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A FAITHFUL HILLE-YOSIDA THEOREM FOR FINITE DIMENSIONAL EVOLUTIONS

BY

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ABSTRACT. As a natural generalization of the classical Hille-Yosida theorem to evolution operators, necessary and sufficient conditions are found for an evolution U acting in R^N so that for each $s > t$, $U(s, t)$ can be uniquely represented as a product integral $\prod_t^s[I + V]^{-1}$ for some additive, accretive generator V. Under these conditions, we further show that $U(\xi, \zeta)$ is differentiable a.e.

I. Introduction. In his landmark paper [7], J. S. Mac Nerney establishes a one-to-one correspondence $\mathcal{E}$ between an evolution class $\mathcal{OM}$ and a generator class $\mathcal{OA}$, where $\mathcal{OM}$ is the set of all functions U taking $\Gamma = \{(s, t)|s > t\}$ into $\mathcal{BL}(X)$, the bounded linear operators on a Banach space X, such that

(a) $U(a, b)U(b, c) = U(a, c)$ for all $a \geqslant b \geqslant c$,

(b) $U(\cdot, \cdot) - I$ is of bounded variation in the norm-topology of X;

while $\mathcal{OA}$ is the set of all functions V from Γ into $\mathcal{BL}(X)$ which satisfy

(c) $V(a, b) + V(b, c) = V(a, c)$ for all $a \geqslant b \geqslant c$,

(d) $V(\cdot, \cdot)$ is of bounded variation in the norm-topology of X.

The invertible map $\mathcal{E}$ from $\mathcal{OA}$ onto $\mathcal{OM}$ is given by

$$\mathcal{E}(V)(a, b) = \prod_b^a [I + V], \qquad V \in \mathcal{OA},$$

$$\mathcal{E}^{-1}(U)(a, b) = \sum_b^a [U - I], \qquad U \in \mathcal{OM}.$$

With property (a) above referred to as the *evolution* property and property (c) referred to as the *generator* property, we can thus say that every evolution U in $\mathcal{OM}$ is *generated* by a unique generator V in $\mathcal{OA}$ through the product integral formula

$$U(a, b) = \prod_b^a [I + V] \quad \text{where } V(a, b) = \sum_b^a [U - I].$$

J. V. Herod and R. W. McKelvey [4] first succeeded in extending Mac Nerney's results to the case of unbounded generators. Their theory is broad enough to include the classical Hille-Yosida theorem, which states a one-to-one correspondence between the class of densely defined m-accretive operators A on a Banach space X and the class of strongly continuous, contractive semigroups $T(t)$ on X,

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where if $T(t)$ is the semigroup associated with A, then $T(t) = \exp(-tA)$, $t \geqslant 0$. In other words, Herod and McKelvey are able to enlarge Mac Nerney's classes $\mathcal{O}\mathcal{A}$ and $\mathcal{O}\mathcal{M}$ to classes $\mathcal{A}$ and $\mathcal{M}$, respectively, where the generator class $\mathcal{A}$ is large enough to contain generators of the form $V(s, t) = (s - t)A$ and the evolution class $\mathcal{M}$ is large enough to contain evolutions of the form

$$U(s, t) = \exp(-(s - t)A).$$

Furthermore, $\mathcal{A}$ and $\mathcal{M}$ are kept in one-to-one correspondence via the invertible map $\mathcal{E} : \mathcal{A} \overset{\text{onto}}{\to} \mathcal{M}$ given by

$$\mathcal{E}(V)(a, b) = \prod_b^a [I + V]^{-1} \quad \text{and} \quad \mathcal{E}^{-1}(U)(a, b) = \sum_b^a [I - U]$$

for V in $\mathcal{A}$ and U in $\mathcal{M}$.

Similar to the classes $\mathcal{O}\mathcal{A}$ and $\mathcal{O}\mathcal{M}$, membership in the classes $\mathcal{A}$ and $\mathcal{M}$ also requires that $V(\cdot, \cdot)$ in $\mathcal{A}$ and $U(\cdot, \cdot) - I$, with U in $\mathcal{M}$, satisfy a bounded variation condition. The Hille-Yosida theorem itself, however, makes no such a priori bounded variation requirement on either the class of strongly continuous, contractive semigroups or on the class of densely defined, accretive operators. Yet, remarkably, one of the dramatic conclusions of this theorem is that every strongly continuous, contractive semigroup $T(t)$ is t-differentiable for all $t > 0$. Herein lies the chief motivation for this paper–placing no a priori bounded variation assumptions on an evolution U, what conclusions can be made about the differentiability of $U(s, t)$?

II. Definitions and results. In the case where U is acting in the finite dimensional Banach space $X = R^N$, under the sup-norm $\|\cdot\|$ with $\|\{x_i\}_{i-1}^N\| = \sup_{1 \leqslant i \leqslant n} |x_i|$, we show in Corollary II that every norm-continuous, contractive evolution $U(s, t)$ is differentiable a.e. in the s and t variables. Such an evolution U will be said to belong to the class $\mathcal{H}_N$. That is, we define $\mathcal{H}_N$ as the class of all functions U from Γ into $\mathcal{B}\mathcal{L}(X)$ which satisfy

(i) $U(s, t)U(t, r) = U(s, r)$ for all $s \geqslant t \geqslant r$,

(ii) $U(s, t)$ is jointly continuous and $U(s, s) = I$ for all s, and

(iii) $\|U(s, t)\| \leqslant 1$ for all $s \geqslant t$.

