A TANGENTIAL CONVERGENCE
FOR BOUNDED HARMONIC FUNCTIONS
ON A RANK ONE SYMMETRIC SPACE

BY

JACEK CYGAN

Abstract. Let $u$ be a bounded harmonic function on a noncompact rank one
symmetric space $M = G/K \approx N^-A, N^-AK$ being a fixed Iwasawa decomposition
of $G$. We prove that if for an $a_0 \in A$ there exists a limit $\lim_{n \to \infty} u(na_0) = c_{a_0}$ as
$n \in N^-$ goes to infinity, then for any $a \in A$, $\lim_{n \to \infty} u(na) = c_0$. For $M =
SU(n, 1)/S(U(n) \times U(1)) = B^n$, the unit ball in $C^n$ with the Bergman metric, this
is a result of Hulanicki and Ricci, and in this case it reads (via the Cayley
transformation) as a theorem on convergence of a bounded harmonic function to a
boundary value at a fixed boundary point, along appropriate, tangent to $\partial B^n$,
surfaces.

0. Introduction. Let $M$ be a noncompact symmetric space of rank one. $M$ can be
expressed as a homogeneous space $G/K$ where $G$ is a semisimple group of
isometries of $M$ and $K$ is a maximal compact subgroup of $G$. Let $\mathfrak{g}, \mathfrak{f}$ denote the Lie
algebras of $G$ and $K$, $B$ the Killing form of $\mathfrak{g}$, and $\mathfrak{p}$ the orthogonal complement of
$\mathfrak{f}$ in $\mathfrak{g}$ relative to $B$. If $\pi: G \to G/K$ denotes the canonical projection, its differential
at the identity, $\pi_*$, identifies the subspace $\mathfrak{p}$ of $\mathfrak{g}$ with $T_o(M)$, the tangent space of
$M$ at the origin $o = \pi(e)$, and the invariant metric $g$ on $M$ can be chosen so that $g_0$
corresponds to the restriction of $B$ to $\mathfrak{p} \times \mathfrak{p}$ under the above identification. We
denote by $\Delta$ the corresponding ($G$-invariant) Laplace-Beltrami operator on $M$. A
function $u \in C^\infty(M)$ is called harmonic if $\Delta u = 0$. Let $a$ be a maximal (one-dimen-
sional) abelian subspace of $\mathfrak{p}$, $\alpha$ and possibly $2\alpha$ in $a^*$, the corresponding system of
positive restricted roots relative to the fixed choice of a "positive part" $a^+$ in $a$. Let
$\mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-2\alpha}$ denote the root spaces corresponding to $-\alpha$ and $-2\alpha$. Then
$n^- = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha}$ is a nilpotent subalgebra of $\mathfrak{g}$ and one has the Iwasawa decom-
position $G = N^-AK$, with $N^- = \exp n^-$, $A = \exp a$. The above decomposition shows that every $p \in M$
can be uniquely written as $p = na \cdot o$ ($n \in N^-$, $a \in A$). We regard the nilpotent group $N^-$ as a boundary for the symmetric space $M$ in the
following sense. The bounded harmonic functions $u$ on $M$ have boundary values
on $N^-$, i.e. $\lim_{n \to \infty} u(na \cdot o) = \varphi(n)$ exists a.e. (relative to the Haar measure on
$N^-$) and $\varphi \in L^\infty(N^-)$. $\log a \to \infty$ is understood with respect to the ordering
induced on $a$ by $a^+$. Moreover,

$$u(na \cdot o) = \varphi \ast P_a(n) = \int_{N^-} \varphi(n_1) P_a(n_1^{-1}n) \, dn_1.$$
The function $P_a(n)$ on $N^- \times A$ is called the Poisson kernel for the symmetric space $M$ and is given by (Helgason [4])

$$P_a(n) = ce^{d/2} \left[ \left( e + \frac{1}{2} Q(X_{-a}) \right)^2 + 2Q(X_{-2a}) \right]^{-d/2},$$

where

$$n = \exp(X_{-a} + X_{-2a}), \quad X_{-a} \in \mathfrak{g}_{-a}, \quad X_{-2a} \in \mathfrak{g}_{-2a};$$

$$e = e^{-\alpha(\log a)}; \quad Q(X) = (X, X)_\theta / 2(m_a + 4m_{2a})$$

with $(X, X)_\theta = -B(X, \theta X)$ for $X \in \mathfrak{g}, \theta$ denoting the Cartan involution associated with the pair $(\mathfrak{g}, \mathfrak{f}); m_a = \dim \mathfrak{g}_{-a}, m_{2a} = \dim \mathfrak{g}_{-2a}, d = m_a + 2m_{2a}$. The constant $c$ is such that the integral of $P_a$ over $N^-$ is equal to 1.

The following theorem on “tangential” convergence for bounded harmonic functions on the Siegel domain

$$D_{r-1} = \left\{ (z_1, \ldots, z_r) \in \mathbb{C}^r : \text{Im} z_r > \sum_{j=1}^{r-1} |z_j|^2 \right\},$$

$r > 2$, (or, equivalently, on $M = SU(r, 1)/S(U(r) \times U(1))$—the complex hyperbolic space) has been obtained by Hulanicki and Ricci [5]. We formulate it below in terms of a homogeneous space $M$.

**Theorem.** Let $u$ be a bounded harmonic function on a noncompact rank one symmetric space $M$. In the notation above, assume that for an $a_0 \in A$, $\lim_{n \to \infty} u(na_0 \cdot o) = c_0$. Then for any $a \in A$, $\lim_{n \to \infty} u(na \cdot o) = c_0$.

Our aim here is to prove the above Theorem and the proof is based on the classification of symmetric spaces. That is, we discuss separately the cases of $M$ being the real, complex (to see how the $M = D_{r-1}$ case fits to our scheme), quaternion and octonion hyperbolic space, which corresponds respectively to $G$ being the classical group $SO(q, r, 1), SU(r, 1), Sp(r, 1)$ and the exceptional group $F_{4(20)}$. Following the Hulanicki-Ricci method, for each case we construct a suitable commutative subalgebra $\mathcal{K}$ of (multi) radial functions in $L^1(N^-)$, to which the Poisson kernel $P_a$ belongs. We describe the set $\mathcal{M}(\mathcal{K})$ of the maximal ideals in $\mathcal{K}$ and check that the Gel'fand transform $\hat{P}_a$ of $P_a$ never vanishes on $\mathcal{M}(\mathcal{K})$. The Theorem may then be stated as a theorem on certain ideals in $L^1(N^-)$ and is a consequence of the Wiener property of the algebra $\mathcal{K}$. To study the algebra $\mathcal{K}$ we use the holomorphically induced (realizations of the irreducible unitary) representations of $N^-$.

