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ISOMORPHISM THEOREMS FOR OCTONION PLANES OVER LOCAL RINGS

BY

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ABSTRACT. It is proved that there is a collineation between two octonion planes over local rings if and only if the underlying octonion algebras are isomorphic as rings. It is shown that every isomorphism between the little or middle projective groups of two octonion planes over local rings is induced by conjugation with a collineation or a correlation of the planes when the local rings contain $\frac{1}{2}$.

Octonion planes over local rings were defined and studied in [**3**]. In this paper we prove two main theorems about such planes. In §1 we show that there is a collineation between two such planes if and only if there is a semilinear algebra isomorphism of the underlying octonion algebras. This was proved for octonion algebras over fields by Faulkner [**6**, p. 20].

Second, we prove that every isomorphism between the little or middle projective groups of two octonion planes over local rings is induced by conjugation with a collineation or a correlation of the planes when 2 is a unit in the local rings. This was proved for octonion division algebras over fields of characteristic $\neq 2$ by Suh [**10**, p. 338] and Veldkamp [**11**, p. 287]. The corresponding result for octonion division algebras over fields of characteristic 2 was proved by Faulkner [**6**, p. 57]. Thus our theorem extends known results to split octonion algebras over fields of characteristic $\neq 2$ and to arbitrary octonion algebras over local rings containing $\frac{1}{2}$. In the case of fields of characteristic $\neq 2$, we further generalize known results by extending our theorem to include isomorphisms between subgroups of the full collineation groups of two planes when each subgroup contains the little projective group. This extension follows directly from the classification of the subgroups of the full collineation group normalized by the little projective group [**3**, Corollary 7.2].

The results of the last paragraph are proved in §§2–6. In §2 we study two kinds of involutions in the little projective group PS: those that fix all the points on a line and a point not connected to the line, and those that fix a four-point. We prove in §3 that every involution in the middle projective group PG is of one of these two kinds. We show in §4 that involutions of the first kind are distinguished in PS by group-theoretic properties, so they are preserved by isomorphisms of little projective groups. In §5 we determine the geometric conditions for two involutions of the

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first kind to commute and for their product to be an involution of the first kind. We apply this criterion in §6 to prove the results of the preceding paragraph.

All notation is as in [3]. $\mathfrak{O}$ is an octonion algebra over a local ring (R, m) with norm $n(x)$, trace $t(x)$, and involution $x \to x^d$. $J = H(\mathfrak{O}_3, \gamma)$ is the quadratic Jordan algebra of Hermitian 3-by-3 matrices over $\mathfrak{O}$. If $x, y \in J$, $N(x)$ is the generic norm of x, $x^\#$ is the adjoint of x, and $x \times y = (x+y)^\# - x^\# - y^\#$. $\Gamma = \Gamma(J)$ is the group of norm semisimilarities of J, $G = G(J)$ is the group of norm similarities, and $S = S(J)$ is the group of norm preserving transformations [3, Definition 1.2]. The octonion plane PJ consists of points $x_* = Rx$ and lines $x^* = Rx$ for $x \in J - mJ$, $x^\# = 0$, with relations:

$x_* | y^*$, x_* "on" y^*, if $V_{x,y} = 0$,
$x_* \sim y^*$, x_* "connected" to y^*, if $T(x, y) \in m$,
$x_* \sim y_*$, x_* "connected" to y_*, if $x \times y \in mJ$,
$x^* \sim y^*$, x^* "connected" to y^*, if $x \times y \in mJ$.

If $x_* \nsim y_*$, $(x \times y)^*$ is the unique line on x_* and y_*; if $x^* \nsim y^*$, $(x \times y)_*$ is the unique point on x^* and y^* [3, Lemma 2.2]. A three-point is an ordered triple of points (a_{1*}, a_{2*}, a_{3*}) such that $a_{1*} \nsim (a_2 \times a_3)^*$, a condition symmetrical in the a_i. A four-point is an ordered quadruple of points such that any three form a three-point. A collineation of two octonion planes consists of a bijection of their points and a bijection of their lines preserving the relations "on" and "connected to". $\phi \in \Gamma(J)$ induces a collineation $P\phi$ of PJ by $P\phi(x_*) = (\phi x)_*$ and $P\phi(y^*) = (\phi^{*-1}y)^*$. If $H \subset \Gamma$, let $PH = \{P\phi | \phi \in H\}$. $P\Gamma$ is the full collineation group of PJ, and $R - m$ is the kernel of the homomorphism $\phi \to P\phi$ taking Γ onto $P\Gamma$ [3, Lemma 3.3 and Theorem 8.4]. Let $\mathfrak{O}'$ and J' be defined analogously over a local ring (R', m').