Corresponding to $\mathcal{H}_N$ we define the generator class $\mathcal{G}_N$ as the set of all functions V from Γ into $\mathcal{B}\mathcal{L}(X)$ which satisfy

(iv) $V(s, t) + V(t, r) = V(s, r)$ for all $s \geqslant t \geqslant r$,

(v) $V(s, t)$ is jointly continuous,[1] and

(vi) $\|[I + V(s, t)]x\| \geqslant \|x\|$ for all x in R^N and $s \geqslant t$.

Property (vi) is called the *accretive* property for V. Thus $\mathcal{H}_N$ shall be called the set of norm-continuous, accretive generators.

We now state the finite dimensional version of the Hille-Yosida theorem applied to evolution operators.

[1]Observe that $V(s, s) = 0$ for all s follows immediately from (iv).

THEOREM I. *There is a reversible function* $\mathcal{E}$ *from the class* $\mathcal{J}_N$ *onto the class* $\mathcal{K}_N$ *such that each of the following is a necessary and sufficient condition for the member* (V, U) *of* $\mathcal{J}_N \times \mathcal{K}_N$ *to belong to* $\mathcal{E}$. *Given* $a > b$,

(i) $U(s, t) = \prod_t^s [I + V]^{-1}$ *for all* $s > t$ *in* $[b, a]$, *and*

(ii) $V(s, t) = \sum_t^s [I - U]$ *for all* $s > t$ *in* $[b, a]$.

The proof of this theorem will be offered after we provide the succeeding lemmas needed to prove the existence of the *product integral* in (i) for V in $\mathcal{J}_N$ and the existence of the *sum integral* in (ii) for U in $\mathcal{K}_N$.

We begin with

LEMMA 1. *If* $V \in \mathcal{J}_N$, *then* V *is of norm-bounded variation on every closed interval in the sense that for each* $a > b$ *there exists a number* $\gamma > 0$ *such that for all partitions* $b = t_0 < t_1 < \cdots < t_n = a$ *of* $[b, a]$, $\sum_{i=1}^n \| V(t_i, t_{i-1}) \| \leqslant \gamma$.

PROOF. If $V(\xi, \zeta) = [V_{ij}(\xi, \zeta)]_{1 \leqslant i,j \leqslant N}$, then given $a \geqslant b$, it will suffice to show by the compactness of the interval $[b, a]$ and the fact that all norms in a finite dimensional linear space are equivalent, that for each $t \in [b, a]$ there exists an open neighborhood of t, N_t, such that for each $1 \leqslant i, j \leqslant N$, there is an increasing function on N_t, $g_{ij}(\cdot)$, which satisfies $|V_{ij}(\xi, \zeta)| \leqslant g_{ij}(\xi) - g_{ij}(\zeta)$ for all $\xi \geqslant \zeta \in N_t$.

In the present sup-norm setting, the accretive property of V may be expressed as

$$\sup_{1 \leqslant i \leqslant N} \left\{ \left| \sum_{j=1}^N \left[\delta_{ij} + V_{ij}(\xi, \zeta) \right] x_j \right| \right\} \geqslant \sup_{1 \leqslant j \leqslant N} \{ |x_j| \} \qquad (')$$

for all $(x_j) \in R^N$ and all $\xi \geqslant \zeta$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Suppose that for $t \in [b, a]$, $\varepsilon_t > 0$ is such that for all $\xi > \zeta$ in $N_t = (t - \varepsilon_t, t + \varepsilon_t)$, if $1 \leqslant i, j \leqslant N$, then $|V_{ij}(\xi, \zeta)| < \frac{1}{3}$. Then by letting $x_k = \delta_{ik}$ in relation (′) we see that $V_{kk}(\xi, \zeta) \geqslant 0$ for all $\xi \geqslant \zeta$ and $k = 1, 2, \ldots, N$. Hence, if we define $g_{kk}(\xi) = V_{kk}(\xi, b)$, then g_{kk} is increasing on $[b, a]$ and $0 \leqslant V_{kk}(\xi, \zeta) \leqslant g_{kk}(\xi) - g_{kk}(\zeta)$ for all $\xi \geqslant \zeta$ in $[b, a]$.

