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TIME-ORDERED OPERATORS. I. FOUNDATIONS FOR AN ALTERNATE VIEW OF REALITY

BY

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ABSTRACT. The purpose of this paper is to present the proper framework for the mathematical foundations of time-ordered operators. We introduce a new mathematical process which we call the chronological process. This process generalizes the notion of a limit and allows us to recapture many properties lost in time-ordering. We then construct time-ordered integrals and evolution operators. We show that under reasonable assumptions, the time-ordered sum of two generators of contraction semigroups is a generator. This result resolves a question that has been debated in physics for forty years.

Introduction. In a series of papers [F1], [F2], [F3], R. P. Feynman developed a new approach to quantum electrodynamics. The idea was to deal directly with the solutions to the equations describing the system rather than the equations themselves. This approach requires an overall space-time point of view. This means that we view a physical event as occurring on a film in which we become more aware of the outcome as more of the film comes into view. This led Feynman to consider time histories (paths) as the primary object of physical concern. He [F4] noticed that altering the manner in which one views operators permits considerable ease in the manipulation of complex expressions. The method developed by Feynman is called time-ordering because time is used as an index to determine the order of operators in a product. At the present time this notion is used in every branch of physics.

For the most part this method has not been taken seriously by mathematicians (see, however, Segal [SE]). This is the first in a series of papers in which we develop a mathematical theory of time-ordered operators. *From a mathematical point of view, Feynman's time-ordering is equivalent to the assumption that time stands still and physical processes flow through it from past to present to future*. This is the alternate view.

In order to construct time-ordered operators, there are two threads that must be brought together; the film on which space-time events can occur and a method which will directly allow time to act as an index identifying the position of operators on the film as they create the path. §1 is devoted to the construction of the film. In §2 we define and study time-ordered operators. We introduce the

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notion of a chronological process and show that it generalizes the notion of a limit. In §3 we identify some fundamental ideas from the theory of semigroups of operators so that we can develop our theory of time-ordered evolution operators in §4. In §4 we show that the time-ordered sum of two generators of contraction semigroups of operators is a generator except on a set of measure zero, assuming a common domain. This result implies that the cause for the divergencies in quantum electrodynamics cannot be attributed to the mathematics and, hence, there is some underlying physical cause. This question has been tossed about for over forty years.

The appendix is devoted to the extension of the Riemann-complete integral of Henstock [**H**] and Kurzweil [**K**] to the operator-valued case. This integral is used in §4. It should be pointed out that in applications the time-ordered integrals are not absolutely integrable even in the bounded operator case. The Riemann complete integral includes the Bochner integral as a special case when it exists.

1. Let $[a, b]$ be a compact set of reals, and for each $t \in [a, b]$ let $B(t)$ be a separable Banach space. Suppose $\{\phi_t\} \in \prod_t B(t)$ and denote by Δ, those sequences $\{\phi_t\}$ such that

$$\sum_t \big| \|\phi_t\| - 1\big| < \infty. \tag{1.1}$$

(We do not use the same requirement as in [**G1**] but see Lemma 3.5 of that paper.) For each $t \in [a,b]$ let $B^*(t)$ be the dual space of $B(t)$ and let Δ^* be the set of all sequences $\{u_t\}$ in $\prod_t B^*(t)$ such that

$$\sum_t \big| \|u_t\|^* - 1\big| < \infty. \tag{1.2}$$

If $\{z_t\}$ is a sequence of complex numbers, we define convergence of $\prod_t z_t$ in the normal way and set $\prod_t z_t = 0$ if it does not converge. Let us define a functional on Δ^* by

$$\Gamma(u) = \sum_{v=1}^{n} \prod_t \langle \phi_t^v, u_t \rangle \tag{1.3}$$

where $u = \{u_t\} \in \Delta^*$, $\{\phi_t^v\} \in \Delta$ for $v = 1, 2, \ldots, n$, and $\langle\ ,\ \rangle$ is the duality bracket. The t-index for norms and duality brackets has been suppressed and should cause no confusion. We denote Γ by

$$\Gamma = \sum_{j=1}^{n} \otimes \phi_t^v \tag{1.4}$$

and define the algebraic tensor product, $\otimes_t B(t)$, by

$$\bigotimes_t B(t) = \left\{ \sum_{v=1}^{n} \bigotimes_t \phi_t^v \,|\, \{\phi_t^v\} \in \Delta, v = 1, 2, \ldots, n \right\}. \tag{1.5}$$

We define two important norms on $\otimes_t B(t)$; let $\|\cdot\|_\lambda$ and $\|\cdot\|_\gamma$ be defined for $\phi = \sum_{v=1}^{n} \otimes_t \phi_t^v$, by

$$\|\phi\|_\lambda = \sup\left\{ \sum_{v=1}^{n} \prod_t |\langle \phi_t^v, u_t\rangle| \,|\, u_t \in B^*(t), \|u_t\| \leq 1 \right\}, \tag{1.6}$$

$$\|\phi\|_\gamma = \inf\left\{ \sum_{k=1}^{m} \prod_t \|\psi_t^k\| \,\Big|\, \sum_{v=1}^{m} \bigotimes_t \phi_t^v = \sum_{k=1}^{n} \bigotimes_t \psi_t^k \right\}. \tag{1.7}$$

THEOREM 1.1. (1) $\|\cdot\|_\lambda$ *and* $\|\cdot\|_\gamma$ *are norms and* $\|\cdot\|_\lambda \leq \|\cdot\|_\gamma$.
(2) *If* $\phi = \otimes_t \phi_t$ *then* $\|\phi\|_\lambda = \|\phi\| = \Pi_t \|\phi_t\|$.

Condition (2) is the definition of the notion of a crossnorm; $\|\cdot\|_\lambda$ is called the least crossnorm and $\|\cdot\|_\gamma$ is known as the greatest crossnorm. (These ideas are due to Shatten [S].) Let us denote them by λ and γ, respectively.

Let us denote by $\otimes_t^\alpha B(t)$ the completion of $\otimes_t B(t)$ with respect to any α with $\lambda \leq \alpha \leq \gamma$. ($\alpha$ is a crossnorm by Theorem 1.1(2)).

DEFINITION 1.2. Suppose $u = \otimes_t u_t$ is in $\otimes_t B^*(t)$; set

$$\|u\|_{\alpha^*} = \sup_{\Sigma_{v=1}^n \otimes_t \phi_t^v} \left(\sum_{v=1}^n \prod_t \langle \phi_t^v, u_t \rangle \Big/ \left\| \sum_{v=1}^n \otimes_t \phi_t^v \right\|_\alpha \right). \tag{1.8}$$

We call α^* the dual norm of α.

THEOREM 1.2. *Suppose* α *is a crossnorm on* $\otimes_t B(t)$ *so that* $\lambda \leq \alpha \leq \gamma$; *then*:
(1) $\gamma^* \leq \alpha^* \leq \lambda^*$;
(2) α^* *is a crossnorm on* $\otimes_t B^*(t)$.

Any crossnorm α whose dual norm α^* is also a crossnorm is known as reasonable.

DEFINITION 1.3. A crossnorm α on $\otimes_t B(t)$ is called uniform if

$$\sup_{\|\phi\|_\alpha \leq 1} \left\{ \left\| \left[\otimes_t T(t) \right] \phi \right\|_\alpha \right\} \leq \prod_t \|T(t)\|$$

where $\{T(t)\}$ is a family of bounded linear transformations with $\Pi_t \|T(t)\| < \infty$.

LEMMA 1.1. λ *and* γ *are uniform.*

DEFINITION 1.4. A reasonable norm α on $\otimes_t B(t)$ is called faithful if the linear map J_λ^α, obtained by extending the identity on $\otimes_t^\alpha B(t)$ to the entire space $\otimes_t^\lambda B(t)$ by continuity, is one-to-one.

This means that if an element ϕ in $\otimes_t^\alpha B(t)$ vanishes on $\otimes_t B^*(t)$, then it is the zero element.

DEFINITION 1.5. Let α be any reasonable crossnorm on $\otimes_t B(t)$. We call the closure of $\otimes_t B(t)$ with respect to α a Banach space of type v and denote it by $\otimes_t^\alpha B(t)$.

From the point of view of time-ordering we consider $\otimes_t^\alpha B(t)$ the set of all possible film for events that occur over the time interval $[a, b]$. This set is too large for almost all applications. The problem is related to the fact that systems of interest are governed by evolution equations subject to initial conditions.

DEFINITION 1.6. Let $\phi = \otimes_t \phi_t$ and $\psi = \otimes_t \psi_t$ be vectors in $\otimes_t^\alpha B(t)$; we say ϕ is equivalent to ψ and write $\phi \equiv \psi$ provided $\Sigma_t \|\psi_t - \phi_t\|^2 < \infty$ and $\Sigma_t \operatorname{Im}\langle \psi_t, J_t \phi_t \rangle < \infty$ where $\{J_t\}$ is a joint family of duality maps for both $\{\psi_t\}$ and $\{\phi_t\}$.