1. Isomorphism of octonion planes. In this section we prove that there is a collineation between two octonion planes if and only if their underlying octonion algebras are isomorphic as rings. Since every collineation between two octonion planes is induced by a norm semisimilarity [3, Theorem 8.4], it suffices to prove that there is a norm semisimilarity of J onto J' if and only if there is a ring isomorphism of $\mathfrak{O}$ onto $\mathfrak{O}'$.

Define a norm semisimilarity (ϕ, σ) of $\mathfrak{O}$ onto $\mathfrak{O}'$ to be an additive group isomorphism $\phi: \mathfrak{O} \to \mathfrak{O}'$, a ring isomorphism $\sigma: R \to R'$, and a unit $\rho' \in R'$ such that $\phi(\alpha x) = \alpha^\sigma \phi x$ and $n'(\phi x) = \rho' n(x)^\sigma$ for $\alpha \in R$ and $x \in \mathfrak{O}$. If ϕ takes $1 \in \mathfrak{O}$ to $1' \in \mathfrak{O}'$, setting $x = 1$ in the last sentence implies that $n'(\phi x) = n(x)^\sigma$ for $x \in \mathfrak{O}$.

LEMMA 1.1. *There is a ring isomorphism of $\mathfrak{O}$ onto $\mathfrak{O}'$ if and only if there is a norm semisimilarity of $\mathfrak{O}$ onto $\mathfrak{O}'$ taking $1 \in \mathfrak{O}$ to $1' \in \mathfrak{O}'$. In fact, any ring isomorphism ϕ of $\mathfrak{O}$ onto $\mathfrak{O}'$ induces a norm semisimilarity (ϕ, σ) taking 1 to 1′ such that $(\phi x)^{d'} = \phi(x^d)$ and $t'(\phi x) = t(x)^\sigma$ for $x \in \mathfrak{O}$.*

PROOF. Let ϕ be a ring isomorphism of $\mathfrak{O}$ onto $\mathfrak{O}'$. $\phi(R1) = R'1'$ [3, Lemma 1.11]. Since $R' \cong R'1'$, we can define a ring isomorphism σ of R onto R' by $r^\sigma 1' = \phi(r1)$, so that ϕ is σ-semilinear. Let $\mathfrak{O}$ have a free basis $1, x_1, \ldots, x_7$ over R.

Since ϕx_i and $1'$ are linearly independent, $t'(\phi x_i)$ is determined by the equation

$$(\phi x_i)^2 - t'(\phi x_i)\phi x_i + n'(\phi x_i)1' = 0.$$

Applying ϕ to the equation $x_i^2 - t(x_i)x_i + n(x_i)1 = 0$ shows that $t'(\phi x_i) = t(x_i)^\sigma$. Since $t'(\phi 1) = 2' = t(1)^\sigma$, $t'(\phi x) = t(x)^\sigma$ for $x \in \mathfrak{O}$. Since $x^d = t(x)1 - x$, $\phi(x^d) = (\phi x)^{d'}$. Applying ϕ to the equation $xx^d = n(x)1$ yields $n'(\phi x) = n(x)^\sigma$.