Now, for fixed $1 \leqslant i \neq j \leqslant N$, let

$$x_p = \begin{cases} 1, & p = i, \\ \dfrac{1}{3}, & p = j, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, for all $\xi \geqslant \zeta \in N_t$

$$\left| \sum_{p=1}^N \left[\delta_{ip} + V_{ip}(\xi, \zeta) \right] x_p \right| = 1 + V_{ii}(\xi, \zeta) + \frac{V_{ij}(\xi, \zeta)}{3}$$
$$\geqslant 1 - \frac{1}{9} = \frac{8}{9}$$

and

$$\left|\sum_{p=1}^{N}\left[\delta_{jp}+V_{jp}(\xi,\zeta)\right]x_p\right|=\left|V_{ji}(\xi,\zeta)+\frac{(1+V_{jj}(\xi,\zeta))}{3}\right|$$

$$\leqslant\frac{1}{3}+\frac{1}{3}+\frac{1}{9}=\frac{7}{9}$$

and for $k\neq i$ and $k\neq j$,

$$\left|\sum_{p=1}^{N}\left[\delta_{kp}+V_{kp}(\xi,\zeta)\right]x_p\right|=\left|V_{ki}(\xi,\zeta)+\frac{V_{kj}(\xi,\zeta)}{3}\right|$$

$$\leqslant\frac{1}{3}+\frac{1}{9}=\frac{4}{9}.$$

Relation (′) now yields that for all $\xi\geqslant\zeta\in N_t$,

$$1+V_{ii}(\xi,\zeta)+\frac{V_{ij}(\xi,\zeta)}{3}\geqslant 1.$$

Likewise, by choosing

$$x_p=\begin{cases}1, & p=i,\\ -\dfrac{1}{3}, & p=j,\\ 0, & \text{otherwise},\end{cases}$$

we obtain $1+V_{ii}(\xi,\zeta)-(V_{ij}(\xi,\zeta)/3)\geqslant 1$. Thus $|V_{ij}(\xi,\zeta)|\leqslant 3V_{ii}(\xi,\zeta)$ for all $\xi\geqslant\zeta\in N_t$. Q.E.D.

The multiplicative counterpart to Lemma 1 is

Lemma 2. *If $U\in\mathcal{H}_N$ then $I-U$ is of norm-bounded variation on every closed interval (in the sense of Lemma 1).*

The proof of this lemma relies on

Lemma 3. *Let $\{A(s,t)\}_{s\geqslant t\in R}$ be a family of operators on the Banach space $X=R^N$ in the* sup-*norm topology, such that*

(i) *$A(s,t)$ is continuous and $A(s,s)=I$ for all s, and*

(ii) *$\|A(s,t)\|\leqslant 1$ for all $s\geqslant t\in R$.*

Then for each t there exists an open neighborhood N_t of t such that for all $\xi\geqslant\zeta\in N_t$

$$0<\det A(\xi,\zeta)\leqslant A_{ii}(\xi,\zeta),\qquad i=1,2,\dots,N.$$

Remark. In the sup-norm topology on $X=R^N$ a linear operator $A=[A_{ij}]$: $X\to X$ has norm $\|A\|=\sup_{1\leqslant i\leqslant N}\sum_{j=1}^{N}|A_{ij}|$.

Proof. The determinant of a matrix is the sum of all possible signed products of its entries, one entry taken from each row and column. Thus the determinant consists of the sum of $N!$ terms, one of which is the product of its diagonal elements. Therefore, $\det A(\xi,\zeta)$ is a continuous function and we conclude from hypothesis (i) that given t there exists $\delta_1>0$ such that for all $\xi\geqslant\zeta\in(t-\delta_1,t+\delta_1)$ both $0<\det A(\xi,\zeta)$ and $0<A_{ii}(\xi,\zeta)$, $i=1,2,\dots,N$, hold.

If for each pair $\xi \geqslant \zeta \in (t - \delta_1, t + \delta_1)$ we define

$$\Delta(\xi, \zeta) = A_{11}(\xi, \zeta)A_{22}(\xi, \zeta) \cdots A_{NN}(\xi, \zeta) - \det A(\xi, \zeta)$$

we have, using hypothesis (ii),

$$\begin{aligned} A_{11}(\xi, \zeta) - \det A(\xi, \zeta) &= A_{11}(\xi, \zeta)(1 - A_{22}(\xi, \zeta)A_{33}(\xi, \zeta) \cdots A_{NN}(\xi, \zeta)) + \Delta(\xi, \zeta) \\ &\geqslant \sum_{i=2}^{N} \frac{A_{11}(\xi, \zeta)}{N}(1 - A_{ii}(\xi, \zeta)) + \Delta(\xi, \zeta) \\ &\geqslant \sum_{i=2}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{A_{11}(\xi, \zeta)}{N} |A_{ij}(\xi, \zeta)| + \Delta(\xi, \zeta). \end{aligned}$$

Now $\Delta(\xi, \zeta)$ is the sum of $N! - 1$ terms, say $\Delta(\xi, \zeta) = \sum_{i=1}^{N!-1} q_i(\xi, \zeta)$, where each $q_i(\xi, \zeta)$ is the signed product of N elements from $A(\xi, \zeta)$ such that no two factors of $q_i(\xi, \zeta)$ come from the same column or from the same row of $A(\xi, \zeta)$. Thus each term $q_i(\xi, \zeta)$ can contain at most $N - 2$ diagonal element factors, or equivalently, must contain at least two off-diagonal element factors. Therefore, each $q_i(\xi, \zeta)$ may be expressed as $q_i(\xi, \zeta) = A_{m_i n_i}(\xi, \zeta)\, P_i(\xi, \zeta)$, where $m_i \neq n_i$ and by hypothesis (i), $\lim_{|\xi - \zeta| \to 0} P_i(\xi, \zeta) = 0$. We may further assume that $m_i \geqslant 2$ above, since each $q_i(\xi, \zeta)$ term contains at least one off-diagonal element not from the first row of $A(\xi, \zeta)$. By grouping terms together with common off-diagonal elements, we obtain

$$\Delta(\xi, \zeta) = \sum_{\substack{2 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant N \\ i \neq j}} \pm A_{ij}(\xi, \zeta) Q_{ij}(\xi, \zeta),$$

where each $Q_{ij}(\xi, \zeta)$ term is the sum of $P_k(\xi, \zeta)$ terms and thus satisfies $\lim_{|\xi - \zeta| \to 0} Q_{ij}(\xi, \zeta) = 0$.