THEOREM 1.3. *The relationship defined above is an equivalence relation which decomposes* $\otimes_t^\alpha B(t)$ *into disjoint equivalence classes.*

PROOF. (See Theorem 4.2(2), (3) of [G1].)

Let $\phi \otimes_t^\alpha B(t)$ be the closure of the linear span of all $\psi \equiv \phi$.

THEOREM 1.4. *$\otimes_t^\alpha B(t)$ is a closed subspace such that, if $\phi \not\equiv \psi$, then*

$$\phi \overset{\alpha}{\underset{t}{\otimes}} B(t) \cap \psi \overset{\alpha}{\underset{t}{\otimes}} B(t) = \{\theta\}$$

where θ is the additive identity on $\otimes_t^\alpha B(t)$. If we restrict t to lie in a countable dense subset of $[a, b]$ then $\phi \otimes_t^\alpha B(t)$ is a separable Banach space.

LEMMA 1.2. *Suppose $\|\phi_t\| = 1$ for all t; then $\phi \otimes_t^\alpha B(t)$ is the closure of the linear span of all $\psi = \otimes_t \psi_t$ such that $\psi_t \neq \phi_t$ occurs for at most a finite number of t.*

DEFINITION 1.7. We call $\phi \otimes_t^\alpha B(t)$ the set of film for all possible space-time realities of *classical type* associated with the family $\{B(t)\}$ and pre-prepared to be in state ϕ on $[a, b]$.

Consider the Schrödinger type evolution equation where $H(t)$ is the selfadjoint generator of a unitary group:

$$i\partial x(t)/\partial t = H(t)x(t), \qquad x(0) = x_0; \tag{1.9}$$

if y is a solution then $e^{i\theta}y$ is also a solution for all real θ. This phenomenon forces us to consider another class of subspaces of $\otimes_t^\alpha B(t)$.

DEFINITION 1.8. $\phi = \otimes_t \phi_t$ is said to be weakly equivalent to $\psi = \otimes_t \psi_t$ if there exists a set $\{z_t\}$ of complex numbers with $|z_t| = 1$ such that $\otimes_t \phi_t \equiv \otimes_t z_t\psi_t$. In this case we write $\phi \equiv^w \psi$.

THEOREM 1.5. *$\phi \equiv^w \psi$ is an equivalence relation which decomposes $\otimes_t^\alpha B(t)$ into disjoint equivalence classes. If we denote by $\phi_w \otimes_t^\alpha B(t)$ the closure of the linear span of all $\psi \equiv^w \phi$, then:*

(1) *$\phi_w \otimes_t^\alpha B(t)$ is a proper closed subspace of $\otimes_t^\alpha B(t)$, and if $\psi \not\equiv^w \phi$, then*

$$\phi_w \overset{\alpha}{\underset{t}{\otimes}} B(t) \cap \psi_w \overset{\alpha}{\underset{t}{\otimes}} B(t) = \{\theta\}.$$

(2)

$$\phi_w \overset{\alpha}{\underset{t}{\otimes}} B(t) = \bigoplus_{\psi \equiv^w \phi} \left[\psi \overset{\alpha}{\underset{t}{\otimes}} B(t)\right] \quad (\textit{direct sum}).$$

DEFINITION 1.9. We call $\phi_w \otimes_t^\alpha B(t)$ the set of film for all possible space-time realities of *modern type* associated with the family $\{B(t)\}$ and pre-prepared to be in state ϕ on $[a, b]$.

2. Time-ordered operators. In this section we introduce a number of new concepts and terms. In order to motivate our approach, let us consider a bounded linear map $\tilde{A}(t)$ from $[a, b]$ into $L(B)$, where $L(B)$ is the space of bounded linear operators on B. We assume B is a fixed Banach space (separable). Define a family $\{A(t) | t \in [a, b]\}$ on $\otimes_t^\alpha B(t)$ ($B(t) = B$ for each t) by

$$A(t) = \bigotimes_{a<s<t} I_s \otimes \tilde{A}(t) \otimes \Big(\bigotimes_{t<s<b} I_s\Big). \tag{2.0}$$

It is easy to see that for $t \neq t'$, $A(t)A(t') = A(t')A(t)$. In this way time can act as a place keeper in any product of operators. Let us suppose that $\tilde{A}(t)$ is continuous so that $\lim_{t \to t_0} \|\tilde{A}(t) - \tilde{A}(t_0)\| = 0$. If we attempt to compute the same limit with $\|A(t) - A(t_0)\|_\alpha$, we find that it is meaningless, that is, the notion of limit in this setting does not exist. This is the price we must pay for time-ordered operators.

In order to remedy this situation we shall introduce a new process which includes the notion of a limit as a special case. This process does not lead to an extended set of operators; it simply allows us access to operators on $L[\otimes_t^\alpha B(t)]$ which are not accessible by algebraic or topological processes. von Neumann **[VN]** showed that the set ($H(t)$ is a Hilbert space) $L^\#[\otimes_t H(t)]$ of bounded operators in $L[\otimes_t H(t)]$ that are accessible by algebraic and topological processes on $\{\overline{L}(H(t)) | t \in [a, b]\}$ is a proper subspace of $L[\otimes_t H(t)]$, where $\overline{L}[H(t)]$ is the natural injection of $L[H(t)]$ into $L[\otimes_t H(t)]$ defined by (2.0).

In what follows we assume that for any pair $(t, t') \in [a, b]$ there is a bijective continuous isometric linear map $\Gamma_t^{t'}: B(t) \to B(t')$ with a continuous inverse which we denote by $\Gamma_{t'}^t$. In most cases $B(t) = B$ for all t so that there is no problem. The cases in which $B(t)$ is not fixed usually occur in relativistic quantum mechanics, but then each $B(t)$ is a separable Hilbert space.

Let $\mathfrak{S}[\otimes_s^\alpha B(s)]$ be the set of closable operators on $\otimes_s^\alpha B(s)$ and let $\mathfrak{S}[B(t)] \to \mathfrak{S}[\otimes_s^\alpha B(s)]$ be those closable operators on $B(t)$ which exist in $\mathfrak{S}[\otimes_s^\alpha B(s)]$ under the natural injection (2.0).

If we restrict $\mathfrak{S}[\otimes_s^\alpha B(s)]$ to the linear operators, then it is a metric space that is not complete (cf. Kato **[K]**). This space is not linear since the sum of two closable operators need not be defined, and if the sum is defined it need not be closable.

Definition 2.1. An exchange operator $E[t, t']$ is a linear operator defined on $\mathfrak{S}[\otimes_s^\alpha B(s)]$ for pairs $t, t' \in [a, b]$ such that:

(1) $E[t, t']$ maps $\mathfrak{S}[B(t')]$ onto $\mathfrak{S}[B(t)]$;

(2) $E[t, s]E[s, t'] = E[t, t']$;

(3) $E[t, t']E[t', t] = I$;

if $s \neq t, t'$, then $E[t, t']A(s) = A(s)$, $A(s) \in \mathfrak{S}[B(s)]$.

It should be noted that $E[t, t']$ is linear in the sense that whenever the sum of two closable operators is defined and closable then $E[t, t']$ maps in the appropriate manner.

In this section we will restrict our consideration to $L[\otimes_t^\alpha B(t)]$. We shall define all concepts in terms of the uniform operator topology. Unless otherwise noted one may replace uniform by strong or weak to obtain the same concept.

Definition 2.2. A family $\{A(t) | t \in [a, b]\}$ of operators in $L[\otimes_t^\alpha B(t)]$ is said to be *chronologically continuous* or c-continuous in the uniform operator topology at t' if there is an exchange operator $E(t, t')$ such that

$$\lim_{t \to t'} \|E[t', t]A(t) - A(t')\|_\alpha = 0 \tag{2.1}$$

(where α is any fixed uniform faithful crossnorm such that $\lambda < \alpha < \gamma$).

Definition 2.3. The family $\{A(t) | t \in [a, b]\}$ is said to be *chronologically differentiable* or c-differentiable at t' if there is an operator $DA(t')$ and an exchange operator $E[t', t]$ such that

$$\lim_{t \to t'} \left\| \frac{E[t', t]A(t) - A(t')}{t - t'} - DA(t') \right\|_\alpha = 0. \tag{2.2}$$

DEFINITION 2.4. The family $\{A(t) | t \in [a, b]\}$ is said to be of *chronological bounded variation* or of c-bounded variation if

$$\sup_P \sum_{j=1}^{n} \| A(t_{j+1}) - E[t_{j+1}, t_j] A(t_j) \|_\alpha < \infty, \tag{2.3}$$

where the sup is over all finite partitions P of $[a, b]$.