1. Nilpotent group $N^-$. Let $F$ denote the field $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or the Cayley numbers $\mathbb{O}$ (octonions); $F_0 = \{ q \in F : q + \overline{q} = 0 \}$, $\overline{q}$ being the usual conjugation in $F = \mathbb{C}, \mathbb{H}, \mathbb{O}$ and $\overline{q} = q$ for $q \in \mathbb{R}$; $\text{Im} q = \frac{1}{2}(q - \overline{q}), \sigma = 2s = \dim_{\mathbb{R}} F$. According to the notation of the previous section, $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ and for the classical $G$ we have (cf., e.g., [3, pp. 348–351])
\[\mathfrak{f} = \left\{ \begin{pmatrix} Z & 0 \\ 0 & p \end{pmatrix} : Z \text{ an } r \times r \text{ skew-Hermitian matrix over } F, \quad p \in F_0, \quad \text{tr } Z = -p \quad \text{in case of } F = C \right\},\]

\[\mathfrak{p} = \left\{ \begin{pmatrix} 0 & 'q \\ 'q & 0 \end{pmatrix} : q \in \mathbb{F}^r = F \times \cdots \times F \right\},\]

\[\alpha = \{ tE_{1,r+1} + tE_{r+1,1} : t \in \mathbb{R} \},\]

where \(E_{kl}\) denotes the \((r + 1) \times (r + 1)\) matrix \((\delta_{ak} \delta_{bl})_{1 \leq a,b \leq r+1}, \ r > 2\). We choose a basis \(H = E_{1,r+1} + E_{r+1,1}\) in \(\alpha\) and fix an ordering so that \(H \in \alpha^+\). Then \(\alpha \in \alpha^*\) such that \(\alpha(H) = 1\) is a positive restricted root, and we have

\[\mathfrak{a}_{-\alpha} = \left\{ \begin{pmatrix} 0 & -\bar{q} & 0 \\ 'q & 0 & 'q \\ 0 & \bar{q} & 0 \end{pmatrix} : q = (q_2, \ldots, q_r) \in \mathbb{F}^{r-1} \right\},\]

\[\mathfrak{a}_{-2\alpha} = \left\{ \begin{pmatrix} p & 0 & p \\ 0 & 0 & 0 \\ \bar{p} & \bar{q} & \bar{p} \end{pmatrix} : p \in F_0 \right\} \quad (= \{0\} \quad \text{for } F = \mathbb{R}).\]

We shall identify \(n^- = \mathfrak{a}_{-\alpha} \oplus \mathfrak{a}_{-2\alpha}\) with \(\mathbb{F}^{r-1} \times F_0\) by the correspondence

\[\begin{pmatrix} p & -\bar{q} & p \\ 'q & 0 & 'q \\ \bar{p} & \bar{q} & \bar{p} \end{pmatrix} \leftrightarrow (q, -p).\]

In these coordinates on \(n^-\), the commutator of \((q, p) = (q_1, \ldots, q_{r-1}, p)\) and \((q', p') = (q'_1, \ldots, q'_{r-1}, p')\) in \(\mathbb{F}^{r-1} \times F_0\) is given by

\[\left[ (q, p), (q', p') \right] = (0, 2 \Im (\bar{q} \cdot q')), \quad (1)\]

where we have put \(\bar{q} \cdot q'\) for \(\sum_{i=1}^{r-1} \bar{q}_i q_i\). We also have the formula (cf., e.g., \([11, \text{p. } 39]\))

\[((q, p), (q', p'))_{\theta} = 4(m_\alpha + 4m_{2\alpha}) \Re (\bar{q} \cdot q' + \bar{p}p'). \quad (2)\]

For the exceptional \(G\) (cf., e.g., \([10, \text{pp. } 522-530]\)), \(\mathfrak{g} = \mathfrak{f}_{4(2)}\) is isomorphic to the Lie algebra \(\text{Der}(\mathfrak{g})\) of derivations of the Jordan algebra \((\mathfrak{g}, \circ)\) of \(3 \times 3\) octonion matrices \(A\) of the form

\[A = \begin{pmatrix} \alpha_1 & a_3 & a_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ -\bar{a}_2 & -\bar{a}_1 & \alpha_3 \end{pmatrix}, \quad a_i \in \mathbb{O}, \quad \alpha_i \in \mathbb{R}, \quad i = 1, 2, 3,\]

with multiplication given by \(A \circ B = \frac{1}{2}(AB + BA)\), \(A, B \in \mathfrak{g}\), \(AB\) denoting the usual matrix multiplication. We have

\[\mathfrak{f} = \{ D \in \text{Der}(\mathfrak{g}) : D(E_{33}) = 0 \},\]

\[\mathfrak{p} = \left\{ D_Q \in \text{Der}(\mathfrak{g}) : Q = \begin{pmatrix} 0 & 'q \\ q & 0 \end{pmatrix}, \quad q \in \mathbb{O}^2 \right\},\]

where \(D_Q(B) = QB - BQ, \quad B \in \mathfrak{g}\).
\[ a = \{ D_q \in \text{Der}(\mathfrak{j}) : Q = tE_{13} + tE_{31}, \ t \in \mathbb{R} \} . \]

We choose \( H = D_Q \) with \( Q = E_{13} + E_{31} \in a^+ \) and \( a \in a^* \) such that \( \alpha(H) = 1 \). Then

\[ a_{-a} = \left\{ D_{Q(q)} : Q(q) = \begin{bmatrix} 0 & -\bar{q} & 0 \\ q & 0 & q \\ 0 & \bar{q} & 0 \end{bmatrix}, \ q \in \mathbb{R} \right\} , \]

\[ a_{-2a} = \left\{ D_{Q(p)} : Q(p) = \begin{bmatrix} p & 0 & p \\ 0 & 0 & 0 \\ \bar{p} & 0 & \bar{p} \end{bmatrix}, \ p \in \mathbb{R} \right\} . \]

Identifying \( D_{Q(q)} + D_{Q(p)} \) in \( a_{-a} \oplus a_{-2a} \) with \((q, -p)\) in \( \mathfrak{g} \times \mathfrak{g}_{0} \), we obtain the same formulas for the commutator and the inner product of \((q, p)\) and \((q', p')\) in \( \mathfrak{g} \times \mathfrak{g}_{0} \) as those given by (1) and (2) above.