Now let δ_2 be such that $0 < \delta_2 \leqslant \delta_1$ and for all $\xi \geqslant \zeta \in (t - \delta_2, t + \delta_2) = N_t$

$$\frac{A_{11}(\xi, \zeta)}{N} \geqslant |Q_{ij}(\xi, \zeta)| \quad \text{for all } i \neq j.$$

Then clearly for all $\xi \geqslant \zeta \in N_t$

$$\sum_{i=2}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{A_{11}(\xi, \zeta)}{N} |A_{ij}(\xi, \zeta)| \geqslant \sum_{i=2}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} |Q_{ij}(\xi, \zeta)|\, |A_{ij}(\xi, \zeta)| \geqslant |\Delta(\xi, \zeta)|.$$

Hence $0 < \det A(\xi, \zeta) \leqslant A_{ii}(\xi, \zeta)$ for all $\xi \geqslant \zeta \in N_t$ and $i = 1$. That a neighborhood of t can be found for which the above relation holds for all $1 \leqslant i \leqslant N$ is evident from the symmetry of the preceding proof with respect to the diagonal elements of $A(\xi, \zeta)$.

We now return to the

Proof of Lemma 2. It will suffice to show, by a usual compactness argument, that with $U(\xi, \zeta) = [U_{ij}(\xi, \zeta)]_{1 \leqslant i,j \leqslant N}$, given $a > b$ and $t \in [b, a]$ there exists an open neighborhood of t, $\mathfrak{N}$, and a function h increasing on $\mathfrak{N}$ such that

$$h(\xi) - h(\zeta) \geqslant \begin{cases} |1 - U_{ij}(\xi, \zeta)|, & i = j, \\ |U_{ij}(\xi, \zeta)|, & i \neq j, \end{cases} \tag{$''$}$$

for all $\xi \geqslant \zeta \in \mathfrak{N}$ and $1 \leqslant i, j \leqslant N$.

In fact, since the family of operators $\{U(s, t)\}_{s \geqslant t}$ satisfies the hypotheses of Lemma 3, we may let $\mathfrak{N} = N_t$, the open neighborhood of t presented in the concluding statement of Lemma 3. Therefore, for $i = 1, 2, \ldots, N$ and $\xi \geqslant \zeta \in \mathfrak{N}$ we have $0 < \det U(\xi, \zeta) \leqslant U_{ii}(\xi, \zeta) \leqslant 1$. Hence

$$0 \leqslant 1 - U_{ii}(\xi, \zeta) \leqslant \mu(\xi, \zeta) - 1, \qquad \xi \geqslant \zeta \in \mathfrak{N}, 1 \leqslant i \leqslant N,$$

where $\mu(\xi, \zeta) = [\det U(\xi, \zeta)]^{-1}$ satisfies

$$\mu(\xi, \zeta)\mu(\zeta, \rho) = \mu(\xi, \rho) \geqslant 1 \quad \text{for all } \xi \geqslant \zeta \geqslant \rho \in \mathfrak{N}.$$

As well, since $\|U(\xi, \zeta)\| \leqslant 1$ for all $\xi \geqslant \zeta$, it follows that $0 \leqslant |U_{ij}(\xi, \zeta)| \leqslant \mu(\xi, \zeta) - 1$ for all $\xi \geqslant \zeta \in \mathfrak{N}$ and $1 \leqslant i \neq j \leqslant N$. We complete the proof by setting $h(\cdot) = \mu(\cdot, b)$. Hence relation ($''$) is satisfied, for if $\xi \geqslant \zeta \in \mathfrak{N}$, then $h(\xi) - h(\zeta) = (\mu(\xi, \zeta) - 1)\mu(\zeta, b) \geqslant \mu(\xi, \zeta) - 1 \geqslant 0$. Q.E.D.

As a final prelude to the proof of Theorem 1, we present

LEMMA 4. *If $V \in \mathcal{J}_N$ and $U \in \mathcal{K}_N$ then for all $a > b$,*

$$\prod_t^s [I + V]^{-1} \quad \text{and} \quad \sum_t^s [I - U]$$

exist and are continuous for all $s \geqslant t \in [b, a]$.