Conversely, let (ϕ, σ) be a norm semisimilarity of $\mathfrak{O}$ onto $\mathfrak{O}'$ such that $\phi 1 = 1'$. Since $n(\phi x) = n(x)^\sigma$ for $x \in \mathfrak{O}$, the equations $t(x) = n(x, 1)$ and $x^d = t(x)1 - x$ imply that $t(\phi x) = t(x)^\sigma$ and $\phi(x^d) = (\phi x)^{d'}$ for $x \in \mathfrak{O}$. $n(x)$ is a nondegenerate quadratic form on $\mathfrak{O}$ [**3**, Definition 1.6]. Take $a \in \mathfrak{O}$ such that $C_1 = R1 + Ra$ is a free module of rank 2 on which n is nondegenerate. Since $a^2 - t(a)a + n(a)1 = 0$, ϕ is a ring isomorphism of C_1 into $\mathfrak{O}'$. By induction, assume we have found a subalgebra C_t of $\mathfrak{O}$ such that C_t is a free R-module of rank 2^t, the restriction of n to C_t is nondegenerate, and there is a σ-semilinear algebra isomorphism ψ_t of C_t into $\mathfrak{O}'$ such that $\psi_t(x^d) = (\psi_t x)^{d'}$. Applying ψ_t to $xx^d = n(x)1$ shows that $n'(\psi_t x) = n(x)^\sigma$. Then $\phi^{-1}\psi_t \in \mathrm{Hom}_R(C_t, \mathfrak{O})$ satisfies $n(\phi^{-1}\psi_t x) = n(x)$ for $x \in C_t$. $\phi^{-1}\psi_t$ extends to $\eta \in \mathrm{Hom}_R(\mathfrak{O}, \mathfrak{O})$ such that $n(\eta x) = n(x)$ for $x \in \mathfrak{O}$, by Witt's cancellation theorem for nondegenerate quadratic forms over local rings [**2**, p. 80]. Then $\phi\eta$ is a σ-semilinear isomorphism of $\mathfrak{O}$ and $\mathfrak{O}'$ such that $n'(\phi\eta x) = n(x)^\sigma$ for $x \in \mathfrak{O}$. If $t < 3$, take $p \in C_t^\perp$ such that $n(p)$ is a unit. $\phi\eta p \in (\psi_t C_t)^\perp$ and $n'(\phi\eta p) = n(p)^\sigma$. Set $C_{t+1} = C_t + C_t p$ and set

$$\psi_{t+1}(x + yp) = \psi_t x + (\psi_t y)(\phi\eta p)$$

for $x, y \in C_t$. The proof of [**3**, Lemma 1.9] shows that C_{t+1} is a free R-module of rank 2^{t+1}, C_{t+1} is a subalgebra of $\mathfrak{O}$, n is nondegenerate on C_{t+1}, and ψ_{t+1} is a σ-semilinear ring isomorphism of C_{t+1} into $\mathfrak{O}'$ such that $\psi_{t+1}(z^d) = (\psi_{t+1}z)^{d'}$ for $z \in C_{t+1}$, completing the induction. ψ_3 is a ring isomorphism of $\mathfrak{O}$ onto $\mathfrak{O}'$, by Nakayama's Lemma [**5**, p. 7]. $\square$

If char $R/m \neq 2$, the next lemma follows from [**4**, Theorem 3.5].

LEMMA 1.2. *Let $\{x_1, \ldots, x_t\}$ span J as an R-module and let $R[\eta_1, \ldots, \eta_t]$ be a polynomial ring. Let $X = \sum \eta_i x_i \in J \otimes_R R[\eta_i]$. Then X^2, X, and 1 are linearly independent over $R[\eta_i]$.*

PROOF. Let h be a positive integer and let $P(h, R)$ be the R-submodule of $R[\eta_i]$ composed of polynomials of total degree at most h. Let $t_1, \ldots, t_d$ be the monic monomials of total degree at most $h - 2$ in $R[\eta_i]$. The images of the t_iX^2, t_iX, and t_i1 in

$$(J \otimes_R P(h, R))/m(J \otimes_R P(h, R)) \cong (J/mJ) \otimes_{R/m} P(h, R/m)$$

are linearly independent over R/m [**6**, p. 10]. Since $J \otimes_R P(h, R)$ is a finite free R-module, the t_iX^2, t_iX, and t_i1 are linearly independent over R as elements of $J \otimes_R P(h, R)$ [**5**, p. 24]. Since $J \otimes_R P(h, R)$ is isomorphic to its image in $J \otimes_R R[\eta_i]$, the lemma follows. $\square$

THEOREM 1.3. *The following conditions are equivalent*:

(1) *There is a ring isomorphism of $\mathfrak{O}$ onto $\mathfrak{O}'$.*

(2) *There is a semilinear algebra isomorphism of an isotope of J onto J'.*

(3) *There is a norm semisimilarity of J onto J'.*

PROOF. (1) $\Rightarrow$ (2). If ϕ is a ring isomorphism of $\mathfrak{O}$ onto $\mathfrak{O}'$, the last sentence of Lemma 1.1 implies that there is σ: $R \to R'$ such that