Writing \( N^- \) as the manifold \( \mathfrak{n}^- \) with the group multiplication given by the Campbell-Hausdorff formula we obtain

**Proposition 1.** The underlying manifold for the nilpotent group \( N^- \) is \( F^{r-1} \times F_0 \) with \( r > 2 \) for \( F = \mathbb{R}, \mathbb{C}, \mathbb{H} \) \((F_0 = \{0\} \text{ if } F = \mathbb{R})\) and with \( r = 2 \) for \( F = \mathbb{O} \). The group law is

\[ (q, p)(q', p') = (q + q', p + p' + \text{Im}(\bar{q} \cdot q')) . \]

The Haar measure on \( N^- \) is the ordinary Lebesgue measure on \( \mathbb{R}^{k} \equiv F^{r-1} \times F_0 \), \( k = r \alpha - 1 \). We normalize it so that the volume of the unit cube in \( \mathbb{R}^{k} \) is 1 and denote by \( dqdp \). The Poisson kernel is given by

\[ \rho_{\exp(iH)}(q, p) = c_{r,F} \frac{d^2}{2}\left( (|q|^2 + \epsilon)^2 + 4|p|^2 \right)^{-d/2}, \]

where \( \epsilon = e^{-i} \), \( d = (r + 1)\sigma - 2 \), \( |q|^2 = \bar{q} \cdot q \), \( c_{r,F} = 2^{-d} \Gamma(r\sigma) \) with \( \sigma = 2s = \text{dim}_{\mathbb{R}} F \).

2. **Holomorphically induced representations of \( N^- \).** The adjoint and coadjoint action of \( N^- \) on \( \mathfrak{n}^- \) and \( \mathfrak{n}^- * \), respectively, is given by

\[ \text{Ad}_{(q,p)}(q'', p'') = (q'', p'' + 2 \text{Im}(\bar{q} \cdot q'')) , \]

\[ \text{Ad}^*_{(q,p)}(q', p') = (q' + 2q\bar{p}', p') , \]

\((q, p) \in N^-, (q'', p'') \in \mathfrak{n}^-, (q', p') \in \mathfrak{n}^- * \), \( q\bar{p}'' = (q_1\bar{p}, \ldots , q_{r-1}\bar{p}) \), and we have identified \( \mathfrak{n}^- * \), the dual space of \( \mathfrak{n}^- \), with \( \mathfrak{n}^- \) by \( \langle \cdot , \cdot \rangle = \langle \cdot , \cdot \rangle_{\mathfrak{g}}/4(m_2 + 4m_3) \).

The single points \((q', 0) \in \mathfrak{n}^- * \) are 0-dimensional orbits of \( \text{Ad}^* \) on \( \mathfrak{n}^- * \) and the corresponding (1-dimensional) representations of \( N^- \) are given by the characters

\[ \chi_{(q,0)}(q, p) = \exp(\sqrt{-1} \text{ Re}(\bar{q} \cdot q)), \quad (q, p) \in N^- . \]

The remaining (maximal dimensional) orbits, for \( F = \mathbb{C}, \mathbb{H} \) and \( \mathbb{O} \), are of the form \( F^{r-1} \times \{p''\}, p'' \neq 0 \), so they are parameterized, e.g., by the functionals \( f = (0, p'') \) \( \in \mathfrak{n}^- * \) with \( p'' \in F_0 \setminus \{0\} \). For such \( f \) and \( (q, p), (q', p') \in \mathfrak{n}^- \) we have

\[ \langle f, [(q, p), (q', p')] \rangle = 2 \text{ Re}((\bar{q}p'') \cdot q') = -2 \text{ Re}(\bar{q} \cdot (q'p'')) , \]
i.e. the operator $R_p: q \rightarrow qp''$ is skew-symmetric on $F^{-1}$ with respect to the $R$-bilinear symmetric form $\langle \cdot, \cdot \rangle$ on $F^{-1}$ given by $\langle q, q' \rangle = 2 \text{Re}(\bar{q} \cdot q')$ is the usual inner product on $R$, $l, i, e, k$ the basis of $H$, and $1, i, j, k, e, ie, je, ke$ the basis of $O (= H + He)$ with the multiplication defined by $(ae)b = (ab)e$, $a(be) = (ba)e$, $(ae)(be) = -\bar{b}a$ for $a, b \in H$). Suppose now that $f = (0, i\lambda) \in \mathbb{N}^*$ with $\lambda$ positive real. In the above bases of $F$ the matrix of $R_{il}$ acting on $F^{-1}$ with $r = 2$ is equal to $\lambda(E_{21} - E_{12})$ for $F = C$, to $\lambda(E_{21} - E_{12}) - \lambda(E_{43} - E_{34})$ for $F = H$ and to $\lambda(E_{21} - E_{12}) - \lambda(E_{43} - E_{34}) - \lambda(E_{65} - E_{56}) + \lambda(E_{87} - E_{78})$ for $F = O$. Put

$$
e_1 = 1/2(1 + \sqrt{-1} i), \quad e_2 = 1/2(\sqrt{-1} j + k), \quad e_3 = 1/2(\sqrt{-1} e + ie), \quad e_4 = 1/2(\sqrt{-1} ke)$$

for elements in $F^C = F + \sqrt{-1} F$—the complexification of $F$. Now define a subspace $W$ of $F^C$ by

$$W = \begin{cases} Ce_1 & \text{if } F = C, \\ Ce_1 + Ce_2 & \text{if } F = H, \\ Ce_1 + Ce_2 + Ce_3 + Ce_4 & \text{if } F = O. \end{cases}$$

Thus

$$b = W^{-1} \times F^C \quad (2.1)$$

is a positive polarization at $f = (0, i\lambda)$ such that

$$b + \bar{b} = n^{-c}, \quad b \cap \bar{b} = \{0\} \times F^C_0, \quad b/\{0\} \times F^C_0 = W, \quad (2.2)$$

where $z = x - \sqrt{-1} y$ for $z = x + \sqrt{-1} y \in \mathbb{N}^* + \sqrt{-1} n^{-} = n^{-c}$. For arbitrary $f = (0, p'')$ with $p'' \in F_0 \times \{0\}$, there exists an orthogonal transformation $\Omega$ on $R^n \approx F$, such that

$$\langle R_{p''}q, q' \rangle = \langle R_{ilp''}^{\Omega}q, \Omega q' \rangle, \quad q, q' \in F^{-1}.$$