REMARK. Given a Banach space $(Y, |\cdot|)$ of operators, $a > b$ and a Y-valued function J defined on $[b, a] \times [b, a]$, the sum integral of J exists on $[b, a]$ if for each $s > t$ in $[b, a]$ there is an element of Y, denoted as $\sum_t^s J$, such that for each $\varepsilon > 0$, a partition S of $[t, s]$ exists where if $T = \{T_i\}_{i=0}^n$ is a refinement of S then $|\sum_{i=1}^n J(T_i, T_{i-1}) - \sum_t^s J| < \varepsilon$. On the other hand, the definition of the product integral of J on $[b, a] - \prod_t^s J$, for $s \geqslant t \in [b, a]$–with the convention that $\prod_{i=1}^n J(T_i, T_{i-1}) = J(T_n, T_{n-1})J(T_{n-1}, T_{n-2}) \cdots J(T_1, T_0)$, is obtained by replacing the summation sign Σ above with the product symbol Π.

INDICATION OF PROOF. We shall have repeated occasion to make use of the operator identity

$$\prod_{i=1}^n A_i - \prod_{i=1}^n B_i = \sum_{i=1}^n \prod_{j=i+1}^n B_j [A_i - B_i] \prod_{j=1}^{i-1} A_j. \tag{$*$}$$

Now, given $a > b$ and $V \in \mathcal{J}_N$ we first show that $\prod_t^s [I - V]$ exists for all $s > t$ in $[b, a]$. In fact, if $\sigma = \{\sigma_i\}_{i=1}^n$ is a partition of $[t, s]$ then, employing ($*$) with

$B_i \equiv I$ gives

$$\left\| \prod_{i=1}^{n} \left[I - V(\sigma_i, \sigma_{i-1}) \right] - \left[I - V(s, t) \right] \right\|$$
$$= \left\| - \sum_{i=1}^{n} V(\sigma_i, \sigma_{i-1}) \prod_{j=1}^{i-1} \left[I - V(\sigma_j, \sigma_{j-1}) \right] + \sum_{i=1}^{n} V(\sigma_i, \sigma_{i-1}) \right\|$$
$$= \left\| \sum_{i=1}^{n} V(\sigma_i, \sigma_{i-1}) \sum_{j=1}^{i-1} V(\sigma_j, \sigma_{j-1}) \prod_{k=1}^{j-1} \left[I - V(\sigma_k, \sigma_{k-1}) \right] \right\|$$
$$\leqslant \sum_{i=1}^{n} \left[g(\sigma_i) - g(\sigma_{i-1}) \right] \sum_{j=1}^{i-1} \left[g(\sigma_j) - g(\sigma_{j-1}) \right] \prod_{k=1}^{j-1} \left[1 + g(\sigma_k) - g(\sigma_{k-1}) \right]$$
$$\leqslant \prod_{i=1}^{n} \left[1 + g(\sigma_i) - g(\sigma_{i-1}) \right] - \left[1 + g(s) - g(t) \right],$$

where g is an increasing function which dominates V on $[b, a]$. Moreover, as is evident from the proof of Lemma 1, g can also be chosen to be continuous. We thus conclude, via relation $(*)$, that if $\tau = \{\tau_i\}_{i=0}^{n}$ is a refinement of σ then

$$\left\| \prod_{i=1}^{m} \left[I - V(\tau_i, \tau_{i-1}) \right] - \prod_{i=1}^{n} \left[I - V(\sigma_i, \sigma_{i-1}) \right] \right\|$$
$$\leqslant \prod_{i=1}^{m} \left[1 + g(\tau_i) - g(\tau_{i-1}) \right] - \prod_{i=1}^{n} \left[1 + g(\sigma_i) - g(\sigma_{i-1}) \right].$$

However, as is easily shown, if $dg(\xi, \zeta)$ is defined to be $g(\xi) - g(\zeta)$ then $\prod_t^s [1 + dg]$ exists and equals $e^{g(s) - g(t)}$. This then implies that $\prod_t^s [I - V]$ exists and is continuous for all $s > t$ in $[b, a]$.

We show next that $\prod_t^s [I + V]^{-1}$ also exists on $[b, a]$ and, in fact, equals $\prod_t^s [I - V]$. For, if $\sigma = \{\sigma_i\}_{i=0}^{n}$ is a partition of $[t, s]$ then by identity $(*)$ and the accretiveness of V we obtain

$$\left\| \prod_{i=1}^{n} \left[I + V(\sigma_i, \sigma_{i-1}) \right]^{-1} - \prod_{i=1}^{n} \left[I - V(\sigma_i, \sigma_{i-1}) \right] \right\|$$
$$= \left\| \sum_{i=1}^{n} \left(\prod_{j=i+1}^{n} \left[I - V(\sigma_j, \sigma_{j-1}) \right] \right) \left[V(\sigma_i, \sigma_{i-1}) \right]^2 \prod_{j=1}^{i} \left[I + V(\sigma_j, \sigma_{j-1}) \right]^{-1} \right\|$$
$$\leqslant \sum_{i=1}^{n} \prod_{j=i+1}^{n} \left[1 + g(\sigma_j) - g(\sigma_{j-1}) \right] \left[g(\sigma_i) - g(\sigma_{i-1}) \right]^2$$
$$\leqslant e^{(g(s) - g(t))} \sum_{i=1}^{n} \left[g(\sigma_i) - g(\sigma_{i-1}) \right]^2$$
$$\leqslant e^{g(s) - g(t)} (g(s) - g(t)) \max_{1 < i < n} \left[g(\sigma_i) - g(\sigma_{i-1}) \right],$$

which goes to zero as σ becomes a finer and finer partition of $[t, s]$.