It is an open question as to the number of exchange operators up to equivalences; our definitions require only one. Since the identity operator is an exchange operator we have the following:

THEOREM 2.1. *If the family* $\{A(t) | t \in [a, b]\}$ *is*:
(1) *continuous, it is c-continuous*;
(2) *differentiable, it is c-differentiable*;
(3) *of bounded variation, it is of c-bounded variation.*

PROOF. Set $E[t, t'] = I$ for all $t, t' \in [a, b]$; note that in (1) and (2) the limits are also the same.

From the above theorem we see that the chronological process includes the limit process as a special case. We shall now construct the *standard exchange operator*.

Let us define a map $G(t;\, t)$ from $\otimes_t^\alpha B(t)$ to an isomorphic copy of itself by (assume $t' < t$)

(Comparison operator)

$$G[t', t] \bigotimes_t \phi_t = \left[\bigotimes_{a < s < t'} \phi_s \otimes \Gamma_t^{t'} \phi_t \otimes \left(\bigotimes_{t' < s < t} \phi_s \right) \otimes \Gamma_{t'}^{t} \phi_{t'} \otimes \left(\bigotimes_{t' < s < b} \phi_s \right) \right], \tag{2.4}$$

so that $G[t', t]$ exchanges the positions of $B(t)$ and $B(t')$.

Before continuing our analysis we generalize a few lemmas due to Ichinose [I1]–[I3]. The proofs are the same so we omit them.

LEMMA 2.1 [I1]. *Let* $\alpha > \lambda$ *be a uniform reasonable norm on* $\otimes_t B(t)$. *If* $Q(t) \in \mathfrak{L}[B(t)]$ *for each* t *and* $\| \otimes_t Q(t) \| = \Pi_t \| Q(t) \| < \infty$, *then the diagram*

$$\begin{array}{ccc} \otimes_t^\alpha B(t) & \xrightarrow{\otimes_t^\alpha Q(t)} & \otimes_t^\alpha B(t) \\ J_\lambda^\alpha \downarrow & & \downarrow J_\lambda^\alpha \\ \otimes_t^\lambda B(t) & \xrightarrow{\otimes_t^\lambda Q(t)} & \otimes_t^\lambda B(t) \end{array}$$

is commutative, where $\otimes_t^\alpha Q(t)$ *and* $\otimes_t^\lambda Q(t)$ *are the continuous extensions of* $\otimes_t Q(t)$ *to the spaces* $\otimes_t^\alpha B(t)$, $\otimes_t^\lambda B(t)$, *respectively, and* J_λ^α *is the natural linear mapping of* $\otimes_t^\alpha B(t)$ *into* $\otimes_t^\lambda B(t)$. *In particular, if* $K[U]$ *denotes the kernel of an operator* U *and* α *is faithful, we have*

$$J_\lambda^\alpha K\left[\bigotimes_t Q(t) \right] = K\left[\bigotimes_t^\lambda Q(t) \right] \cap J_\lambda^\alpha \left[\bigotimes_t B(t) \right]. \tag{2.5}$$

LEMMA 2.2 [13]. *If the family* $\{Q(t)\}$ *is a topological isomorphism for each* t *and* α *is a uniform reasonable crossnorm, then the continuous extension of* $\otimes_t Q(t)$ *to the entire space* $\otimes_t^\alpha B(t)$ *is a topological isomorphism.*

LEMMA 2.3. *Let* $\tilde{A}(t)$ *and* $\tilde{A}(t')$ *be densely defined closed linear operators on* $B(t)$ *and* $B(t')$ *with nonempty resolvent sets* $\rho(\tilde{A}(t))$, $\rho(\tilde{A}(t'))$ *such that it does not occur that one of the extended spectra* $\sigma_e(\tilde{A}(t))$ *and* $\sigma_e(\tilde{A}(t'))$ *contains* 0 *while the other contains* ∞. *Assuming* α *is a uniform faithful reasonable norm on* $\otimes_t^\alpha B(t)$, *we have that the closures of the graphs of* $(t < t')$ $A(t)A(t')$ *and* $A(t')A(t)$ *coincide, where*

$$A(t') = \bigotimes_{a<s<t'} I_s \otimes \tilde{A}(t') \otimes \Big(\bigotimes_{t'<s<b} I_s \Big).$$

Furthermore, the relationship

$$A(t)A(t') = A(t')A(t) = \bigotimes_{a<s<t} I_s \otimes \tilde{A}(t) \otimes \Big(\bigotimes_{t<s<t'} I_s \Big) \otimes \tilde{A}(t') \otimes \bigotimes_{t'<s<b} I_s$$

is maintained in the closure.

THEOREM 2.2. $G[t, t']$ *extends to an isometry onto an isomorphic copy of* $\otimes_t^\alpha B(t)$ *and*:

(1) $G(t, t')G(t', t) = I$.

(2) *The extension is one-to-one.*

(3) $G(t, t')G(t', t'') = G(t, t'')$.

PROOF. (1) is obvious and (2) follows from Lemma 2.2; (3) follows from the definition of $G(,)$.

DEFINITION 2.5. For any linear operator $U \in \mathfrak{L}[\otimes_t^\alpha B(t)]$ we define the standard exchange operator by

$$E[t, t']U = G[t', t]UG[t, t']. \tag{2.6}$$

THEOREM 2.3. $E[t, t']$ *is an exchange operator.*

DEFINITION 2.6. A linear operator $\tilde{A}(t)$ defined on $B(t)$ will be called space continuous in the uniform topology at t' if

$$\lim_{t \to t'} \|\Gamma_t^{t'} \tilde{A}(t) \Gamma_{t'}^{t} - \tilde{A}(t')\| = 0 \tag{2.7}$$

where the norm is on $B(t')$. We also define right and left space continuity in the obvious manner.

It should be noted that if $B(t') = B$ for all t' then space continuity is the same as our ordinary notion.

THEOREM 2.4. *Suppose* $\tilde{A}(t)$ *is right or left space continuous*; *define* $A(t)$ *by*

$$A(t) = \overline{\bigotimes_{a<s<t} I_s \otimes \tilde{A}(t) \otimes \Big(\bigotimes_{t<s<b} I_s \Big)}; \tag{2.8}$$

then

(1) $A(t)A(t') = A(t')A(t)$;

(2) *the family* $\{A(t) | t \in [a, b]\}$ *is c-right or left continuous according to* $\tilde{A}(t)$.

PROOF. (1) is clear; see Lemma 2.3. For (2) let

$$\phi = \sum_{v=1}^{n} \bigotimes_{t} \phi_t^v \in \bigotimes_{t}^{\alpha} B(t);$$

then

$$\begin{aligned}\|E[t', t]A(t)\phi - A(t')\phi\|_\alpha &\leq \sum_{v=1}^{n} \left\| E[t', t]A(t)\left[\bigotimes_t \phi_t^v\right] - A(t') \bigotimes_t \phi_t^v \right\|_\alpha \\ &\leq \sum_{v=1}^{n} M^v(t') \|\Gamma_t^{t'} \tilde{A}(t) \Gamma_{t'}^{t} \phi_{t'}^v - \tilde{A}(t')\phi_{t'}^v\| \\ &\leq \sum_{v=1}^{n} M^v \|\Gamma_t^{t'} \tilde{A}(t) \Gamma_{t'}^{t} - \tilde{A}(t')\|\end{aligned}$$

where

$$M^v(t') = \prod_{t \neq t'} \|\phi_t^v\| \quad \text{and} \quad M^v = \prod_{t} \|\phi_t^v\| + 1.$$

The last term converges to zero as $t \to t'$ in the appropriate manner. Since ϕ is in a dense subset of $\otimes_t^\alpha B(t)$ and α is faithful, we are done.

DEFINITION 2.7. Any family of operators (not necessarily bounded) $\{A(t) | t \in [a, b]\}$ satisfying (2.8) is called a family of time-ordered operators.

DEFINITION 2.8. A linear operator $\tilde{A}(t)$ defined on $B(t)$ will be called space differentiable at t' in the uniform topology provided there is an operator $D\tilde{A}(t')$ such that

$$\lim_{t \to t'} \left\| \frac{\Gamma_t^{t'} \tilde{A}(t) \Gamma_{t'}^{t} - \tilde{A}(t')}{t - t'} - D\tilde{A}(t') \right\| = 0.$$

THEOREM 2.5. *Suppose $A(t)$ is space differentiable at t' and let $A(t)$ be the time-ordered version of $A(t)$ as in* (2.8); *then*:

(1) *$A(t)$ is c-continuous at t'*;

(2) *$A(t)$ is c-differentiable at t' and*

$$DA(t') = \overline{\bigotimes_{a < s < t'} I_s \otimes D\tilde{A}(t') \otimes \Big(\bigotimes_{t' < s < b} I_s \Big)}. \tag{2.9}$$

PROOF. The proof follows the lines of Theorem 2.4 and, hence, is omitted.