Hence,

$$\langle f, [(q, p), (q', p')] \rangle = \langle (0, i|p'\rangle, [(\Omega q, p), (\Omega q', p')] \rangle,$$

and $b' = \Omega b$, with $b$ as in (2.1), is a positive polarization at $f$ and $b'$ satisfies (2.2) with $W' = \Omega W$ instead of $W$. Here $\Omega(q + \sqrt{-1} q')$ is understood as $\Omega q + \sqrt{-1} \Omega q'$, $q, q' \in F^{-1}$, and $\Omega q = (\Omega q_1, \ldots, \Omega q_{r-1}) \in F^{-1}$. As in $[1, pp. 158–162]$ one obtains that the space $\mathcal{K}(f, b)$ of the representation $\rho(f, b)$ corresponding to the chosen $f$ and $b'$ may be realized as a space of complex $C^\infty$ functions $\psi$ on the complex space $\mathcal{W}'$, square integrable with respect to the measure $\exp(-\sqrt{-1} \langle f, [Y, \bar{Y}] \rangle)dyd\bar{Y}$ ($dyd\bar{Y}$ denoting the Lebesgue measure) on $\mathcal{W}'$ and satisfying the following functional equation:

$$[\tau(\sqrt{-1} X)\psi](\bar{Y}) = \sqrt{-1} [\tau(X)\psi](\bar{Y}), \quad X, Y \in W',$$
where \([\tau(\vec{X})\psi](\vec{Y}) = (d/dt)\psi(\vec{Y} + t\vec{X})]_{t=0}\). The representation \(\rho\) is given by

\[
\rho(f, b)(\exp(\vec{X}_0 + X_0 + Z_0))\psi(\vec{X}) = \exp\left(\sqrt{-1} \left< f, [X_0, \vec{X}] \right> - \frac{\sqrt{-1}}{2} \left< f, [X_0, \vec{X}_0] \right>\right) \chi(f(Z_0)) \cdot \psi(\vec{X} - \vec{X}_0),
\]

(2.3)

where \(\vec{X}_0 + X_0 + Z_0\) is in \(\mathfrak{n}_-\) for \(X_0 \in \mathcal{W}'\), \(Z_0 \in \mathcal{F}_0\); \(\chi(Z_0) = \exp(\sqrt{-1} \left< f, Z_0 \right>)\).

Passing to the complex coordinates \((z_1, \ldots, z_{d(r-1)})\) on \(\mathcal{W}'\), according to the identifications

\[
\begin{align*}
\vec{X} = (z_1\bar{\varepsilon}_1, \ldots, z_{r-1}\bar{\varepsilon}_1) & \leftrightarrow (z_1, \ldots, z_{r-1}) \quad \text{for } \mathfrak{F} = \mathbb{C}, \\
\vec{X} = (z_1\bar{\varepsilon}_1 + z_2\bar{\varepsilon}_2, \ldots, z_{2(r-1)}\bar{\varepsilon}_1 + z_{2(r-1)}\bar{\varepsilon}_2) & \leftrightarrow (z_1, \ldots, z_{2(r-1)}) \quad \text{for } \mathfrak{F} = \mathbb{H}, \\
\vec{X} = (z_1\bar{\varepsilon}_1 + z_2\bar{\varepsilon}_2 + z_3\bar{\varepsilon}_3 + z_4\bar{\varepsilon}_4) & \leftrightarrow (z_1, \ldots, z_4) \quad \text{for } \mathfrak{F} = \mathbb{O},
\end{align*}
\]

we have

\[
\sqrt{-1} \left< f, [X, \vec{X}'] \right> = |p''|^2 \cdot z' \cdot \bar{z}', \quad X = (\vec{X})^{-1}, \quad \vec{X}, \vec{X}' \in \mathcal{W}',
\]

\[
f = (0, p''), \quad z = (z_1, \ldots, z_{d(r-1)}) \in \mathbb{C}^{(r-1)}.
\]

Rewriting (2.3) in these coordinates we obtain

**Proposition 2.1.** *All the inequivalent irreducible unitary representations of \(N^-\) fall into two classes:

(a) a family of 1-dimensional characters \(\chi_{q'}\) parameterized by \(q' \in \mathbb{F}_r^{-1}\) and given by

\[
\chi_{q'}(q, p) = \exp(\sqrt{-1} \Re(\bar{q} \cdot q)), \quad (q, p) \in N^-;
\]

(b) a family of infinite-dimensional representations \(\rho_{p'}\) parameterized by \(p' \in \mathcal{F}_0 \setminus \{0\}\). The Hilbert space \(\mathcal{H}_{p'}\) of the representation \(\rho_{p'}\) consists of holomorphic functions \(\psi\) on \(\mathbb{C}^{(r-1)}\), such that

\[
\|\psi\|^2_{p'} = \int_{\mathbb{C}^{(r-1)}} |\psi(z)|^2 \exp(-|p''| |z|^2) \, dz \, d\bar{z} < \infty,
\]

with

\[
\int_{\mathbb{C}^{(r-1)}} dz \, d\bar{z} = \prod_{j=1}^{(r-1)} 2d \Re z_j d \Im z_j.
\]

The action of \(\rho_{p'}\) on \(\psi \in \mathcal{H}_{p'}\) is given by

\[
(\rho_{p'}(q, p)\psi)(z) = \exp(\sqrt{-1} \Re(\bar{p'}p) + |p''|(|\bar{z}_0 \cdot z - \frac{1}{2} |z_0|^2|))\psi(z - z_0),
\]

\(z \in \mathbb{C}^{(r-1)}, (q, p) \in N^-\) with \(q = q(z_0), z_0 \in \mathbb{C}^{(r-1)}\), where

\[
q(z) = \Omega(P(z_1, \ldots, z_r), \ldots, P(z_{d(r-2)}+1, \ldots, z_{d(r-1)}))
\]

and \(P(z_{d(l-1)}+1, \ldots, z_{d}), l = 1, \ldots, r - 1\), is defined as

\[
\Re z_l + i \Im z_l \quad \text{for } \mathfrak{F} = \mathbb{C},
\]

\[
\Re z_{2l-1} + i \Im z_{2l-1} + j \Im z_{2l} + k \Re z_{2l} \quad \text{for } \mathfrak{F} = \mathbb{H},
\]

\[
\Re z_1 + i \Im z_1 + j \Im z_2 + k \Re z_2 + e \Im z_3
\]

\[+ ie \Re z_3 + je \Re z_4 + ke \Im z_4 \quad \text{for } \mathfrak{F} = \mathbb{O};
\]
Im(a + \sqrt{-1} b) = b, a, b \in \mathbb{R}. The functions
\[ \psi_n^p(z) = (2\pi)^{-m/2}|p'|(n!)^{m/2}y^{-1/2}z^n, \quad z \in \mathbb{C}^n, \]
\( n = (n_1, \ldots, n_m) \in \mathbb{N}^m \), with \( n! = n_1! \cdots n_m! \), \( z^n = z_1^{n_1} \cdots z_m^{n_m} \), \( |n| = n_1 + \cdots + n_m \), \( m = s(r - 1) \), form an orthonormal basis of \( \mathcal{H}_p \) as \( n \) runs over \( \mathbb{N}^m \).