Next, for $U \in \mathcal{H}_N$ we show that $\Sigma_t^s[I - U]$ exists and is continuous on $[b, a]$. Given $s > t$, let $\sigma = \{\sigma_i\}_{i=0}^n$ be a partition of $[t, s]$. Then

$$\begin{aligned}\left\| \sum_{i=1}^{n} \left[I - U(\sigma_i, \sigma_{i-1}) \right] - \left[I - U(s, t) \right] \right\| \\ &= \left\| \sum_{i=1}^{n} \left[I - U(\sigma_i, \sigma_{i-1}) \right] \left[I - U(\sigma_{i-1}, t) \right] \right\| \\ &\leqslant \sum_{i=1}^{n} \left[h(\sigma_i) - h(\sigma_{i-1}) \right] \left[h(\sigma_{i-1}) - h(t) \right] \\ &= \sum_{i=1}^{n} h(\sigma_{i-1}) \left[h(\sigma_i) - h(\sigma_{i-1}) \right] - h(t) \left[h(s) - h(t) \right],\end{aligned}$$

where h is an increasing function which dominates $I - U$ on $[b, a]$, and, as is clear from the proof of Lemma 2, can be taken to be continuous. Therefore, if $\tau = \{\tau_i\}_{i=1}^m$ is a refinement of σ then

$$\begin{aligned}\left\| \sum_{i=1}^{m} \left[I - U(\tau_i, \tau_{i-1}) \right] - \sum_{i=1}^{n} \left[I - U(\sigma_i, \sigma_{i-1}) \right] \right\| \\ &\leqslant \sum_{i=1}^{m} h(\tau_{i-1}) \left[h(\tau_i) - h(\tau_{i-1}) \right] - \sum_{i=1}^{n} h(\sigma_{i-1}) \left[h(\sigma_i) - h(\sigma_{i-1}) \right]\end{aligned}$$

whence it follows that $\Sigma_t^s[I - U]$ exists and is continuous on $[b, a]$ and for all $s > t \in [b, a]$

$$\left\| \sum_t^s \left[I - U \right] - \left[I - U(s, t) \right] \right\| \leqslant \int_t^s h \, dh - h(t) \left[h(s) - h(t) \right]. \qquad (**)$$

Q.E.D.

We now provide the

PROOF OF THEOREM I. Given $V \in \mathcal{G}_N$, first observe that $\mathcal{E}(V) = \Pi[I + V]^{-1}$ is in $\mathcal{H}_N$, since the evolution property (i) and the contraction property (iii) are inherited directly from the corresponding properties on the finite product approximations to the product integral, while the continuity property (ii) was demonstrated in Lemma 4. Likewise, if $U \in \mathcal{H}_N$, we may show $\Sigma[I - U] \in \mathcal{G}_N$. Note, first of all that the generator property (iv) and the accretive property (vi) are inherited from the corresponding properties on the finite sum approximations to the sum integral. Specifically, with respect to (vi), given $s > t$, if $\{\sigma_i\}_{i=0}^n$ is a partition of $[t, s]$ and $x \in X$ then

$$\left\| \left(\sum_{i=1}^{n} \left[I - U(\sigma_i, \sigma_{i-1}) \right] + I \right) x \right\| \geqslant (n + 1)\|x\| - n\|x\| = \|x\|.$$

As for the continuity property (v), this follows from Lemma 4.

In order to verify the equivalence of conditions (i) and (ii) in Theorem I and thereby establish $\mathcal{E}$ as a bijection with $\mathcal{E}^{-1}(U) = \Sigma[I - U]$, we introduce a third condition (Δ), taken from relation (**) of Lemma 4, which we claim is equivalent to both condition (i) and to condition (ii). Namely, for $(V, U) \in \mathcal{G}_N \times \mathcal{H}_N$ and $a > b$

there is a continuous, increasing function h on $[b, a]$ such that for all $s > t$ in $[b, a]$

$$\|V(s, t) - I + U(s, t)\| \leqslant \int_t^s h\, dh - h(t)[h(s) - h(t)]. \tag{Δ}$$

Clearly, relation $(**)$ of Lemma 4 gives us that condition (ii) of Theorem I implies condition (Δ). Lemma 4 also provides us with the result $\prod[I + V]^{-1} = \prod[I - V]$, for $V \in \mathcal{J}_N$, which shall be useful in establishing succeeding implications.

Suppose condition (i) holds. Given $s > t$ and $\{\sigma_i\}_{i=1}^n$ a partition of $[t, s]$ we proceed to show that condition (Δ) holds.

$$\begin{aligned}
&\left\| V(s, t) - I + \prod_{i=1}^n [I - V(\sigma_i, \sigma_{i-1})] \right\| \\
&\quad = \left\| \sum_{i=1}^n V(\sigma_i, \sigma_{i-1}) \left(I - \prod_{j=1}^{i-1} [I - V(\sigma_j, \sigma_{j-1})] \right) \right\| \\
&\quad = \left\| \sum_{i=1}^n V(\sigma_i, \sigma_{i-1}) \sum_{j=1}^{i-1} V(\sigma_j, \sigma_{j-1}) \prod_{k=1}^{j-1} [I - V(\sigma_k, \sigma_{k-1})] \right\| \\
&\quad \leqslant e^{[g(a)-g(b)]} \left\{ \sum_{i=1}^n g(\sigma_{i-1})[g(\sigma_i) - g(\sigma_{i-1})] - g(t)[g(s) - g(t)] \right\},
\end{aligned}$$

where g is a continuous, increasing function which dominates V. Thus (Δ) holds with $h = e^{[g(a)-g(b)]}g$.