$$\sum \alpha_i e_i + a_i[jk] \to \sum \alpha_i^\sigma e_i + \phi a_i k[jk]$$

is a σ-semilinear algebra isomorphism of $H(\mathfrak{O}_3, 1)$ and $H(\mathfrak{O}_3', 1')$. (2) follows, since $Y \to Y\gamma$ is an isomorphism of the γ-isotope $H(\mathfrak{O}_3, 1)^{(\gamma)}$ onto $H(\mathfrak{O}_3, \gamma)$. (2) $\Rightarrow$ (3). Let $J^{(u)}$ be the u-isotope of J, $u \in J$ invertible, and let (ϕ, σ) be a semilinear algebra isomorphism of $J^{(u)}$ onto J'. Let $X \in J^{(u)} \otimes_R R[\eta_i]$ be as in Lemma 1.2 and extend (ϕ, σ) to a semilinear algebra isomorphism of $J^{(u)} \otimes_R R[\eta_i]$ onto $J' \otimes_{R'} R'[\eta_i]$. X satisfies a monic polynomial over $R[\eta_i]$ of degree three with constant term $N(u)N(X)$ [9, p. 500]. Applying ϕ to this polynomial and taking the corresponding polynomial for $\phi(X)$ shows that $\phi(X)$ satisfies monic polynomials of degree three with constant terms $N(u)^\sigma N(X)^\sigma$ and $N'(\phi X)$. Applying Lemma 1.2 to $\phi X = \sum \phi(x_i)\eta_i$ gives $N(u)^\sigma N(X)^\sigma = N'(\phi X)$. Specializing $\eta_1 = 1$ and $\eta_i = 0$ for $i \geqslant 2$ gives $N(u)^\sigma N(x_1)^\sigma = N'(\phi x_1)$. Since x_1 is an arbitrary element of J, ϕ is a norm semisimilarity. (3) $\Rightarrow$ (1). Let (ϕ, σ) be a norm semisimilarity of J onto J'. We can assume that $\phi(Re_i) = R'e_i'$, since $S(J')$ is transitive on three-points [3, Proposition 2.1]. Then $\phi(\mathfrak{O}[jk]) = \mathfrak{O}'[jk]$, by the proof of [3, Lemma 3.2]. Since $N(e_1 + x[23]) = -\gamma_2\gamma_3 n(x)$ for $x \in \mathfrak{O}$, define a norm semisimilarity ψ of $\mathfrak{O}$ onto $\mathfrak{O}'$ by $(\psi x)[23] = \phi(x[23])$. $\psi 1$ is invertible, since $n(\psi 1)$ is a unit. $x \to (\psi 1)^{-1}\psi x$ is a norm semisimilarity of $\mathfrak{O}$ onto $\mathfrak{O}'$ taking 1 to 1′. We are done by Lemma 1.1. $\square$

2. Involutions of the first and second kinds. Henceforth we assume that 2 *is a unit in* R. In this section we study two types of involutions (elements of order two) in PG. Define an *involution of the first kind* to be an involution in PG fixing a point a_* and all points on a line b^*, where $a_* \nsim b^*$. Define an *involution of the second kind* to be an involution in PG fixing a four-point.

We claim that PS is transitive on pairs $a_* \nsim b^*$. There are c_* and d_* on b^* such that $c_* \nsim d_*$, since PS is transitive on lines [3, Proposition 2.1]. $a_* \nsim b^* = (c \times d)^*$, so (a_*, c_*, d_*) is a three-point. The claim follows, since PS is transitive on three-points [3, Proposition 2.1].

Define $\zeta_{e_{i*}, e_i^*} \in S$ to be 1 on $Re_i + J_0(e_i)$ and -1 on $J_{1/2}(e_i)$.

PROPOSITION 2.1. *If $a_* \nsim b^*$, there is a unique involution $P\zeta_{a_*, b^*} \in PG$ fixing a_* and all points on b^*. If $P\phi \in P\Gamma$,*

$$P\phi P\zeta_{a_*, b^*} P\phi^{-1} = P\zeta_{P\phi a_*, P\phi b^*}.$$

Involutions of the first kind form a conjugacy class of PS.

PROOF. Since PS is transitive on pairs $a_* \nsim b^*$ and $\zeta_{e_{1*}, e_1^*} \in S$, it suffices to prove that $P\zeta_{e_{1*}, e_1^*}$ is the unique involution in PG fixing e_{1*} and all points on e_1^*. Let $P\psi \in PG$ be another such involution. Since $P\psi$ fixes e_{2*}, we can replace ψ by a scalar multiple and assume that ψ fixes e_2. $P\psi^2 = 1$, so ψ^2 is scalar multiplication [3, Lemma 3.3]. Then $\psi^2 = 1$, since ψ fixes e_2. Since ψ fixes each Re_i, it fixes $\mathfrak{O}[23]$ [3, Lemma 3.2]. Because ψ fixes e_2 and $R(e_2 + a[23] + \gamma_2\gamma_3 n(a)e_3)$ for all $a \in \mathfrak{O}$, ψ is the identity map on $J_0(e_1)$. It follows that ψ is multiplication by $\tau \in R - m$ on $J_{1/2}(e_1)$, where $\psi e_1 = \tau^2 e_1$ [3, Lemma 5.1]. $\psi^2 = 1$ implies that $\tau^2 = 1$. $\tau = \pm 1$, since 2 is a unit in R. Since $\psi \neq 1$, $\tau = -1$ and $\psi = \zeta_{e_{1*}, e_1^*}$. □