We also note the following symmetry properties of \( x_q^p \) and \( \rho_p^e \) relative to the orthogonal and the unitary transformations.

**Proposition 2.2.** (a) Let \( a_1, \ldots, a_{r-1} \in O(\sigma, \mathbb{R}) \); then for \( q' \in F_r^{-1} \),
\[ x_q^p(a_1q_1, \ldots, a_{r-1}q_{r-1}, p) = x_{a_1q_1, \ldots, a_{r-1}q_{r-1}}(q, p), \quad (q, p) \in N^- . \]

(b) Let \( u_1, \ldots, u_{r-1} \in U(s) \); then for \( p' \in F_0 \setminus \{0\} \),
\[ \rho_p^{-1}(u) \rho_p(q, p) \rho_p^{-1}(u) = \rho_p(q, p), \quad (q, p) \in N^- \]
with \( q = q(z_0) \), \( z_0 \in \mathbb{C}^{s(r-2)} \) and \( q^u = q(uz_0) \) with \( uz = (u_1(z_1, \ldots, z_s), \ldots, u_{r-1}(z_{r-2}+1, \ldots, z_{s(r-2)})); ((\rho_p^e)\psi)(z) = \psi(uz) \) for \( \psi \in \mathcal{H}_p \); \( u = (u_1, \ldots, u_{r-1}) \), \( \sigma = 2s = \text{dim}_R F \).

3. Algebra of multiradial functions.

**Definition (cf. Geller [2]).** We say that a function \( F \) on \( N^- = F_r^{-1} \times F_0 \), \( F = \mathbb{C}, H, O \), is multiradial if there is a function \( f \) on \( \mathbb{R}^{(r-1)} \times F_0 \) such that
\[ F(q, p) = f(|q_1|, \ldots, |q_{r-1}|, p), \quad (q, p) \in N^- . \]

**Proposition 3.1.** Let \( \mathcal{E} \) denote the space of multiradial functions in \( L^1(N^-) \). Then \( \mathcal{E} \) is a commutative closed *-subalgebra of \( L^1(N^-) \) and \( \mathcal{E} \) is symmetric.

**Proof.** 1°. If \( F, G \in \mathcal{E} \) then \( F \ast G \in \mathcal{E} \). For we have
\[ F \ast G(q', p') = \int f(|q_1|, \ldots, |q_{r-1}|, p) \times g(|q'_1 - q_1|, \ldots, |q'_{r-1} - q_{r-1}|, p' - p - \text{Im}(\bar{q} \cdot q')) dq dp . \]
(3.1)

Substituting \( q = ((q'_1/|q'_1|)\bar{q}_1, \ldots, (q'_{r-1}/|q'_{r-1}|)\bar{q}_r) \) we get (since \((ab)b = a(bb)\) for \( a, b \in F\))
\[ \int f(|\bar{q}_1|, \ldots, |\bar{q}_{r-1}|, p) g(|q'_1| |1 - \bar{q}_1/|q'_1||, \ldots, |q'_{r-1}| |1 - \bar{q}_{r-1}/|q'_{r-1}||, p' - p - \text{Im}\left(\sum_{i=1}^{r-1} \bar{q}_i |q'_i|\right)) d\bar{q} dp , \]
i.e. \( F \ast G \) is multiradial. Obviously \( \mathcal{E} \) is closed.

2°. \( \mathcal{E} \) is commutative (cf. Kaplan and Putz [6, p. 377]). Under the orthogonal change of variables
\[ q_l \mapsto q''_l = q_l \cdot 2 \text{Re}(\bar{q}_l q'_l)/|q'_l|^2 - q_l, \quad l = 1, \ldots, r - 1, \]

one has \(|q_i' - q_i''| = |q_i' - q_i|\) and \(\text{Im}(\bar{q}_i q_i') = -\text{Im}(\bar{q}_i' q_i)\). Thus (3.1) is equal to
\[
\int f(|q_i''|, \ldots, |q_{r-1}'|, p) \times \left( |q_i' - q_i|, \ldots, |q_{r-1}' - q_{r-1}|, p' - p - \sum_{l=1}^{r-1} \text{Im}(\bar{q}_l q_l') \right) dq'' dp
\]
\[
= \int_{\mathbb{N}} F(q'', p) G((q', p')(q'', p)^{-1}) dq'' dp
\]
\[
= G \ast F(q', p').
\]

3°. Since \(L^1(\mathbb{N}^-)\) is symmetric (Leptin [8, p. 205]), its *-subalgebra \(\mathcal{C}\) is also symmetric.

**Proposition 3.2.** For \(F \in \mathcal{C}\) and \(u = (u_1, \ldots, u_{r-1}) \in U(s) \times \cdots \times U(s)\), the operators \(\rho_p(F) = \int_{\mathbb{N}} F(q, p)\rho_p(q, p) dq dp\) and \(A_u\) commute on \(\mathcal{K}_p\).

**Proof.** By Proposition 2.2(b),
\[
A_u^{-1} \int \rho_p(q(z_0), p) F(q(z_0), p) dq(z_0) dp A_u
\]
\[
= \int \rho_p(q(z_0), p) F(q(z_0), p) dq(z_0) dp.
\]

Since
\[
q_i(u z_0) = \Omega(P(u_i Z_i)) = (\Omega(P u_i P^{-1} \Omega^{-1}))(\Omega P Z_i)
\]
with \(Z_l = (z_{l(1-1)+1}, \ldots, z_{l0})\), \(l = 1, \ldots, r - 1\), and since \(\Omega(P u_i P^{-1} \Omega^{-1})\) is an orthogonal transformation on \(\mathbb{R}^n \cong F\), and \(\Omega P Z_i = q_i(z_0)\), the last integral is equal to \(\rho_p(F)\).

**Remark.** For \(F = \mathbb{R}\), the corresponding group \(\mathbb{N}^-\) is \(\mathbb{R}^{r-1}\), so the algebra \(L^1(\mathbb{N}^-)\) is already commutative, and, as in the case \(M = \mathbb{R}^n \times \mathbb{R}^+\) with the Euclidean metric [5], we consider \(\mathcal{C} = L^1(\mathbb{N}^-)\).