Next, assuming condition (Δ), if $(V, U) \in \mathcal{J}_N \times \mathcal{H}_N$ and $\{\sigma_i\}_{i=0}^n$ is a partition of an interval $[t, s]$ then for some continuous, increasing function we have

$$\begin{aligned}
\left\| V(s, t) - \sum_{i=1}^n [I - U(\sigma_i, \sigma_{i-1})] \right\| &\leqslant \sum_{i=1}^n \|V(\sigma_i, \sigma_{i-1}) - I + U(\sigma_i, \sigma_{i-1})\| \\
&\leqslant \int_t^s h\, dh - \sum_{i=1}^n h(\sigma_{i-1})[h(\sigma_i) - h(\sigma_{i-1})]
\end{aligned}$$

whence, $V(s, t) = \sum_t^s [I - U]$, and thus condition (Δ) implies condition (ii).

Finally, we show that (ii) $\to$ (i). Since we have seen that (ii) $\to (\Delta)$, there exists a continuous, increasing function h on $[b, a]$ such that given $s > t$ and $\{\sigma_i\}_{i=0}^n$ a partition $[t, s]$,

$$\begin{aligned}
&\left\| \prod_{i=1}^n [I - V(\sigma_i, \sigma_{i-1})] - U(t, s) \right\| \\
&\quad = \left\| \sum_{i=1}^n \prod_{j=i+1}^n U(\sigma_j, \sigma_{j-1})[I - V(\sigma_i, \sigma_{i-1}) - U(\sigma_i, \sigma_{i-1})] \prod_{j=1}^{i-1} [I - V(\sigma_j, \sigma_{j-1})] \right\| \\
&\quad < e^{[g(a)-g(b)]} \left\{ \int_t^s h\, dh - \sum_{i=1}^n h(\sigma_{i-1})[h(\sigma_i) - h(\sigma_{i-1})] \right\},
\end{aligned}$$

where $U \in \mathcal{H}_N$, $V = \sum[I - U]$ and g dominates V on $[b, a]$. Q.E.D.

Corollary I. *If $V \in \mathcal{J}_N$ then the partial derivatives of $V(\xi, \zeta)$ exist a.e.*

PROOF. We write $V(\xi, \zeta) = [V_{ij}(\xi, \zeta)]_{1 \leq i,j \leq N}$. Now given $a > b$, $V_{ij}(\xi, \zeta)$ can be represented on $[b, a]$ as $V_{ij}(\xi, \zeta) = g(\xi) - g(\zeta)$ where $g(\cdot) = V_{ij}(\cdot, b)$ is of bounded variation, hence is differentiable a.e., on $[b, a]$. Q.E.D.

COROLLARY II. *If $U \in \mathcal{H}_N$ then $\partial U(\xi_1, \xi_2)/\partial \xi_i$ exists a.e., $i = 1, 2$, and satisfies*

$$\frac{\partial U(\xi_1, \xi_2)}{\partial \xi_i} = \begin{cases} \dfrac{-\partial V(\xi_1, \xi_2)}{\partial \xi_1} U(\xi_1, \xi_2) \text{ a.e.}, & i = 1, \\ -U(\xi_1, \xi_2) \dfrac{\partial V(\xi_1, \xi_2)}{\partial \xi_2} \text{ a.e.}, & i = 2, \end{cases} \tag{∇}$$

where $V \in \mathcal{J}_N$ is the generator for U.

PROOF. By the identities

$$\frac{U(\xi + \delta, \zeta) - U(\xi, \zeta)}{\delta} = \frac{(U(\xi + \delta, \xi) - I)}{\delta} U(\xi, \zeta)$$

and

$$\frac{U(\xi, \zeta + \delta) - U(\xi, \zeta)}{\delta} = U(\xi, \zeta + \delta) \frac{(I - U(\zeta + \delta, \zeta))}{\delta},$$

which hold for all $\xi > \zeta$ and $\delta \in (0, \xi - \zeta)$, it will suffice to establish for all $\xi > \zeta$

$$\lim_{\delta \downarrow 0} \frac{U(\xi + \delta, \xi) - I}{\delta} = \frac{-\partial^+ V(\xi, \zeta)}{\partial \xi} \quad \text{a.e.}, \tag{1}$$

$$\lim_{\delta \downarrow 0} \frac{I - U(\zeta + \delta, \zeta)}{\delta} = \frac{-\partial^+ V(\xi, \zeta)}{\partial \zeta} \quad \text{a.e.} \tag{2}$$

in order to prove the corollary holds at least with right-hand derivatives appearing in (∇).