DEFINITION 2.9. The operators $\tilde{A}(t)$, $t \in [a, b]$, are said to be of space bounded variation in the uniform topology if

$$\sup_{P} \sum_{j=1}^{n} \|\Gamma_{t_j}^{t_{j+1}} \tilde{A}(t_j) \Gamma_{t_{j+1}}^{t_j} - \tilde{A}(t_{j+1})\| < \infty$$

where each norm is on the space $B(t_{j+1})$.

THEOREM 2.6. *If the operators $\tilde{A}(t)$, $t \in [a, b]$, are of space bounded variation then the time-ordered versions are of c-bounded variation.*

PROOF. The proof follows the lines of Theorem 2.4 and, hence, is omitted.

If we note that

$$\|E[\bar{t}', t']A(t')\|_\alpha = \|A(t')\|_\alpha$$

and

$$\left\|E[\bar{t}, t']\left(E[t', t]A(t) - E[t', \bar{t}]A(\bar{t})\right)\right\|_\alpha = \left\|E[\bar{t}, t]A(t) - A(\bar{t})\right\|_\alpha,$$

hence,

$$\left\|E[t', t]A(t) - E[t', \bar{t}]A(\bar{t})\right\|_\alpha = \left\|E[\bar{t}, t]A(t) - A(\bar{t})\right\|_\alpha. \tag{2.10}$$

We now have the following result.

THEOREM 2.7. *The family $\{A(t), t \in [a, b]\}$ is c-continuous at $\bar{t}$ if and only if for arbitrary $t' \in [a, b]$, we have*

$$\lim_{t \to \bar{t}} \left\|E[t', t]A(t) - E[t', \bar{t}]A(\bar{t})\right\|_\alpha = 0. \tag{2.11}$$

Results of a similar nature may be obtained in the c-differentiability and c-bounded variation case. At this point it should be clear that all point notions in analysis have a counterpart in this setting.

3. Preliminaries. In this section we assume that for each $t \in [0, T]$, $\tilde{A}(t)$ is the infinitesimal generator of a strongly continuous contraction semigroup on $B(t)$ with domain $D(t)$. We assume that our family $\Gamma_t^{t'}: D(t) \to D(t')$ is an isometric topological isomorphism in the norm graph closure. We denote $D = \otimes_t^\alpha D(t)$, where α is a uniform faithful reasonable norm and D is the norm graph closure of $\otimes_t D(t)$ with respect to α.

Assumption. We assume that for any $\bar{\tau}$ and family $\{t_j\}$ in $[0, T]$, the operator $\sum_{j=1}^n \Gamma_{t_j}^{\bar{\tau}} \tilde{A}(t_j) \Gamma_{\bar{\tau}}^{t_j}$ is the generator of a strongly continuous contraction semigroup of operators on $B(\bar{\tau})$.

We shall need the following results from the theory of semigroups of operators. The reader is referred to Butzer and Berens **[B-B]** and Gill **[G1]** for proofs as indicated.

LEMMA 3.1. *Let $\tilde{T}_t(\eta)$ be the contraction semigroup on $B(t)$ generated by $\tilde{A}(t)$. Set $\tilde{A}^\eta(t) = (\tilde{T}_t(\eta) - I)/\eta$, $\eta > 0$; then $\tilde{A}^\eta(t)$ is a bounded linear operator for $\eta > 0$ and $\tilde{A}(t) = \text{s-lim}_{n \to 0} \tilde{A}^\eta$.*

PROOF. Cf. **[B-B]**.

LEMMA 3.2. $\text{s-lim}_{\eta \to 0} \Gamma_\tau^t \tilde{A}^\eta(\tau) \Gamma_t^\tau = \Gamma_\tau^t \tilde{A}(\tau) \Gamma_t^\tau$.

PROOF. Note that for any $\phi_t \in D(t)$, then $\Gamma_t^\tau \phi_t \in D(\tau)$.

LEMMA 3.3. *There exists a contraction semigroup $T_t(\eta)$ defined on $\otimes_s^\alpha B(s)$ such that for $\phi = \otimes \phi_s$ we have:*

(1)

$$T_t(\eta)\phi = \bigotimes_{0<s<t} \phi_s \otimes \tilde{T}_t(\eta)\phi_t \otimes \Big(\bigotimes_{t<s<T} \phi_s\Big);$$

(2)

$$\text{s-lim}_{\eta \to 0} (T_t(\eta) - I)/\eta = A(t) \quad (\textit{time-ordered version of } \tilde{A}(t));$$

(3) *$A^\eta(t) = (T_t(\eta) - I)/\eta$ is a bounded operator;*

(4) *all the above operators are unique.*

PROOF. The proofs of (1), (2), (4) are in **[G1]**. (3) follows from the definition of $A^{\eta}(t)$ and Lemma 3.1.

We now use Hille's first exponential formula to define (cf. **[B-B]**)

$$\exp\{A(t)\} = \operatorname*{s-lim}_{\eta\to 0} \exp\{A^{\eta}(t)\}. \tag{3.1}$$

LEMMA 3.4. *For any $\bar{\tau} \in [0, T]$ we have*

$$E[\bar{\tau}, t]\exp\{A(t)\} = \exp\{E[\bar{\tau}, t]A(t)\}. \tag{3.2}$$

PROOF.

$$\begin{aligned}[E[\bar{\tau}, t]A^{\eta}(t)]^2 &= G[\bar{\tau}, t]A^{\eta}(t)G[t, \bar{\tau}]G[\bar{\tau}, t]A^{\eta}(t)G[t, \bar{\tau}]\\ &= G[\bar{\tau}, t][A^{\eta}(t)]^2G[t, \bar{\tau}] = E[\bar{\tau}, t][A^{\eta}(t)]^2.\end{aligned}$$

It follows from induction that

$$[E[\bar{\tau}, t]A^{\eta}(t)]^k = E[\bar{\tau}, t][A^{\eta}(t)]^k;$$

hence

$$\exp\{E[\bar{\tau}, t]A^{\eta}(t)\} = E[\bar{\tau}, t]\exp\{A^{\eta}(t)\}.$$

Taking strong limits provides the result.

4. Time-ordered integrals and evolution operators. In this section we construct a general class of time-ordered operators. This class contains operators associated with almost all problems of interest. In the development of our integration theory, we use the Riemann complete integral as developed by Kurzweil **[K]** and Henstock **[H]**. We present a slight extension of their theory to the operator-valued case in the appendix and the reader is referred there for details.

Let δ: $[0, T] \to (0, \infty)$, and suppose P is a K-H partition for δ, $P = \{t_0, \tau_1, t_2, \ldots, \tau_n, t_n\}$. Let $P' = \{t'_0, \ldots, t'_k\}$ be an ordinary partition of $[0, T]$ such that $\{t_0, t_1, \ldots, t_n\} \subset P'$; suppose $\tau'_q \in [t'_{q-1}, t'_q)$ is arbitrary and $t'_{q-1} = t_{j-1} < t_j < \cdots < t_m = t'_q$ is a typical subinterval of P'. Reindex P' by replacing $q - 1$ by $j - 1$ and q by m, so that we now write $[t'_{q-1}, t'_q)$ as $[t_{j-1}, t_m)$. Define $Q^{\eta}_{\delta}[P'|P]$ and $Q_{\delta}[P'|P]$ by

$$Q^{\eta}_{\delta}[P'|P] = \sum_{j=1}^{n} \Delta t_j E[\tau'_j, \tau_j]A^{\eta}(\tau_j), \tag{4.1}$$

$$Q_{\delta}[P'|P] = \sum_{j=1}^{n} \Delta t_j E[\tau'_j, \tau_j]A(\tau_j), \tag{4.2}$$

where $\tau'_l = \tau'_q$ for $j \leqslant l \leqslant m$.

DEFINITION 4.1. We say that the family $\{A(t)|t \in [0, T]\}$ has a time-ordered integral $Q[P']$ with information concentrated on P', if for every $\varepsilon > 0$ there exists a function δ: $[0, T] \to (0, \infty)$ and an exchange operator $E[\ ,\ ]$ such that whenever P is a K-H partition for δ then

$$\|Q_{\delta}[P'|P]\phi - Q[P']\phi\|_{\alpha} < \varepsilon \quad \text{for all } \phi \in \bigotimes_{s}^{\alpha} D(s).$$

It is not hard to show that if our integral exists then[2]

$$Q[P']\phi = \sum_{q=1}^{k} \int_{t'_{q-1}}^{t'_q} E[\tau'_q, \tau]A(\tau)\, r(d\tau), \tag{4.3}$$

which may be written as

$$Q[P']\phi = \sum_{q=1}^{k} \Delta t'_q \left\{ \frac{1}{\Delta t'_q} \int_{t'_{q-1}}^{t'_q} E[\tau'_q, \tau]A(\tau)\, r(d\tau) \right\}.$$

In this form we see that each term is an average over the interval $[t'_{q-1}, t'_q]$, and the exchange operator concentrates all information contained in $A(\tau)$ at the τ'_qth place.