**4. Multiplicative linear functionals on \(\mathcal{C}\).** Let \(\Phi\) be a nonzero multiplicative linear functional on \(\mathcal{C}\). Since \(\mathcal{C}\) is a symmetric *-subalgebra of \(L^1(\mathbb{N}^-)\), there exist an irreducible *-representation \(\pi\) of \(L^1(\mathbb{N}^-)\) and a unit vector \(\xi\) in the Hilbert space \(\mathcal{K}_\pi\) such that
\[
\pi(F)\xi = \Phi(F)\xi \quad \text{for all } F \text{ in } \mathcal{C}.
\]
If \(\mathcal{K}_\pi\) is one dimensional, then
\[
\pi(F)\xi = \int_{\mathbb{N}} F(q, p)\chi_q(q, p) dq dp \xi
\]
for some \(q' \in \mathbb{F}^{r-1}\), and by Proposition 2.2(a), if \(q'\) and \(q''\) in \(\mathbb{F}^{r-1}\) are such that \(|q'_l| = |q''_l|, l = 1, \ldots, r - 1\), the \(\Phi's\) corresponding by (4.1) and (4.2) to \(\chi_q'\) and \(\chi_q''\) are identical. If \(\pi \approx \rho_p\), then by Proposition 3.2, \(\rho_p(F)\) and \(A_u\) commute. Now for \(\psi(z) = \psi_1(Z_1) \ldots \psi_{r-1}(Z_{r-1})\) with \(z = (Z_1, \ldots, Z_{r-1})\), \(Z_i = (z_{i(l-1)+1}, \ldots, z_{l0})\), \(l = 1, \ldots, r - 1\), we have
\[
(A_u \psi)(z) = \psi_1(u_1 Z_1) \ldots \psi_{r-1}(u_{r-1} Z_{r-1}).
\]
Thus putting \( \psi_0(z) = Z_n \) with \( n_l = (n_{l1}, \ldots, n_{ls}) \in \mathbb{N}^s \), we note that \( A_u \) preserves the finite-dimensional subspaces of \( \mathcal{H}_p \), namely the spaces \( \mathcal{H}^n = \bigotimes_{l=1}^{r-1} \mathcal{H}^{n_l} \), where \( n = (|n_1|, \ldots, |n_{r-1}|) \in \mathbb{N}^{r-1} \). \( \mathcal{H}^{n_l} \) is the space of homogeneous polynomials in \( z_{(l-1), \ldots, z_s} \) of degree \( |n_l| \). Moreover, \( \mathcal{H}_p = \bigoplus_n \mathcal{H}^n \) — an orthogonal direct sum over \( n \in \mathbb{N}^{r-1} \). We also note that \( A_u \) restricted to \( \mathcal{H}^n \) is equal to \( \bigotimes_{l=1}^{r-1} T^{n_l}(u_1, \ldots, u_{r-1}) \) with \( T^k \), \( k = |n_l| \), being the representation of \( U(s) \) on \( \mathcal{H}^k \) given by \( (T^k_u \psi)(Z) = \psi(u^{-1}Z) \). Since \( T^k \) is irreducible (cf., e.g., [13, pp. 204–209]), the representations \( T^n = \bigotimes \mathcal{H}^{n_l} \) of \( U(s) \times \cdots \times U(s) \), \( r-1 \) copies of \( U(s) \), act irreducibly on \( \mathcal{H}^n \), and \( T^n \approx T^m \) iff \( n = m \). Hence, by Schur’s Lemma, every intertwining operator \( S \) for \( \bigoplus_n T^n \) on \( \mathcal{H}_p \) is of the form \( S = \bigoplus_n c_n(S)1d_{x_n} \). In particular, each \( \rho_p(F) \) with \( F \in \mathcal{C} \) is such. It follows from (4.1) that \( \Phi(F) \) is equal to one of the constants \( c_n(\rho_p(F)) \), \( n \in \mathbb{N}^{r-1} \). Conversely, for every fixed \( n \), the mapping \( F \mapsto c_n(\rho_p(F)) \) defines a multiplicative linear functional on \( \mathcal{C} \). Now we shall derive explicit formulas for the constants \( c_n \) above. Since, e.g.,

\[
c_n(\rho_p(F)) = (\rho_p(F) \psi_n^F \psi_n^F)_{x_n}.
\]

with \( n' = (n_1, 0, \ldots, 0; n_2, 0, \ldots, 0; \ldots; n_{r-1}, 0, \ldots, 0) \in (\mathbb{N}^r)^{-1} \), we calculate the integral, see Proposition 2.1(b),

\[
\int_{C^{(r-1)}} [\rho_p(F)(\psi_n^F)](z) \overline{\psi_n^F}(z) \exp(-|p'||z|^2) \, dz \, d\overline{z}, \tag{4.3}
\]

which in expanded form is equal to (with \( k = s(r-1) \))

\[
(2\pi)^{-k(n!)}^{-1}|p'|^{|n|+k}
\times \int_{C^s} \int_{F_0} F(q(z_0),\rho)\exp\left(\sqrt{-1} \text{Re}(\bar{p}'p) + |p'|(\bar{z}_0 - \frac{1}{2}|z_0|^2)\right)
\times (z - z_0)^{n'} \bar{z}^{n'} \exp(-|p'||z|^2) \, dq(z_0) \, dp \, dz \, d\overline{z}. \tag{4.4}
\]

The integral

\[
\int_{C^{(r-1)}} (z - z_0)^{n'} \bar{z}^{n'} \exp(-|p'|(z^2)) \exp(|p'|\bar{z}_0 \cdot z) \, dz \, d\overline{z} \tag{4.5}
\]

is equal to

\[
(2\pi/|p'|)^{s(r-1)} r^{-1} \prod_{l=1}^{r-1} 2\pi n_l! |p'|^{-n_l-1} \sum_{j=0}^{n_l} (-|p'| |z_0^{(l-1)+1}|^2)^j \left(\frac{n_l}{j}\right)(j!)^{-1}
\]

\[
= (2\pi)^{s(r-1)} |p'|^{-|n|-s(r-1)} n! \prod_{l=1}^{r-1} L_n(|p'| |z_0^{(l-1)+1}|^2) \tag{4.5a}
\]

with \( L_n \) being the Laguerre polynomial. (4.5a) is obtained (see [5]) by substituting the binomial formula for \( (z - z_0)^n \), developing \( \exp(-|p'|\bar{z}_0 \cdot z) \) in a power series and integrating this series term by term using the orthogonality relations for the
functions $z^n$ in $\mathcal{H}_\nu$. Substituting (4.5a) in (4.4) we obtain that (4.3) is equal to