With respect to (1), given $\xi \in R$ and $\delta > 0$ let $\sigma = \{\sigma_i\}_{i=1}^n$ be a partition of $[\xi, \xi + \delta]$ and let g be an increasing function which dominates V on $[\xi, \xi + \delta]$. Then, for all $\zeta < \xi$ we have

$$\begin{aligned} &\left\| \frac{\prod_{i=1}^n [I - V(\sigma_i, \sigma_{i-1})] - I}{\delta} + \frac{V(\xi + \delta, \zeta) - V(\xi, \xi)}{\delta} \right\| \\ &\quad \leq \frac{1}{\delta} \sum_{i=1}^n g(\sigma_i) - g(\sigma_{i-1}) \max_{1 \leq i \leq n} \left\| I - \prod_{j=1}^{i-1} [I - V(\sigma_j, \sigma_{j-1})] \right\| \\ &\quad \leq \frac{g(\xi + \delta) - g(\xi)}{\delta} \max_{1 \leq i \leq n} \| I - U(\sigma_{i-1}, \xi) \| + e_\delta \\ &\quad \leq \frac{g(\xi + \delta) - g(\xi)}{\delta} \max_{\rho \in [\xi, \xi + \delta]} \| I - U(\rho, \xi) \| + e_\delta, \end{aligned}$$

where $\lim_{\delta \downarrow 0} e_\delta = 0$ uniformly in n. By first letting the partition σ become finer and finer (i.e. $n \to \infty$), and then letting $\delta \downarrow 0$, since g is differentiable a.e., we conclude (1) holds true. Similar arguments give equation (2), and the completion of the proof of (∇) now continues along these same lines.

REMARKS. (1) The preceding corollary is decidedly not true in the case $X = C^N$. In fact, with $N = 1$, let $U(\xi, \zeta) = \exp(i(g(\xi) - g(\zeta)))$, where g is a continuous, nowhere differentiable, real-valued function. Then U is a norm-continuous, contractive evolution on X, but clearly $U(\xi, \zeta)$ has no partial derivatives.

(2) The results of this paper do not necessarily hold if the sup-norm, used throughout in conjunction with the contractive and accretive conditions, is replaced by an arbitrary norm on R^N. Consider, for example,

$$U(\xi, \zeta) = \begin{bmatrix} \cos(g(\xi) - g(\zeta)) & \sin(g(\xi) - g(\zeta)) \\ -\sin(g(\xi) - g(\zeta)) & \cos(g(\xi) - g(\zeta)) \end{bmatrix}$$

acting in Euclidean R^2, where g is as in (1) above. It is easily verified that U is an evolution and that for each $s > t$, $U(s, t)$ is unitary. Therefore, U is a Euclidean norm-continuous, contractive evolution for which Corollary II fails. In fact, it is possible here for g to be oscillating so rapidly that even $\Sigma_t^s[I - U]$ does not exist, thus also ruling out Theorem I. A sufficient condition for Theorem I to hold true for the present example is $\int_t^s (dg)^2 = 0$, in which case $V(s, t) = \Sigma_t^s[I - U]$ can be shown equal to $(g(s) - g(t))\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.

(3) In [6], Y. Kōmura defines the generator $A(t)$ of a strongly continuous, contractive evolution U set in an arbitrary real reflexive Banach space E to be the strong limit of $h^{-1}[U(t + h, t) - I]$ as $h \to 0^+$. He is then able to characterize those evolutions U which have densely defined generators $A(t)$ such that for x in an appropriate subset of E, $A(\cdot)x$ is strongly measurable and $\|A(\cdot)x\|$ is monotone nonincreasing. While Kōmura's results hold in a much more general setting than that of the present paper, his class of evolutions is, however, very restrictive. For example, those strongly continuous, contractive evolutions given by $U(\xi, \zeta) = e^{-(f(\xi) - f(\zeta))I}$ where $f', f'' > 0$, would not be included since U has generator $A(\xi) = f'(\xi)I$ and therefore $\|A(\cdot)x\|$ is monotone increasing. From the perspective of the present paper, [6] is grounded too deeply in semigroup theory (consider, for example, the choice of definition for the generator of an evolution) and does not attempt to exploit the duality between multiplicative and additive functions via the powerful tools of product and sum integrals.

References

1. J. V. Herod, *A product integral representation for an evolution system*, Proc. Amer. Math. Soc. **27** (1971), 549–556.

2. ______, *A pairing of a class of evolution systems with a class of generators*, Trans. Amer. Math. Soc. **157** (1971), 247–260.

3. ______, *Generators for evolution systems with quasicontinuous trajectories*, Pacific J. Math. **53** (1974), 153–161.

4. J. V. Herod and R. W. McKelvey, *A Hille-Yosida theory for evolutions*, Israel J. Math. **36** (1980), 13–40.

5. E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

6. Y. Kōmura, *On linear evolution operators in reflexive Banach spaces*, J. Math. Soc. Japan **17** (1970), 529–542.

7. J. S. Mac Nerney, *Integral equations and semigroups*, Illinois J. Math. **7** (1963), 148–173.

8. ______, *A linear initial-value problem*, Bull. Amer. Math. Soc. **69** (1963), 314–329.

9. K. Yosida, *Functional analysis*, Springer-Verlag, New York, 1978.

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