THEOREM 4.1. *If the family* $\{A^{\eta}(t) | t \in [0, T]\}$ *is c-continuous or of c-bounded variation on* $[0, T]$, *then the integral* (4.3) *exists in this case.*

PROOF. *Part* I. Assume c-continuity; by Theorem A.1 it suffices to prove the result in the Riemann sense. Let $\bar{P}$, $\bar{\bar{P}}$ be arbitrary partitions of $[0, T]$ and set $P = \bar{P} \cup \bar{\bar{P}}$, $\delta = \min(\bar{\delta}, \bar{\bar{\delta}})$; then

$$\left\| Q^{\eta}_{\bar{\delta}}[P'|\bar{P}] - Q^{\eta}_{\bar{\bar{\delta}}}\left[P'|\bar{\bar{P}}\right] \right\|_{\alpha} < \| Q^{\eta}_{\bar{\delta}}[P'|\bar{P}] - Q^{\eta}_{\delta}[P'|P] \|_{\alpha} + \left\| Q^{\eta}_{\bar{\bar{\delta}}}\left[P'|\bar{\bar{P}}\right] - Q^{\eta}_{\delta}[P'|P] \right\|_{\alpha}.$$

Reindexing as at the beginning of this section leads to

$$\left\| Q^{\eta}_{\bar{\delta}}[P'|\bar{P}] - Q^{\eta}_{\bar{\bar{\delta}}}\left[P'|\bar{\bar{P}}\right] \right\|_{\alpha} < \sum_{j=1}^{n} \Delta t_j \| E[\tau'_j, \bar{\tau}_j]A^{\eta}(\bar{\tau}_j) - E[\tau'_j, \tau_j]A^{\eta}(\tau_j) \|_{\alpha}$$
$$+ \sum_{j=1}^{n} \Delta t_j \left\| E\left[\tau'_j, \bar{\bar{\tau}}_j\right] A^{\eta}\left(\bar{\bar{\tau}}_j\right) - E[\tau'_j, \tau_j]A^{\eta}(\tau_j) \right\|_{\alpha}. \tag{4.4}$$

We now use the fact that $[0, T]$ is compact, along with c-continuity, to conclude that there is a $\gamma > 0$ such that as soon as $|r - s| < \gamma$,

$$\| E[\tau', r]A^{\eta}(r) - E[\tau', s]A^{\eta}(s) \|_{\alpha} < \varepsilon/2T \quad \text{for all } \tau' \in [0, T].$$

If we combine this with inequality (4.4), we have

$$\left\| Q^{\eta}_{\bar{\delta}}[P'|\bar{P}] - Q^{\eta}_{\bar{\bar{\delta}}}\left[P'|\bar{\bar{P}}\right] \right\|_{\alpha} < \varepsilon.$$

This means that the family $\{Q^{\eta}_{\delta}[P'|P]\colon P \supset P'\}$ is Cauchy; it follows that $Q^{\eta}_{\delta}[P'|P] \to Q^{\eta}[P']$ as $\delta \to 0$.

Part II. If we assume the family is of c-bounded variation, then by (4.4) we have

$$\left\| Q^{\eta}_{\bar{\delta}}[P|\bar{P}] - Q^{\eta}_{\bar{\bar{\delta}}}\left[P'|\bar{\bar{P}}\right] \right\|_{\alpha} < \left[\left[\left(\bar{\delta} + \bar{\bar{\delta}} \right) V_0^T \right] A^{\eta}(\tau) \right],$$

which once again shows that the limit exists.

THEOREM 4.2. $Q^{\eta}[P']$ *converges strongly to*

$$Q[P'] = \sum_{q=1}^{k} \int_{t'_{q-1}}^{t'_q} E[\tau'_q, \tau]A(\tau)\, r(d\tau) \tag{4.5}$$

on $\otimes^{\alpha}_{s} D(s)$.

[2] $r(d\tau)$ is the Riemann complete measure. See the appendix.

PROOF. It suffices to consider the case $k = 1$ and $\phi = \otimes_s \phi_s \in \otimes_s^\alpha D(s)$.

$$\left\|Q^\eta[P']\phi - Q^{\bar\eta}[P']\phi\right\|_\alpha = \left\|\int_{t_0'}^{t_1'} \{E[\tau_1', \tau]A^\eta(\tau)\phi - E[\tau_1', \tau]A^{\bar\eta}(\tau)\phi\}\, r(d\tau)\right\|_\alpha$$
$$\leq \Delta t_1' \sup_{\tau\in[t_0', t_1')} \left\|E[\tau_1', \tau]A^\eta(\tau)\phi - E[\tau_1', \tau]A^{\bar\eta}(\tau)\phi\right\|_\alpha$$
$$\leq \Delta t_1' M \sup_{\tau\in[t_0', t_1')} \left\|\Gamma_\tau^{\tau_1'}\tilde A^\eta(\tau)\Gamma_{\tau_1'}^{\tau}\phi_{\tau_1'} - \Gamma_\tau^{\tau_1'}\tilde A^{\bar\eta}(\tau)\Gamma_{\tau_1'}^{\tau}\phi_{\tau_1'}\right\|_{\tau_1'}$$

where τ_1' means norm in $B(\tau_1')$ and $M = \|\phi\|_\alpha + 1$. Using Lemma 3.4 we have

$$\lim_{\eta,\bar\eta\to 0} \left\|Q^\eta[P']\phi - Q^{\bar\eta}[P']\phi\right\|_\alpha = 0,$$

so that $Q^\eta[P']$ has a strong limit on $\otimes_s^\alpha D(s)$ which we denote by $Q[P']$.

It should be noted that $Q[P']$ is densely defined; however it is not clear that Q is closed, nor is it clear that Q is the generator of a strongly continuous contraction semigroup of operators. We shall dispose of this issue shortly; however we will need the following:

LEMMA 4.1. *$Q_\delta[P'|P] \to^s Q[P']$ on $\otimes_s^\alpha D(s)$ (see (4.2)).*

PROOF. Again it suffices to assume $k = 1$. Let $\phi \in \otimes_s^\alpha D(s)$.

$$\left\|Q[P']\phi - Q_\delta[P'|P]\phi\right\|_\alpha = \left\|\int_{t_0'}^{t_1'} E[\tau_1', \tau]A(\tau)\phi\, r(d\tau) - \sum_{j=1}^n \Delta t_j E[\tau_j', \tau_j]A(\tau_j)\phi\right\|_\alpha$$

where P is a partition of $[t_0', t_1')$, $\tau_j' = \tau_1'$ for all j. This means that

$$\left\|Q[P']\phi - Q_\delta[P'|P]\phi\right\|_\alpha \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} r(d\tau)\left\|E[\tau_j', \tau]A(\tau)\phi - E[\tau_j', \tau_j]A(\tau_j)\phi\right\|_\alpha$$
$$\leq \Delta t_1' \sup_{\tau,\tau_j\in[t_0', t_1')} \left\|E[\tau_j', \tau]A(\tau)\phi - E[\tau_1', \tau_j]A(\tau_j)\phi\right\|_\alpha.$$

The right-hand side approaches zero as $\delta \to 0$.

THEOREM 4.3. *$Q[P']$ has a closure which is the generator of a strongly continuous contraction semigroup of operators.*

PROOF. Recall that an operator Q is said to be dissipative if for any ϕ and duality map J for ϕ we have $\mathrm{Re}\langle Q\phi, J\phi\rangle \leq 0$, where $\phi \in D(Q)$ (cf. Pazy **[P]**). Let $\phi \in \otimes_s^\alpha D(s)$ and suppose J is any duality map for ϕ; then

$$\mathrm{Re}\langle Q[P']\phi, J\phi\rangle = \mathrm{Re}\langle Q_\delta[P'|P]\phi, J\phi\rangle + \mathrm{Re}\langle Q[P']\phi - Q_\delta[P'|P]\phi, J\phi\rangle. \tag{4.6}$$

Note that if we assume $k = 1$ then

$$Q_\delta[P'|P] = \sum_{j=1}^n \Delta t_j E[\tau_1', \tau_j]A(\tau_j)$$
$$= \bigotimes_{0<s<\tau_1'} I_s \left(\sum_{j=1}^n \Delta t_j \Gamma_{\tau_j}^{\tau_1'}\tilde A(\tau_j)\Gamma_{\tau_1'}^{\tau_j}\right) \otimes \left(\bigotimes_{\tau_1'<s<T} I_s\right).$$