$$\int_{F^{-1} \times F_0} F(q(z_0), p) \exp\left( \sqrt{-1} \ Re(\tilde{p}' p) - \frac{1}{2} |p'| |z_0|^2 \right)$$

$$\times \prod_{l=1}^{r-1} L_n\left( |p'| |z_{l+(l-1)}|^{2} \right) \ dq(z_0) \ dp$$

$$= \int_0^\infty dt_1 \cdots \int_0^\infty dt_{r-1} \left( \int_{F_0} f(t_1, \ldots, t_{r-1}, p) \exp\left( \sqrt{-1} \ Re(\tilde{p}' p) \right) \ dp \right)$$

$$\times \exp\left( -\frac{1}{2} |p'|^{2} + \cdots + t_{r-1}^{2} \right) \prod_{l=1}^{r-1} t_{l}^{a_{l}-1} g_{l},$$

with $f$ as in (3.0) and $g_l$ given by

$$g_{l} = \int_{S(\sigma - 1)} L_n\left( |p'| |z_{l}|^{2} \right) dS(q(Z)),$$

$$Z = (z_1, \ldots, z_r) \in \mathbb{C}^r, \quad |Z| = t_l,$$

$S(\sigma - 1)$ being the unit sphere in $\mathbb{F}$. Since here $q(Z) = \Omega(PZ)/t_l$, with $P$ as in Proposition 2.1(b) and $\Omega \in O(\sigma, \mathbb{R})$, in order to compute $g_l$ one has to calculate the integrals

$$\int_{S(\sigma - 1)} |z_1(q)|^{2j} dS(q), \quad j = 0, \ldots, n_l,$$  

(4.6)

where $|z_1(q)|^{2} = t_l^2((q_1)^2 + (q')^2)$ with $q_1, q', \ldots$ denoting the coordinates of $q$ in the (standard) basis $\{1, i, \ldots \}$ of $\mathbb{F}$ over $\mathbb{R}$. Now (4.6) is equal to

$$\int_0^{\pi/2} \cos^{2j} \theta \cos \theta \sin^{a-3} \theta \ d\theta \cdot 2\pi \cdot 2\pi^{s-1}[ (s - 2)! ]^{-1} t_l^a$$

$$= 2\pi^{t_l^a} t_l^a j! / (j + s - 1)!.$$

We summarize the results of this section in the following:

**Proposition 4.** The multiplicative linear functionals on $\mathcal{H}$ fall into two classes:

(a) the functionals corresponding to $(r - 1)$-tuples $(t_1, \ldots, t_{r-1})$ of nonnegative real numbers and given by

$$F \mapsto F(t_1, \ldots, t_{r-1}) = \int_{F^{-1} \times F_0} F(q, p) \exp\left( \sqrt{-1} \ Re(\tilde{q}' q) \right) dq \ dp$$

with $q' \in \mathbb{F}^{-1}$ arbitrary provided $(|q_1|, \ldots, |q_{r-1}|) = (t_1, \ldots, t_{r-1})$.

(b) the functionals corresponding to pairs $(p', n) \in F_0 \setminus \{0\} \times \mathbb{N}^{r-1}$ and given by

$$F \mapsto F(p', n) = (2\pi^s)^{-1} \int_{\mathbb{R}^{r-1}_+} \exp\left( -\frac{|p'|}{2} (t_1^2 + \cdots + t_{r-1}^2) \right) \prod_{l=1}^{r-1} L_n\left( |p'| |t_l|^2 \right) t_l^{a_{l}-1}$$

$$\times \left( \int_{F_0} f(t_1, \ldots, t_{r-1}, p) \exp\left( \sqrt{-1} \ Re(\tilde{p}' p) \right) \ dp \right) dt_1 \cdots dt_{r-1},$$
where \( f(|q_1|, \ldots, |q_r|, p) = F(q_1, \ldots, q_r, p) \) and
\[
L_k^{(m)}(x) = \sum_{j=0}^{k} \frac{(-x)^j}{(j+m)!} \binom{k}{j} \\
= \left[ (k+m)! \right]^{-1} x^{-m} e^{-x} (d^k/dx^k)(x^{k+m} e^{-x}).
\]

5. Nonvanishing of the Gel'fand transform of \( P_a \).

**Lemma 1.** For \( 1 < m < 2k + \frac{1}{2}, k \geq \frac{3}{2} \) and \( Q > 0 \), the following formula holds:
\[
\int_{\mathbb{R}^m} \frac{\exp(\sqrt{-1} x_0 \cdot x)}{(Q^2 + 4|x|^2)^k} \, dx \quad = \quad 2^{-m} m^{-1/2} \frac{\Gamma(k - m/2)}{\Gamma(k)} \frac{Q^{m-2k}}{\Gamma(k)\Gamma(k - (m-1)/2)} \\
\times \int_0^\infty e^{-(r/2)Q^2} ((t+1)^2 - 1)^{(k(m-1))/2} \, dt,
\]
for \( r = |x_0| \neq 0 \).

**Proof.** For \( x_0 = 0 \), the integral is equal to the “area” of the unit sphere in \( \mathbb{R}^m \)
(= 2 when \( m = 1 \)) times \( \int_{\mathbb{R}} r^{m-1} (Q^2 + 4r^2)^{-k} \, dr \) and we substitute \( r = r'Q/2 \).
For \( x_0 \neq 0 \), the function \( 4((r^2 + x^2))^{-k} \) is radial on \( \mathbb{R}^m \), hence its Fourier transform (5.1) is equal to, see, e.g., [9, p. 155],
\[
4^{-k}(2\pi)^{m/2} r^{-(m-2)/2} \int_0^\infty \left( \frac{1}{2} Q^2 + r^2 \right)^{-k} J_{(m-2)/2}(rt) t^{m/2} \, dt,
\]
\[
m > 1, \quad k > 3/2.
\]
Combining now the Sonine formula [12, p. 434, (2)],
\[
\int_0^\infty x^{r+1} J_r(ax) \, dx = \frac{a^n a^{r-\mu}}{2^n \Gamma(\mu + 1)} K_{\mu-r}(ak),
\]
valid when \( -1 < \text{Re}(\nu) < 2 \text{Re}(\mu) + \frac{3}{2} \), with the following expression for the function \( K \) [12, p. 172, (4)],
\[
K_\nu(z) = \frac{\Gamma(\frac{1}{2})(\frac{1}{2} z)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt}(t^2 - 1)^{\nu-1/2} \, dt,
\]
valid for \( \text{Re}(\nu + \frac{1}{2}) > 0, |\arg z| < \pi/2 \), we obtain (5.1).