Proposition 2.2 *If $P\phi \in P\Gamma$ and $a_* \nsim b^*$, then $P\phi$ commutes with $P\zeta_{a_*, b^*}$ if and only if $P\phi$ fixes a_* and b^*.*

Proof. If $P\phi$ fixes a_* and b^*, Proposition 2.1 implies that $P\phi$ commutes with $P\zeta_{a_*, b^*}$. Conversely, assume that $P\phi$ commutes with $P\zeta_{a_*, b^*}$. We can assume that $a_* = e_{1*}$ and $b^* = e_1^*$, by Proposition 2.1. $\phi\zeta_{e_{1*}, e_1^*}\phi^{-1} = \alpha\zeta_{e_{1*}, e_1^*}$ for $\alpha \in R - m$. J is the direct sum of eigenspaces for ζ_{e_{1*}, e_1^*} of ranks 11 and 16, so $\alpha = 1$ and ϕ commutes with ζ_{e_{1*}, e_1^*}. Thus ϕ preserves the 1-eigenspace $Re_1 + J_0(e_1)$ of ζ_{e_{1*}, e_1^*}. If $x \in Re_1 + J_0(e_1)$, $x \in J - mJ$, and $x^\# = 0$, it follows that either $x \in Re_1$ or $x \in J_0(e_1)$. Let $x_1 = \gamma_2 e_2 + 1[23] + \gamma_3 e_3$, $x_2 = e_2$, and $x_3 = e_3$. Since at most one of the x_i can satisfy $\phi x_i \in Re_1$, there are $j \neq k$ such that $\phi x_j, \phi x_k \in J_0(e_1)$. Then

$$P\phi e_1^* = P\phi(x_j \times x_k)^* = (\phi x_j \times \phi x_k)^* = e_1^*.$$

Since $J_0(e_1) = \sum Rx$ such that $x_* | e_1^*$, ϕ preserves $J_0(e_1)$. As above, either $\phi e_1 \in Re_1$ or $\phi e_1 \in J_0(e_1)$, since ϕ preserves $Re_1 + J_0(e_1)$. Since ϕ preserves $J_0(e_1)$, $\phi e_1 \in Re_1$. Thus $P\phi$ fixes e_{1*} and e_1^*. □

A subalgebra Q of $\mathfrak{O}$ is called a quaternion subalgebra if Q is a free R-module of rank 4 and the restriction of $n(x)$ to Q is nondegenerate. $\mathfrak{O} = Q \oplus Q^\perp$ and we define $\tau_Q \in \mathrm{End}_R(\mathfrak{O})$ to be 1 on Q and -1 on $Q^\perp$. τ_Q is an algebra automorphism of period two, by the proof of [3, Lemma 1.9]. Define an algebra automorphism ζ_Q of J by

$$\zeta_Q(\sum \alpha_i e_i + \sum a_i[jk]) = \sum \alpha_i e_i + \sum \tau_Q(a_i)[jk].$$

As in [7, p. 66], we note that every algebra automorphism τ of $\mathfrak{O}$ of order two has the form τ_Q for a quaternion subalgebra Q. To see this, let Q be the 1-eigenspace of τ and let P be the -1-eigenspace. Since 2 is a unit in R, $\mathfrak{O} = Q \oplus P$, whence Q and P are free R-modules [5, p. 24]. $n(\tau x) = n(x)$ for $x \in J$ [Lemma 1.1]. It follows that $n(P, Q) = 0$, so n is nondegenerate on Q and P. Since $P \neq 0$, it contains an element whose norm is a unit. Multiplication by this element interchanges Q and P, so Q has rank 4. Then Q is a quaternion subalgebra and $\tau = \tau_Q$.