By assumption $\Sigma_{j=1}^{n} \Delta t_j \Gamma_{\tau_j}^{\tau_j'} A(\tau_j) \Gamma_{\tau_1'}^{\tau_j}$ is a generator of a contraction semigroup for all $\tau_1' \in [0, T]$ so that $Q_\delta[P'|P]$ is dissipative. Using this result in (4.6) we get

$$\mathrm{Re}\langle Q[P']\phi, J\phi\rangle \leq \mathrm{Re}\langle Q[P']\phi - Q_\delta[P'|P]\phi, J\phi\rangle;$$

letting $\delta \to 0$ reveals that $\mathrm{Re}\langle Q[P']\phi, J\phi\rangle \leq 0$, which shows that $Q[P']$ is dissipative. Since all dissipative densely defined operators are closable, it follows that Q has a densely defined dissipative closure which we again denote by Q. In order to complete our proof, we need to show that the range of $\lambda - Q$ is dense. Since Q_δ is a generator, for any $\phi \in \otimes_s^\alpha B(s)$, $R(\lambda, Q_\delta)\phi \in \otimes_s^\alpha D(s)$, where $R(\lambda, Q_\delta)$ is the resolvent operator; furthermore $[\lambda - Q_\delta]R(\lambda, Q_\delta)\phi = \phi$ for all $\delta > 0$. Letting $\delta \to 0$ and noting that $R(\lambda, Q_\delta) \to^s R(\lambda)$, we conclude that $(\lambda - Q)R(\lambda)\phi = \phi$ for all $\phi \in \otimes_s^\alpha B(s) \Rightarrow \{R(\lambda)\phi | \phi \in \otimes_s^\alpha B(s)\}$ is in the domain of $\lambda - Q$ for $\lambda > 0$, so that $R(\lambda) = R(\lambda, Q)$ and Q is a generator of a strongly continuous contraction semigroup.

In the last theorem we used the fact that $R(\lambda, Q_\delta)$ converges strongly as $\delta \to 0$. We provide a proof of this result below.

Lemma 4.2. *For each $\lambda \to 0$ there is a bounded operator $R(\lambda)$ and*

$$R(\lambda)\phi = \operatorname*{s-lim}_{\delta \to 0} R(\lambda, Q_\delta)\phi, \quad \phi \in \bigotimes_s^\alpha B(s).$$

Proof.

$$\|R(\lambda, Q_\delta)\phi - R(\lambda, Q_{\bar\delta})\phi\|_\alpha = \left\| \int_0^\infty e^{-\lambda t}\{U_\delta[P', t]\phi - U_{\bar\delta}[P', t]\phi\}\, dt \right\|_\alpha$$
$$\leq \frac{1}{\lambda} \sup_{(0,\infty)} \|U_\delta[P', t]\phi - U_{\bar\delta}[P', t]\phi\|_\alpha \leq \frac{2\|\phi\|\alpha_2}{\lambda}.$$

Letting $\delta, \bar\delta \to 0$ shows that the family $\{R(\lambda, Q_\delta) | \delta > 0\}$ is Cauchy. The remaining part of the proof is trivial.

Theorem 4.4. *If the family $\{A(t) | t \in [0, T]\}$ is strongly right c-continuous at t, then*

$$\operatorname*{s-lim}_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} E[\bar\tau, \tau] A(\tau)\, r(d\tau) = E[\bar\tau, t] A(t^+) \quad \textit{for each } \bar\tau \in [0, T].$$

Proof. For any $\phi \in \otimes_s^\alpha D(s)$ we have

$$\left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} E[\bar\tau, \tau] A(\tau)\phi\, r(d\tau) - E[\bar\tau, t] A(t^+)\phi \right\|_\alpha$$
$$\leq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} r(d\tau) \|E[\bar\tau, \tau] A(\tau)\phi - E[\bar\tau, t] A(t^+)\phi\|_\alpha$$
$$< \sup_{\tau \in [t, t+\varepsilon)} \|E[\bar\tau, \tau] A(\tau)\phi - E[\bar\tau, t] A(t^+)\phi\|.$$

Letting $\varepsilon \to 0$ provides our result.

Evolution operators. Let $Q[P'|P]$ be as in (4.1) and set

$$U_\delta^n[P'|P] = \exp\{Q_\delta^n[P'|P]\}. \tag{4.7}$$

THEOREM 4.5. *If the family* $A(t)$, $t \in [0, T]$, *is c-continuous or of c-bounded variation, then*

$$\lim_{\delta \to 0} U_\delta^\eta[P'|P] = U^\eta[P'] = \exp\{Q^\eta[P']\}. \tag{4.8}$$

Furthermore,

$$\lim_{\varepsilon \to 0} \frac{\exp\{\int_t^{t+\varepsilon} E[\bar{\tau}, \tau] A^\eta(\tau)\, r(d\tau)\} - I}{\varepsilon} = E[\bar{\tau}, t] A^\eta(t^+)$$

for each point of right c-continuity of $\{A^\eta(s) | s \in [0, T]\}$ *for each* $\bar{\tau} \in [0, T]$.

PROOF. The proof follows from Theorems 4.1 and 4.4 along with the definition of $\exp\{\ \}$. (Recall that the only type of discontinuities of a function of bounded variation are jump.) (See Theorem 4.7.)

THEOREM 4.6. $U^\eta[P'] \to^s U[P'] = \exp\{Q[P']\}$.

PROOF. Let $\phi \in \otimes_s^\alpha D(s)$; then

$$U^\eta[P']\phi - U^{\bar{\eta}}[P']\phi = \int_0^1 \frac{d}{dv}\left[U^{\bar{\eta}}[P', 1-v]\, U^\eta[P', v]\right]\phi\, dv \tag{4.9}$$

where $U[P', v] = \exp\{vQ^\eta[P']\}$. Differentiating (4.9) we have

$$U^\eta[P']\phi - U^{\bar{\eta}}[P']\phi = \int_0^1 U^{\bar{\eta}}[P', 1-v]\, U^\eta[P', v]\left[Q^\eta[P']\phi - Q^{\bar{\eta}}[P']\phi\right] dv;$$

hence

$$\begin{aligned} &\left\| U^\eta[P']\phi - U^{\bar{\eta}}[P']\phi \right\|_\alpha \\ &\quad < \sup_{0<v<1} \left\| U^{\bar{\eta}}[P', 1-v]\, U^\eta[P', v] \right\|_\alpha \times \left\| Q^\eta[P']\phi - Q^{\bar{\eta}}[P']\phi \right\|_\alpha. \end{aligned}$$

Letting $\eta, \bar{\eta} \to 0$ and using Theorem 4.2 shows that $\{U^\eta[P'] | \eta > 0\}$ is Cauchy for $\phi \in \otimes_s^\alpha D(s)$ and, hence, converges strongly to a limit on $\otimes_s^\alpha D(s)$. Since $\otimes_s^\alpha D(s)$ is dense and $U[P']$ is bounded, the Banach-Steinhaus theorem shows that $U^\eta[P']\phi \to^s U[P']\phi$ for $\phi \in \otimes_s^\alpha B(s)$.

THEOREM 4.7. *Suppose the family* $\{A(t) | t \in [0, T]\}$ *is strongly c-continuous or of strong c-bounded variation; then*

$$\operatorname*{s-lim}_{\varepsilon \to 0} \frac{1}{\varepsilon}\left\{\exp\left[\int_t^{t+\varepsilon} E[\bar{\tau}, \tau] A(\tau)\, r(d\tau)\right] - I\right\} = E[\bar{\tau}, t] A(t^+).$$

PROOF. Set

$$U_1 = \exp\{\varepsilon E[\bar{\tau}, t] A(t^+)\}, \qquad U_2 = \exp\left\{\int_t^{t+\varepsilon} E[\bar{\tau}, \tau] A(\tau)\, r(d\tau)\right\}$$

for any $\phi \in \otimes_s^\alpha D(s)$. We have

$$\begin{aligned}
\|U_1\phi - U_2\phi\|_\alpha &\leq \int_0^1 \left\| \frac{d}{dv} U_2[1-v]U_1[v]\phi \right\|_\alpha dv \\
&\leq \int_0^1 dv \|U_2[1-v]U_1[v]\|_\alpha \\
&\quad \cdot \left\| \varepsilon E[\bar{\tau}, t]A(t^+)\phi - \int_t^{t+\varepsilon} E[\bar{\tau}, \tau]A(\tau)\phi\, r(d\tau) \right\|_\alpha \\
&\leq \int_t^{t+\varepsilon} r(d\tau) \|E[\bar{\tau}, \tau]A(\tau)\phi - E[\bar{\tau}, t]A(t^+)\phi\|_\alpha \\
&\leq \varepsilon \sup_{[t, t+\varepsilon)} \|E[\bar{\tau}, \tau]A(\tau)\phi - E[\bar{\tau}, t]A(t^+)\phi\|_\alpha .
\end{aligned}$$

This means that $\lim_{\varepsilon\to 0}\varepsilon^{-1}\|U_1\phi - U_2\phi\|_\alpha = 0$. We now use the uniqueness of the generator of a contraction semigroup to complete our proof.