**Lemma 2.** For \( \epsilon > 0, m > 0, x \in \mathbb{R}^n \),
\[
(\epsilon + |x|^2)^{-m} = (4\pi)^{-n/2} \Gamma(m)^{-1} \int_{\mathbb{R}^n} \exp(-\sqrt{-1} x \cdot y) \\
\times \left( \int_0^\infty t^{m-1-m/2} e^{-t} e^{-|y|^2/4t} \, dt \right) \, dy,
\]
i.e. \( (\epsilon + |x|^2)^{-m} \) is a Fourier transform of a positive function in \( L^1(\mathbb{R}^n) \).
Proof. Combine
\[
(e + |x|^2)^{-m} = \Gamma(m)^{-1} \int_0^\infty t^{m-1} e^{-(e + |x|^2)t} \, dt, \quad m > 0,
\]
with
\[
\exp(-|x|^2t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp(-|y|^2/4t) \exp(-\sqrt{-1} x \cdot y) \, dy,
\]
and note that the obtained double integral is absolutely convergent.

Proposition 5. For every \( a \in A \), \( \hat{P}_a \) does not vanish on \( \mathfrak{M}(\mathfrak{A}) \)--the maximal ideal space of \( \mathfrak{A} \).

Proof. (a) For the points \((t_1, \ldots, t_{r-1}) \in \mathfrak{M}(\mathfrak{A})\), integrating over \( F_0 \) in the formula of Proposition 4(a), according to Lemma 1 (with \( x_0 = 0 \)) we get
\[
\hat{P}_a(t_1, \ldots, t_{r-1}) = c \int_{\mathbb{R}^{r-1}} \frac{\exp(\sqrt{-1} \text{Re}(\bar{q} \cdot q))}{(e + |q|^2)^{d+1-a}} \, dq,
\]
with
\[
c = 2^{1-\sigma} \pi^{d/2} c_{d\pi} \Gamma(\frac{d}{2} - s + 1/2) \Gamma(\frac{d}{2})^{-1},
\]
and this is positive by Lemma 2 and the Fourier inversion formula.

(b) For the points \((p', n) \in \mathfrak{M}(\mathfrak{A})\) with \( p' \in F_0 \setminus \{0\} \), \( n = (n_1, \ldots, n_{r-1}) \in \mathbb{N}^{r-1} \), we use the formula from Proposition 4(b) for \( \hat{P}_a(p', n) \). Applying Lemma 1 to the integral over \( F_0 \) there (with \( Q = e + t_1^2 + \cdots + t_{r-1}^2 \), \( m = \sigma - 1 \), \( x_0 = p' \), \( k = d/2 \)), then interchanging the order of integration from \( dt_1 \ldots dt_{r-1} \) to \( dt_{r-1} \ldots dt_1 \), making the change of variables \((|p'|t_1^2, \ldots, |p'|t_{r-1}^2) = (x_1, \ldots, x_{r-1})\), and finally applying (4.7), we obtain
\[
\hat{P}_a(p', n) = c \int_0^\infty e^{-|p'|^2/2((t + 1)^2 - 1)} \prod_{i=1}^{r-1} \mathcal{J}_i(t) \, dt,
\]
with
\[
c = 2^{d(r-1)} (e|p'|)^{d/2} \Gamma(d/2)^{-1} \exp(-|p'|/2)
\]
and
\[
\mathcal{J}_i(t) = ((n_i + s - 1)!)^{-1} \int_0^\infty e^{-tx/2}(d^n/dx^n)(x^{n+s-1}e^{-x}) \, dx.
\]
Integrating by parts get
\[
\mathcal{J}_i(t) = ((n_i + s - 1)!)^{-1} \int_0^\infty e^{-tx/2}x^{n+s-1}e^{-x} \, dx \cdot (t/2)^{n_i}
\]
\[
= (t/2)^{n_i}(t/2 + 1)^{-(n_i + s)}.
\]
Thus \( \hat{P}_a(p', n) \) is positive.

Remark. For \( M \) being the real hyperbolic space, i.e. for \( F = \mathbb{R} \), we have \( \mathbb{N}^- = \mathbb{R}_d \), \( \mathfrak{M}(L(\mathbb{N}^-)) = \tilde{\mathbb{N}}^- \), \( P_a(X_{-a}) = c_{d\mathbb{R}} e^{d/2}(e + |q|^2)^{-d} \) and, by Lemma 2, \( \hat{P}_a > 0 \) on \( \tilde{\mathbb{N}}^- \).
6. **Theorem on ideals in** $L^1(N^-)$. Since the algebras @ we consider here have the same qualitative properties as the one considered in [5], similar facts can be proved about them. In particular, the following statement about ideals in $L^1(N^-)$ is a consequence of the Wiener property of @ and existence of the approximate identity for $L^1(N^-)$ in @ (the dilations $\delta_\tau$, $\tau > 0$, on $N^-$ used in the construction of the approximate identity are given by $\delta_\tau(q, p) = (\tau^{-1/2}q, \tau^{-1}p)$).

**Proposition 6.** If $\mathcal{I}$ is a proper closed right ideal in $L^1(N^-)$, then there is a $\Phi$ in $\mathcal{M}(\mathcal{A})$ such that $F(\Phi) = 0$ for every $F \in \mathcal{I} \cap \mathcal{A}$.

7. **Proof of the Theorem** [5]. The Theorem follows now from Proposition 6, for if we put

$$\mathcal{I} = \left\{ f \in L^1(N^-) : \lim_{N^- \ni n \to \infty} q^* f(n) = c_0 \int_{N^-} f(n) \, dn \right\},$$

with $q \in L^\infty(N^-)$ being the boundary value of the bounded harmonic function $u$ on $M$, then $P_{a_0} \in \mathcal{I} \cap \mathcal{A}$ and $\hat{P}_{a_0} \neq 0$ on $\mathcal{M}(\mathcal{A})$, so $\mathcal{I} = L^1(N^-)$. Hence $P_a \in \mathcal{I}$ for every $a$ in $A$.

**Added in proof.** Meanwhile Korányi [14] described the Gel'fand space, as well as the related Plancherel formula, for the commutative algebra $\mathcal{A}$ of *biradial* functions in $L^1(N^-)$, i.e. the functions $F$ such that

$$F(q, p) = f(|q|, |p|), \quad (q, p) \in N^-,$$

for some $f$ on $\mathbb{R}_+ \times \mathbb{R}_+$, cf. §3. His approach uses neither the classification of symmetric spaces nor the representations of nilpotent groups.

**References**


Institute of Mathematics, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland