γ then Γ_x will denote the subgraph of Γ whose vertices are those connected (by an edge) to x.

The graph $\Gamma(\mathfrak{D})$. Let $\mathfrak{D} = j^X$, the class of 3-transpositions of X. The graph $\Gamma(\mathfrak{D})$ has vertex set $\mathfrak{D}$ and $\{d, e\}$ is an edge for $d, e \in \mathfrak{D}$ if and only if $1 \neq ed = de$.

The graph $\Gamma(\mathfrak{D}, D_0, \Lambda_d)$. Let $\pi\colon D_0^X \to D_0^X\pi$ be a bijection with $D_0^X \cap D_0^X\pi = \varnothing$. Put $\Delta = D_0^X \cup D_0^X\pi$. Fischer **[30]** defines the graph $\Gamma = \Gamma(\mathfrak{D}, D_0, \Lambda_d)$ in the following way: Γ has vertex set $\{\pi\} \cup \mathfrak{D} \cup \Delta$ with edges as given below:

$\{\pi, \gamma\}$ is an edge if and only if $\gamma \in \mathfrak{D}$;

$\{d, e\}$ is an edge, $d, e \in \mathfrak{D}$, if and only if $1 \neq de = ed$;

$\{d, D_0^x\}$ is an edge, $d \in \mathfrak{D}$, $x \in X$, if and only if $d \in D_0^x$;

$\{d, D_0^x\pi\}$ is an edge, $d \in \mathfrak{D}$, $x \in X$, if and only if $\{d, D_0^x\}$ is an edge.

Fix $d \in \mathfrak{D} \cap D_0$; let Γ_d^* be the graph with vertex set Γ_d and edges as defined above plus a set of edges Λ_d with vertices in Δ so that $\Gamma_d^* \cong \Gamma(\mathfrak{D})$.

Λ_d is a set of edges in Γ.

For each $g \in D_0$, define the map $g\mu$ on Γ by

$$\pi(g\mu) = \pi; \quad d(g\mu) = d^g \qquad (d \in \mathfrak{D});$$

$$\delta(g\mu) = \begin{cases} D_0^{xg} & \text{if } \delta = D_0^x \in \Delta, \\ D_0^{xg}\pi & \text{if } \delta = D_0^x\pi \in \Delta. \end{cases}$$

$\{\delta_1, \delta_2\}$ is an edge, $\delta_1, \delta_2 \in \Delta$ if and only if there exists $g \in D_0$ with $\{\delta_1(g\mu), \delta_2(g\mu)\} \in \Lambda_d$.

The graph $\Gamma = \Gamma(\mathfrak{D}, D_0, \Lambda_d)$ is a central extension of $\Gamma(\mathfrak{D})$ with respect to D_0, Λ_d. In **[30]** Fischer proves that the definition of Γ is independent of the choice of $d \in \mathfrak{D}$ [**30**, 19.1.4], $\Gamma_d \cong \Gamma(\mathfrak{D})$ [**30**, 19.1.2], Γ has transitive group of automorphisms isomorphic to $M(24)$ [**30**, 19.1.5; 19.2.7] and the central extension is unique [**30**, §20]. Further $D_0\mu \subseteq \operatorname{Aut}\Gamma$ [**30**, 10.6.3].

The graph $\Gamma(G)$. The vertex set for $\Gamma(G)$ is $\{Xg \mid g \in G\}$ while $\{Xg, Xy\}$ is an edge if and only if $X^g \cap X^y \cong C \cong Z_2 \cdot M(22)$. We will use below the fact that $N_G(X) = X$. (Without loss $c \in C_X(z)$ and as $C_H(c)/\langle c, z\rangle \cong U_4(3)$ while $C_H(c) \cap X/\langle c, z\rangle \cong PSp_4(3)$, we have $N_G(C_X(z)) = C_X(z)$. The Frattini argument yields $N_G(X) = X$.)

Our aim, of course, is to prove that $\Gamma(G) \cong \Gamma$. Let $X \cap X^s = C$ (with $Z(C) = \langle j\rangle$) and $X^r \cap X = D$, with $N_X(D) = \langle d, D\rangle = D_0$. We note that for $x \in X$, $C \cap C_X(j^x) \cong C$, $(Z_2 \times Z_2) \cdot U_6(2)$ (if $1 \neq j \cdot j^x = j^x \cdot j$) or $\Omega_7(3)$ and $D_0 \cap D_0^x \cong D_0$, $D_4(2) \cdot \Sigma_3$ or a soluble group [**30**, 18.3.16].

Let $\alpha\colon \{\pi\} \cup \mathfrak{D} \to \{X\} \cup \{Xsx \mid x \in X\}$ be defined by

$$\pi\alpha = X; \qquad j^x\alpha = Xsx \quad \text{for } x \in X.$$

Note that $\langle j^x\rangle = Z(C^x) = Z(X^s \cap X)^x = Z(X^{sx} \cap X)$.

By definition $\{\pi, j^x\}$ is an edge and $\{X, Xsx\} = \{\pi\alpha, j^x\alpha\}$ is an edge for all $x \in X$. Suppose $\{Xsx, Xsy\}$ is an edge $(x, y \in X)$ so that $X^{sx} \cap X^{sy} \cong C$. If

[2] In this section, results of the form [**30**, 19.1.4] refer to Lemma 19.1.4 of [**30**], etc.

$[j^x, j^y] \neq 1$ then $\Omega^* = X^{sx} \cap X^{sy} \cap X \cong \Omega_7(3)$, and by Lemma 1, $C_G(\Omega^*) = \langle j^x, j^y \rangle$. However if $\langle e \rangle = Z(X^{sx} \cap X^{sy})$, $e \in C_G(\Omega^*)$, which is a contradiction. Thus $[j^x, j^y] = 1$. Conversely, suppose $\{j^x, j^y\}$ is an edge so that $[j^x, j^y] = 1$. Then $(X^{sx} \cap X) \cap (X^{sy} \cap X) \cong (Z_2 \times Z_2) \cdot U(2)$, whence $X^{sx} \cap X^{sy} \cong C$ as required. Therefore $\{Xsx, Xsy\}$ is an edge in $\Gamma(G)_X$ if and only if $\{j^x, j^y\}$ is an edge in Γ_π. In particular, we have proved that $\Gamma(G)_X \cong \Gamma(\mathfrak{D})$ and, as G is transitive on $\Gamma(G)$, $\Gamma(G)_{Xg} \cong \Gamma(\mathfrak{D})$ for all $g \in G$.

For convenience take $j = d \in D_0$ and extend the definition of α to Γ_d by taking $\alpha|_{\Gamma_d}$ to be an isomorphism from Γ_d to $\Gamma(G)_{Xs}$. (This is possible because of the way Λ_d is chosen in the definition of Γ.) As in Lemma 3 take $X^r \cap X = D$ (for some $r \in G$). If $D_0^x \in \Gamma_d$ for $x \in G$ we claim that $\{D_0^x\alpha, D_0^x\pi\alpha\} = \{Xrx, Xrdx\}$. Note that $\{D_0, D_0\pi\}$ is uniquely determined by the set $(\mathfrak{D} \cap D_0)$ of involutions joined to D_0, $D_0\pi$. Let $e = d^g$ so that $e\alpha = Xsg$. Since α is an isomorphism we must show therefore that $\{e, D_0\}$ is an edge if and only if both $\{Xsg, Xrx\}$ and $\{Xsg, Xrdx\}$ are edges.

If $\{Xsg, Xrx\}$ is an edge, $X^{sg} \cap X^{rx} \cong C$ from which we get $X \cap X^{sg} \cap X^{rx} \cong \Omega_7(3)$. Since $X \cap X^{rdx} = X \cap X^{rx}$ the possible intersections of conjugates of D_0 in X yield $X^{sg} \cap X^{rdx} \cong C$ also. The same argument yields the converse so we have $\{Xsg, Xrx\}$ is an edge if and only if $\{Xsg, Xrdx\}$ is an edge. Since $C_X(e) \cap X^{rx} \subseteq X \cap X^{sg} \cap X^{rx}$, the argument of Lemma 5 yields that if $X^{sg} \cap X^{rx} \cong C$ then $e \in D_0^x$ ($D^x = X^{rx} \cap X$). Suppose now that $e \in D_0^x$. Then $C_X(e) \cap D^x \cong \Omega_7(3)$ and so $X \cap X^{sg} \cap X^{rx}$ contains a subgroup isomorphic to $\Omega_7(3)$. As above this yields that $X^{sg} \cap X^{rx} \cong C$. This completes the proof that $\{D_0^x\alpha, D_0^x\pi\alpha\} = \{Xrx, Xrdx\}$. Without loss we may take $D_0\alpha = Xr$, $D_0\pi\alpha = Xrd$.

Let β denote the representation of G on the cosets of X, i.e. $Xy(g\beta) = Xyg$ for $y, g \in G$. For each $e = d^g \in D_0 \cap \mathfrak{D}$, $g \in D$, define

$$\alpha_e\colon \Gamma_e \to \Gamma(G)_{Xsg} \quad \text{by } \alpha_e = (g^{-1}\mu)\alpha(g\beta).$$

From the definition we have $\pi\alpha_e = \pi\alpha = X$, and if $d^x \in \Gamma_e$, then $d^x\alpha_e = d^x\alpha = Xsx$. We show that the definition of α_e is independent of g. Since μ, β are homomorphisms it is enough to show that for $g \in C_D(d)$, $(g^{-1}\mu)\alpha(g\beta) = \alpha$ on Γ_d. However this follows by Fischer [**30**, 10.5] as $(g^{-1}\mu)\alpha(g\beta)\alpha^{-1}$ fixes $(\Gamma_d \cap \Gamma_\pi) \cup \{D_0, D_0\pi\}$ (we need here that $g \in D$).

We next prove that if $\gamma \in \Gamma_e \cap \Gamma_f$ (for some $f \in \mathfrak{D} \cap D_0$) then $\gamma\alpha_e = \gamma\alpha_f$. Again, as μ, β are homomorphisms, it is enough to show that if $\gamma \in \Gamma_d \cap \Gamma_e$, $\gamma\alpha = \gamma\alpha_e$. Suppose $\gamma \in \{D_0^x, D_0^x\pi\}$, so that $\langle d, e \rangle \subset D_0^x$. As the definition of α_e is independent of the choice of g ($e = d^g$) we take $g = ed \in D^x$. Thus $\gamma(g^{-1}\mu) = \gamma$ and $(\gamma\alpha)(g\beta) = \gamma\alpha$, whence $\gamma\alpha = \gamma\alpha_e$ as required.

Since $D_0^x \cap D_0 \cap \mathfrak{D} \neq \varnothing$ for all $x \in X$ [**30**, 19.1.5], we can define $\alpha\colon \Gamma \to \Gamma(G)$ by $\alpha|_{\Gamma_e} = \alpha_e$ for each e in $D_0 \cap \mathfrak{D}$. Each α_e is an isomorphism, whence α is an isomorphism and $\Gamma \cong \Gamma(G)$.

LEMMA 8. *We have* $G \cong M(24)'$.

PROOF. As we have proved $\Gamma \cong \Gamma(G)$, G is isomorphic to a subgroup of $\operatorname{Aut}\Gamma = M(24)$. The lemma follows as both G, $M(24)'$ are simple groups of order $\frac{1}{2}|M(24)|$.

References

1. B. Fischer, *Finite groups generated by 3-transpositions*, Invent. Math. **13** (1971), 232–246.

2. G. Glauberman, *Central elements in core-free groups*, J. Algebra **4** (1966), 403–420.

3. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.

4. D. Gorenstein and K. Harada, *A characterization of Janko's two new simple groups*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **16** (1970), 331–406.

5. ______, *On finite groups with Sylow 2-subgroups of type* $\hat{A}_n$, $n = 8, 9, 10, 11$, J. Algebra **19** (1971), 185–227.

6. ______, *Finite groups whose 2-subgroups are generated by at most 4 elements*, Mem. Amer. Math. Soc. No. 147 (1974).

7. D. Gorenstein and J. H. Walter, *On finite groups with dihedral Sylow 2-subgroups*, Illinois J. Math. **6** (1962), 553–593.

8. R. Griess, *Schur multipliers of finite groups of Lie type*, Trans. Amer. Math. Soc. **183** (1973), 355–421.

9. ______, *Schur multipliers of some sporadic simple groups*, J. Algebra **32** (1974), 445–466.

10. K. Harada, *Finite simple groups with short chains of subgroups*, J. Math. Soc. Japan **20** (1968), 655–672.

11. D. Hunt, *A characterization of the finite simple group* $M(22)$, J. Algebra **21** (1972), 103–112.

12. ______, *A characterization of the finite simple group* $M(23)$, J. Algebra **26** (1973), 431–439.

13. B. Huppert, *Endliche Gruppen*. I, Springer-Verlag, Berlin, 1967.

14. Z. Janko, *A characterization of the finite simple group* $PSp_4(3)$, Canad. J. Math. **19** (1967), 872–894.

15. ______, *A characterization of the simple group* $G_2(3)$, J. Algebra **12** (1969), 360–371.

16. D. Parrott, *On Thompson's simple group*, J. Algebra **46** (1977), 389–404.

17. K. W. Phan, *A characterization of the finite simple group* $L_4(3)$, J. Austral. Math. Soc. **10** (1969), 51–76.

18. ______, *A characterization of the finite simple group* $U_4(3)$, J. Austral. Math. Soc. **10** (1969), 77–94.

19. M. Suzuki, *On characterizations of linear groups*. I, Trans. Amer. Math. Soc. **92** (1959), 191–204.

20. H. Wielandt, *Beziehungen zwischen den Fixpunktzahlen von Automorphismengruppen einer endlichen Gruppe*, Math. Z. **73** (1960), 146–158.

21. J. Olsson, *Odd order extensions of some orthogonal groups*, J. Algebra **28** (1974), 573–596.

22. R. Solomon, *Finite groups with Sylow 2-subgroups of type* $\Omega(7, q)$, $q \equiv \pm 3 \pmod 8$, J. Algebra **28** (1974), 174–181.

23. J. H. Conway, *Three lectures on exceptional groups*, Finite Simple Groups (M. B. Powell and G. Higman, editors), Academic Press, London, 1971.

24. D. Held, *A characterization of the alternating groups of degrees eight and nine*, J. Algebra **7** (1967), 218–237.

25. D. C. Hunt, *Character tables of certain finite simple groups*, Bull. Austral. Math. Soc. **5** (1971), 1–42.

26. ______, *The character table of Fischer's simple group*, $M(23)$, Math. Comp. **28** (1974), 660–661.

27. R. Solomon, *Finite groups with Sylow 2-subgroups of type* $\mathfrak{A}_{12}$, J. Algebra **24** (1973), 346–378.

28. R. G. Stanton, *The Mathieu groups*, Canad. J. Math. **3** (1951), 164–174.

29. J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. **74** (1968), 383–437.

30. B. Fischer, Lecture notes, Univ. of Warwick, Warwick, England.

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