Let $\{A_1(t)|t \in [0, T]\}$ and $\{A_2(t)|t \in [0, T]\}$ be two families of time-ordered generators of contraction semigroups of operators obtained from operators $\{\tilde{A}_1(t)|t \in [0, T]\}$ and $\{\tilde{A}_2(t)|t \in [0, T]\}$ which are generators on $B(t)$ for each t and have the same domain $D(t)$. We assume that the conditions in §1 are valid with respect to the family $\{\Gamma_t^{t'}|t, t' \in [0, T]\}$. Construct $U_1[P']$ and $U_2[P'']$ for arbitrary P', P'', where

$$U_1[P'] = \exp\left\{ \sum_{j=1}^{n'} \int_{t'_{j-1}}^{t'_j} E[\tau'_j, \tau]A_1(\tau)\, r(d\tau) \right\}, \tag{4.10}$$

$$U_2[P''] = \exp\left\{ \sum_{j=1}^{n''} \int_{t''_{j-1}}^{t''_j} E[\tau''_j, \tau]A_2(\tau)\, r(d\tau) \right\}. \tag{4.11}$$

We now note that except for a finite number of points we have

$$U_1[P']U_2[P''] = U_2[P'']U_1[P']; \tag{4.12}$$

hence, if we define $U[P', P'']$ by

$$\begin{aligned}
U[P', P''] = \exp\Bigg\{ &\sum_{j=1}^{n'} \int_{t'_{j-1}}^{t'_j} E[\tau'_j, \tau]A_1(\tau)\, r(d\tau) \\
&+ \sum_{j=1}^{n''} \int_{t''_{j-1}}^{t''_j} E[\tau''_j, \tau]A_2(\tau)\, r(d\tau) \Bigg\},
\end{aligned}$$

we obtain the following:

THEOREM 4.8. *$U[P', P'']$ exists except on a set of $r[\ ]$ measure zero.*

Let us now assume that $\otimes_s^\alpha B(s)$ is reflexive so that any bounded set is weakly sequentially compact. We then consider the sequence $\{U[P'_k, P''_k]|\ |P'_k| \to 0, |P''_k| \to 0, k \to \infty\}$.

THEOREM 4.9. *The above sequence has a subsequence, which converges weakly (a.e.) for all* $\phi \in \otimes_s^\alpha B(s)$,

$$U[T, 0] = \exp\left\{\int_0^T [A_1(\tau) + A_2(\tau)]\, r(d\tau)\right\} \tag{4.13}$$

and we call it the weak time-ordered evolution operator associated with the families $\{A_1(t) | t \in [0, T]\}$ *and* $\{A_2(t) | t \in [0, T]\}$.

Conclusion. Theorem 4.8 says that, in the weak sense, the sum of two time-ordered families of generators of a contraction semigroup is a generator except on a set of measure zero. It is not hard to see that the exceptional set is countable.

It should be noted that in some sense weak convergence is unrevealing. A simple analysis shows that whatever the subsequence for which convergence is obtained, the individual terms are unrelated to each other. Recall that a version of Mazur's theorem (cf. Yosida [Y]) states that if $U[P_k]\phi \to^w U\phi$, then, given $\varepsilon > 0$, there exists a convex combination of reals (e.g. $\{\lambda_i\} \ni \sum_{i=1}^N \lambda_i = 1$) and

$$\left\| U\phi - \sum_{i=1}^{N} \lambda_i U[P_i]\phi \right\|_\alpha < \varepsilon.$$

That is, we may use convex combinations of reals to obtain a strong approximation. Another view of these convex combinations is that of weighted probabilities on the first N members of the sequence. A physical interpretation of (4.3) reveals that both the number of partition points in P' and the placement of the τ_q''s in the interval $[t_{q-1}', t_q')$ are discrete and continuous random variables, respectively. It turns out that U can be obtained as an expected value over the joint probability measure for both random variables. This can be done in the uniform operator topology without the assumption that $\otimes_s^\alpha B(s)$ is reflexive. We shall present results in this direction in an upcoming publication where we present a rigorous definition and analysis of Feynman type integrals.

In another direction, Theorem 4.8 tells us that perturbation expansions should lead to correct results. It is well known that quantum electrodynamics has a problem of divergencies with the perturbation expansion in the second order and subsequent terms. It has long been debated that the cause is due to the mathematics (cf. Bjorken and Drell [**B-D**]). It turns out that the divergencies are at least partly caused by the violation of the time energy uncertainty relation in the use of infinitesimal time ($d\tau$) in the evolution integral. The reader is referred to Gill [**G2**] for details.

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APPENDIX

Riemann complete integral. This appendix provides a slight modification and generalization of the work of Kurzweil [**K**] and Henstock [**H**]. Our approach is fashioned after that of Beck [**B**], who so kindly introduced the writer to efforts of Kurzweil in this area.

Throughout our discussion, $[a, b]$ will be a fixed compact interval in R, and B will be a fixed Banach space. Let $\delta(t)$ be any function from $[a, b]$ into $(0, \infty)$ and let $P = \{t_0, \tau_1, t_1, \tau_2, \ldots, \tau_n, t_n\}$, where $a = t_0 < \tau_1 < t_1 < \cdots < \tau_n < t_n = b$.

DEFINITION A.1. We call P a Kurzweil-Henstock partition for δ (or a K-H partition for δ) providing that t_j and $t_{j+1} \in (\tau_{j+1} - \delta(\tau_{j+1}), \tau_{j+1} + \delta(\tau_{j+1}))$ for $j = 0, 1, 2, \ldots, n-1$.

LEMMA A.1. *Let $\delta(t)$ be any function from $[a, b]$ into $(0, \infty)$. Then there exists a K-H partition for δ.*

PROOF. The set $\{(\tau - \delta(\tau), \tau + \delta(\tau)) | \tau \in [a, b]\}$ is an open cover for $[a, b]$ and, hence, contains a finite irreducible subcover. Let $\tau_1, \ldots, \tau_n \in [a, b]$ be ordered by the index such that the set $\{(\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)) | \tau_j \in [a, b], j = 1, 2, \ldots, n\}$ is an irreducible subcover, also

$$\tau_j + \delta(\tau_j) < \tau_{j+1} + \delta(\tau_{j+1}) \quad \text{and} \quad \tau_j - \delta(\tau_j) < \tau_{j+1} - \delta(\tau_{j+1}).$$

Adding, we have $\tau_j < \tau_{j+1}$, hence $t_j \in \tau_j < t_j < \tau_{j+1}, j = 1, 2, \ldots, n-1$. Set $t_0 = a$ and $t_n = b$. It is clear that t_j and $t_{j+1} \in (\tau_{j+1} - \delta(\tau_{j+1}), \tau_{j+1} + \delta(\tau_{j+1}))$.

LEMMA A.2. *Let δ_1 and δ_2 be functions from $[a, b]$ to $(0, \infty)$ and suppose that for each $t \in [a, b]$, $\delta_1(t) < \delta_2(t)$. Then if P is a K-H partition for δ_1, it is one for δ_2.*

DEFINITION A.2. Let $\tilde{A}(t)$ be a bounded operator-valued function from $[a, b]$ defined on B. We say that the Riemann complete integral of $\tilde{A}(t)$ with respect to μ exists if there is a bounded operator Q^μ, such that for each $\varepsilon > 0$, there exists a function δ from $[a, b]$ into $(0, \infty)$ such that, whenever the family $P = \{t_0, \tau_1, \ldots, \tau_n, t_n\}$ is a K-H partition, then (Note: In this case $\mu(t_j), \mu(t_{j-1}) \in (\mu(\tau_j) - \delta(\tau_j), \mu(\tau_j) + \delta(\tau_j))$.)

$$\left\| \sum_{j=1}^{n} \Delta\mu_j \tilde{A}(\tau_j) - Q^\mu \right\| < \varepsilon$$

and we write

$$Q^\mu = (\mathrm{RC}) \int_a^b \tilde{A}(t)\, d\mu.$$

THEOREM A.1. *Suppose the Riemann integral of $\tilde{A}(t)$ exists with respect to μ in any topology. Then the Riemann complete integral of $\tilde{A}(t)$ exists with respect to μ in that topology and they are equal.*

If we use Theorem A.1 we obtain the following theorem for Riemann complete integrals directly from the Riemann counterpart:

THEOREM A.2. *If $\tilde{A}_1(t)$ and $\tilde{A}_2(t)$ both have RC-integrals, then so does their sum and*

$$(\mathrm{RC}) \int_a^b [\tilde{A}_1(t) + \tilde{A}_2(t)]\, d\mu = (\mathrm{RC}) \int_a^b \tilde{A}_1(t)\, d\mu + (\mathrm{RC}) \int_a^b \tilde{A}(t)\, d\mu.$$

If γ is any element of the field then

$$(\mathrm{RC}) \int_a^b \gamma \tilde{A}_1(t)\, d\mu = \gamma (\mathrm{RC}) \int_a^b \tilde{A}_1(t)\, d\mu.$$

THEOREM A.3. *Suppose $\mu(t)$ is continuous and of bounded variation, $\tilde{A}_n(t)$ is a sequence of bounded operator-valued functions converging uniformly to $\tilde{A}(t)$ in the uniform operator topology, and* (RC) $\int_a^b \tilde{A}_n(t)\, d\mu(t)$ *exists for each n; then* (RC) $\int_a^b \tilde{A}(t)\, d\mu(t)$ *exists and* (RC) $\int_a^b \tilde{A}_n(t)\, d\mu(t) \to$ (RC) $\int_a^b \tilde{A}(t)\, d\mu(t)$ *in the uniform operator topology.*

THEOREM A.4. *Let $\tilde{A}(t)$ be Bochner integrable with respect to the measure μ, where μ is continuous in t; then $\tilde{A}(t)$ has a Riemann complete integral with respect to μ on $[a, b]$ and*

$$(\mathrm{RC})\int_a^b \tilde{A}(t)\, d\mu = (\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu.$$

PROOF. We shall prove our theorem in parts.

Part I. Assume $\tilde{A}(t) = \chi_E(t)\tilde{A}$, where E is a measurable subset of $[a, b]$, $\tilde{A}$ is a bounded operator and $\chi_E(t)$ satisfies

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \notin E. \end{cases}$$

In this case we want to show that (RC) $\int_a^b \tilde{A}(t)\, d\mu(t) = \mu[E]\tilde{A}$. Let $\varepsilon > 0$ be given and let D be a compact set contained in E. Let $F \subset [a, b]$ be an open set containing E such that $\mu[F \setminus D] < \varepsilon/\|\tilde{A}\|$; define δ as the function on $[a, b]$ to $(0, \infty)$ such that:

$$\begin{aligned} \delta(t) &= d\big(t, [a, b] \setminus F\big), \quad t \in E, \\ &= d(t, D), \quad t \in [a, b] \setminus E, \end{aligned}$$

where $d(\ ,\ )$ is the distance function on R (reals).

Let $\{t_0, \tau_1, t_2, \ldots, \tau_n, t_n\}$ be a KH-partition for δ; then for $j = 1, 2, 3, \ldots, n$, if $\tau_j \in E$ then $(t_{j-1}, t_j) \subset F$, and hence,

$$\left\| \sum_{j=1}^{n} \tilde{A}(\tau_j)\Delta\mu_j - \tilde{A}\mu(F) \right\| = \|\tilde{A}\| \left[\mu(F) - \sum_{\tau_j \in E} \Delta\mu \right].$$

On the other hand, if $\tau_j \notin E$ then $(t_{j-1}, t_j) \cap D = \varnothing$ (empty set), and it then follows that

$$\left\| \sum_{j=1}^{n} \tilde{A}(\tau_j)\Delta\mu_j - \tilde{A}\mu(D) \right\| = \|\tilde{A}\| \left[\sum_{\tau_j \in E} \Delta\mu_j - \mu(D) \right].$$

Combining these results we obtain

$$\begin{aligned} \left\| \sum_{j=1}^{n} \tilde{A}(\tau_j)\Delta\mu_j - \tilde{A}\mu(E) \right\| &= \|\tilde{A}\| \left| \sum_{\tau_j \in E} \Delta\mu_j - \mu(E) \right| \\ &< \|\tilde{A}\|\big[\mu(F) - \mu(E)\big] < \|\tilde{A}\|\big[\mu(F) - \mu(D)\big] = \|\tilde{A}\|\mu(F \setminus D) < \varepsilon. \end{aligned}$$

Part II. Suppose $\tilde{A}(t)$ is a countable-valued bounded operator function; then

$$\tilde{A}(t) = \sum_{k=1}^{\infty} \tilde{A}_k \chi_{E_k}(t).$$

By definition, $\tilde{A}(t)$ is integrable (Bochner) if and only if $\|A(t)\|$ is integrable (Lebesgue) and

$$(\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu = \sum_{k=1}^{\infty} \tilde{A}_k \mu[E_k].$$

Furthermore (cf. Hille and Phillips **[HP]**)

$$(\mathrm{L})\int_a^b \|\tilde{A}(t)\|\, d\mu = \sum_{k=1}^{\infty} \|\tilde{A}_k\| \mu(E_k).$$

As the partial sums converge uniformly to $\tilde{A}(t)$, we use Theorem 1.2 to conclude that $(\mathrm{RC})\int_a^b \tilde{A}(t)\, d\mu$ exists and

$$(\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu = (\mathrm{RC})\int_a^b \tilde{A}(t)\, d\mu.$$

Part III. Let $\tilde{A}(t)$ be an arbitrary, bounded, Bochner integrable, operator-valued function on $[a, b]$ which is uniformly measurable; by definition there exists a sequence of countably-valued bounded operator functions $\{\tilde{A}_n(t)\}$ on $[a, b]$ which converges to $A(t)$ in the uniform operator topology such that

$$\lim_{n\to\infty} (\mathrm{L})\int_a^b \|\tilde{A}_n(t) - \tilde{A}(t)\|\, d\mu = 0$$

and

$$(\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu = \lim_{n\to\infty} (\mathrm{B})\int_a^b \tilde{A}_n(t)\, d\mu.$$

Since the $\tilde{A}_n(t)$ are countably valued,

$$(\mathrm{B})\int_a^b \tilde{A}_n(t)\, d\mu = (\mathrm{RC})\int_a^b \tilde{A}_n(t)\, d\mu,$$

so that

$$(\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu = \lim_{n\to\infty} (\mathrm{RC})\int_a^b \tilde{A}_n(t)\, d\mu.$$

We will be done if we can show that (RC) $\int_a^b \tilde{A}(t)\, d\mu$ exists. We note that $\|\tilde{A}_n(t) - \tilde{A}(t)\| = f_n(t)$ is a numerically valued function. We can then use a result of Henstock [H] to conclude that (RC) $\int_a^b f_n(t)\, d\mu$ exists and

$$(\mathrm{RC})\int_a^b f_n(t)\, d\mu \to 0, \qquad n \to \infty.$$

Let $\varepsilon > 0$ be given; choose n_0 so large that

$$\left\|(\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu - (\mathrm{RC})\int_a^b \tilde{A}_{n_0}(t)\, d\mu\right\| < \frac{\varepsilon}{4}$$

and

$$(\mathrm{RC})\int_a^b f_{n_0}(t)\, d\mu < \frac{\varepsilon}{4};$$

choose δ_1 so that whenever $\{t_0, \tau_1, \ldots, \tau_n, t_n\}$ is a K-H partition for δ_1 then

$$\left\|(\mathrm{RC})\int_a^b \tilde{A}_{n_0}(t)\, d\mu - \sum_{j=1}^{n} \tilde{A}_{n_0}(\tau_j)\Delta\mu_j\right\| < \frac{\varepsilon}{4};$$

choose δ_2 so that whenever $\{t_0, \tau_1, \ldots, \tau_n, t_n\}$ is a K-H partition for δ_2 then

$$\left|\sum_{j=1}^{n} f_{n_0}(\tau_j)\Delta\mu_j - (\mathrm{RC})\int_a^b f_{n_0}(t)\, d\mu\right| < \frac{\varepsilon}{4};$$

set $\delta = \delta_1 \wedge \delta_2$ so that by Lemma 1.2, if $\{t_0, \tau_1, t_1, \ldots, \tau_n, t_n\}$ is a K-H partition for δ then it is also one for δ_1 and δ_2; hence,

$$\begin{aligned}\left\|(\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu - \sum_{j=1}^{n} \tilde{A}(\tau_j)\Delta\mu_j\right\| &< \left\|(\mathrm{B})\int_a^b \tilde{A}(t)\, d\mu - (\mathrm{RC})\int_a^b \tilde{A}_{n_0}(t)\, d\mu\right\| \\ &\quad + \left\|(\mathrm{RC})\int_a^b \tilde{A}_{n_0}(t)\, d\mu - \sum_{j=1}^{n} \tilde{A}_{n_0}(\tau_j)\Delta\mu_j\right\| \\ &\quad + \left|\sum_{j=1}^{n} f_{n_0}(\tau_j)\Delta\mu_j - (\mathrm{RC})\int_a^b f_{n_0}(t)\, d\mu\right| \\ &\quad + (\mathrm{RC})\int_a^b f_{n_0}(t)\, d\mu < \varepsilon.\end{aligned}$$

DEFINITION A.3. Let us denote by $r(E)$, the set function defined on the Borel sets of $[a, b]$ by

$$r[E] = (\mathrm{RC})\int_a^b \chi_E(t)\, dt \tag{A.1}$$

where $\chi_E(t)$ is the characteristic function of the set E. We call $r[\ ]$ the Riemann complete measure of the set E. Theorem 1.4 shows that $r[E]$ is the Lebesgue measure of the set E, however, in general, $r[E]$ is finitely additive.

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