Proposition 2.3. *Every involution of the second kind is conjugate in PG to $P\zeta_Q$ for some quaternion algebra Q of $\mathfrak{O}$.*

Proof. Let $P\zeta$ be an involution of the second kind. Let $J_1 = H(\mathfrak{O}_3, 1)$ and define a norm similarity $\psi: J \to J_1$ by $\psi(X) = X\gamma^{-1}$. $P\psi\zeta\psi^{-1}$ is an involution of the second kind in $PG(J_1)$. There is $\phi \in G(J_1)$ such that $P\phi\psi\zeta\psi^{-1}\phi^{-1}$ fixes e_{1*}, e_{2*}, e_{3*}, and $(\sum e_i + \sum 1[jk])_*$, since $PG(J_1)$ is transitive on four-points [3, Lemma 8.1]. The proof of [3, Theorem 8.4] shows that there is a ring automorphism τ of $\mathfrak{O}$ such that $P\phi\psi\zeta\psi^{-1}\phi^{-1} = P\eta_\tau$, where η_τ is the norm semisimilarity of J_1 applying τ to each coordinate. η_τ is a scalar multiple of $\phi\psi\zeta\psi^{-1}\phi^{-1}$ [3, Lemma 3.3], so η_τ is linear and τ is an algebra automorphism. $P\eta_\tau^2 = 1$, so η_τ^2 is scalar multiplication. Since τ fixes $1 \in \mathfrak{D}$, $\tau^2 = 1$. As above, $\tau = \tau_Q$ for a quaternion subalgebra Q of $\mathfrak{D}$. $\eta_\tau = \zeta_Q$ and

$$P(\psi^{-1}\phi\psi)\zeta(\psi^{-1}\phi^{-1}\psi) = P\psi^{-1}\zeta_Q\psi,$$

where $\psi^{-1}\phi\psi \in G(J)$ and $P\psi^{-1}\zeta_Q\psi$ is the involution in $PG(J)$ applying τ_Q coordinatewise. □

3. Classification of involutions in *PG*. We prove in this section that every involution in PG is of the first or second kind. (It is immediate that every involution in PG belongs to PS, but it is convenient to work in PG since G is closed under scalar multiplication.)

LEMMA 3.1. *If $R = F$ is a field, there is no involution $P\phi \in PG$ such that $P\phi x_* \sim x_*$ for all $x_* \in PJ$.*

PROOF. Assume such $P\phi$ exists. We claim that $P\phi$ fixes either a point or a line. Suppose that $P\phi$ does not fix e_{1*}. Since F is a field, PS is transitive on pairs of connected points [6, p. 38]. Replacing $P\phi$ by a conjugate, we can assume that $P\phi e_{1*} = (a[12])_*$. Applying $P\phi$ to the equation $(a[12])_* \sim e_{2*}$ gives $e_{1*} \sim P\phi e_{2*}$. Since $e_{2*} \sim P\phi e_{2*}$ as well, $P\phi e_{2*} = (b[12])_*$. Then

$$P\phi e_{3*} = P\phi(e_1 \times e_2)^* = (a[12] \times b[12])^* = (-\gamma_1\gamma_2 n(a, b)e_3)^*,$$

so $P\phi$ fixes e_{3*}. Thus $P\phi$ fixes a point or a line.

We can assume that either $\phi^2 = 1$ or $(\phi^{*-1})^2 = 1$, by the proof of Proposition 2.1. Since $\phi \to \phi^{*-1}$ is a group isomorphism of $P\Gamma$, $\phi^2 = 1$ in either case. Since $P\phi e_{1*} \sim e_{1*}$, $\phi e_1 = \alpha e_1 + c[12] + d[31]$, where $n(c) = 0 = n(d)$ and $dc = 0$. Set $f_1 = e_1 + \phi e_1$ and $f_2 = e_1 - \phi e_1$. $f_i^\# = 0$ and $\phi f_i = \pm f_i$. Since F is not of characteristic two, the e_1 coefficient of at least one of the f_i is nonzero. Thus $P\phi$ fixes a point f_{i*} such that the e_1 coefficient of f_i is nonzero.

Replacing $P\phi$ by a conjugate, we can assume that $P\phi$ fixes e_{2*} [3, Proposition 2.1]. Applying the last paragraph again shows that $P\phi$ fixes a point $f_{i*} \not\sim e_{2*}$. Conjugating $P\phi$, we can assume that it fixes e_{2*} and e_{3*}. Again by the preceding paragraph, $P\phi$ fixes a point f_{i*} such that (f_{i*}, e_{2*}, e_{3*}) is a three-point.

Hence we can assume that $P\phi$ fixes (e_{1*}, e_{2*}, e_{3*}) and $\phi^2 = 1$. Since $\phi e_i = \pm e_i$, we can replace ϕ by $-\phi$ if necessary and assume that ϕ fixes at least two of the e_i. Replacing ϕ by a conjugate, we can assume that it fixes e_1, e_2, and Fe_3.

ϕ induces $\sigma \in \mathrm{End}_F(\mathfrak{D})$ by $\phi(a[12]) = a^\sigma[12]$ [3, Lemma 3.2]. $\sigma^2 = 1$, so $\mathfrak{D} = \mathfrak{D}_1 \oplus \mathfrak{D}_{-1}$ where $\mathfrak{D}_i$ is the i-eigenspace of σ. $n(a^\sigma) = n(a)$ for $a \in \mathfrak{D}$, since $(\phi x)^\# = 0$ for $x = e_1 + a[12] + \gamma_1\gamma_2 n(a)e_2$ and ϕ fixes e_1 and e_2. It follows that the $\mathfrak{D}_i$ are orthogonal and the restriction of n to $\mathfrak{D}_{-1}$ is nondegenerate. If $\mathfrak{D}_{-1} \neq 0$, there is $a \in \mathfrak{D}_{-1}$ such that $n(a)$ is nonzero; taking $x = e_1 + a[12] + \gamma_1\gamma_2 n(a)e_2$ gives $x \times \phi x \neq 0$, a contradiction. Thus $\mathfrak{D}_{-1} = 0$ and ϕ is the identity on $J_0(e_3)$. Since ϕ fixes Fe_3, $P\phi = P\zeta_{e_{3*}, e_3^*}$ [Proposition 2.1]. Then $x_* \not\sim P\phi x_*$ for $x = e_2 + 1[23] + \gamma_2\gamma_3 e_3$, a contradiction. □

LEMMA 3.2. *There is no involution $P\phi \in PG(J)$ inducing the identity in $PG(J/mJ)$.*

PROOF. Assume such $P\phi$ exists. Only a finite number of elements of R are required to express the multiplication in J and the action of ϕ in terms of a given free basis $\{x_i\}$ of J. These elements generate a Noetherian subring R' of R. Replacing R' by its localization at $R' \cap m$, we can assume that R' is local. Replacing R by R', J by $\Sigma R'x_i$ and ϕ by its restriction, we can assume that R is Noetherian. Tensoring J with the completion of R, we can assume that R is complete local Noetherian. The fact that $P\phi$ is an involution is preserved under tensoring, since $P\phi^2 = 1$ if and only if ϕ^2 is multiplication by $\beta \in R - m$ [3, Lemma 3.3]. Let ϕ induce multiplication by $\alpha_1 \in R - m$ on J/mJ. Setting $\alpha_{i+1} = \alpha_i + (2\alpha_i)^{-1}(\beta - \alpha_i^2)$ gives $\alpha_i^2 \equiv \beta$ and $\alpha_{i+1} \equiv \alpha_i \pmod{m^i}$ by induction. $\alpha = \lim \alpha_i$ satisfies $\alpha^2 = \beta$. J is the direct sum of eigenspaces for $\pm\alpha$. Since ϕ induces multiplication by α_1 on J/mJ, Nakayama's Lemma implies that the α-eigenspace equals J [5, p. 7]. Thus $P\phi = 1$, a contradiction. □

LEMMA 3.3. *If $P\phi \in PG$ is an involution, there are a_* and d_* in PJ such that $(a_* P\phi a_*, d_*, P\phi d_*)$ is a four-point.*

PROOF. $P\phi$ induces an involution $P\phi_1 \in PG(J/mJ)$ [Lemma 3.2]. There is $a_{1*} \in P(J/mJ)$ such that $P\phi_1 a_{1*} \nsim a_{1*}$ [Lemma 3.1]. Since $PS(J/mJ)$ is transitive on points and the canonical map $PS(J) \to PS(J/mJ)$ is surjective [3, Corollary 6.5], there is $a_* \in PJ$ whose image in $P(J/mJ)$ is a_{1*}. $a_* \nsim P\phi a_*$, since $a_{1*} \nsim P\phi a_{1*}$. There is a line b^* on a_* such that $P\phi a_* \nsim b^*$ (since we can assume that $a_* = e_{1*}$ and $P\phi a_* = e_{2*}$). We repeatedly apply [3, Lemma 8.2] and its dual. $P\phi b^* \nsim b^*$, since $P\phi a_* | P\phi b^*$ and $P\phi a_* \nsim b^*$. Let $c_* = (b \times \phi b)_*$. $c_* \nsim a_*$, since $a_* \nsim P\phi b^*$. There is $d_* | b^*$ such that $d_* \nsim a_*$ and $d_* \nsim c_*$ (since we can assume that $a_* = e_{1*}$, and $c_* = e_{2*}$, so $b^* = e_{3*}$). $d_* \nsim P\phi b^*$, else $P\phi b^* \sim (d \times c)^* = b^*$. $P\phi a_* \nsim b^* = (a \times d)^*$ and $d_* \nsim P\phi b^* = (\phi a \times \phi d)^*$, and applying $P\phi$ gives $a_* \nsim (\phi a \times \phi d)^*$ and $P\phi d_* \nsim (a \times d)^*$. □

THEOREM 3.4. *Every involution $P\phi \in PG$ is of the first or second kind.*

PROOF. Let $P\phi \in PG$ be an involution. Let

$$\begin{aligned}
z_1 &= \gamma_1 e_1 + \gamma_2 e_2 + 1[12],\\
z_2 &= \gamma_1 e_1 + \gamma_2 e_2 - 1[12],\\
z_3 &= \gamma_1 e_1 + \gamma_3 e_3 + 1[31],\\
z_4 &= \gamma_1 e_1 + \gamma_3 e_3 - 1[31],\\
z_5 &= \gamma_2 e_2 + \gamma_3 e_3 + 1[23],\\
z_6 &= \gamma_2 e_2 + \gamma_3 e_3 - 1[23].
\end{aligned}$$

$(z_1 \times z_2)_* = e_{3*}$, $(z_3 \times z_4)_* = e_{2*}$, and $(z_5 \times z_6)_* = e_{1*}$, so $(z_1^*, z_2^*, z_3^*, z_4^*)$ is the dual of a four-point. [3, Lemma 8.2] implies that $((z_1 \times z_3)_*, (z_1 \times z_4)_*, (z_2 \times z_3)_*, (z_2 \times z_4)_*)$ is a four-point. There are a_* and d_* in PJ such that

$(a_*, P\phi a_*, d_*, P\phi d_*)$ is a four-point [Lemma 3.3]. Since PG is transitive on four-points, we can replace $P\phi$ by a conjugate and assume that these two four-points are equal [3, Lemma 8.1]. Computation shows that $(z_1 \times z_3) \times e_1 = \gamma_1 z_6$, so $(z_1 \times z_3)_*$ is on z_6^*, and z_1^*, z_3^*, and z_6^* are concurrent. Applying $P\zeta_{e_{3*},e_3^*}$ shows that z_1^*, z_4^*, and z_5^* are concurrent, applying $P\zeta_{e_{2*},e_2^*}$ shows that z_2^*, z_3^*, and z_5^* are concurrent, and applying $P\zeta_{e_{1*},e_1^*}$ shows that z_2^*, z_4^*, and z_2^*, z_4^*, and z_6^* are concurrent, as in Figure 1. Let

$$y_* = (e_3 \times z_2)_* = (\gamma_2 e_1 + \gamma_1 e_2 + 1[12])_*.$$

$P\phi$ interchanges a_* and d_* with $P\phi a_*$ and $P\phi d_*$ respectively, so $P\phi$ interchanges z_3^* and z_5^* with z_4^* and z_6^* respectively and fixes z_1^* and z_2^*. Then $P\phi$ fixes all e_{i*} and e_i^* and hence y_*. We can assume that ϕ fixes e_1 and that $\phi^2 = 1$, by the proof of Proposition 2.1.

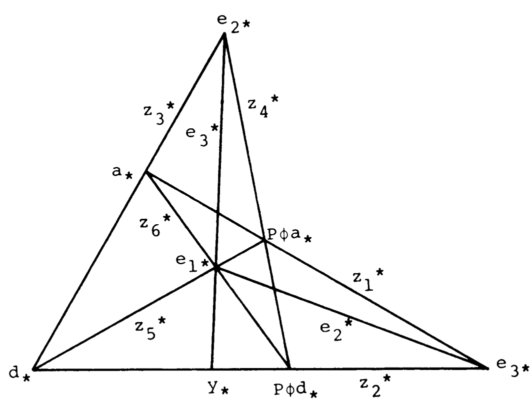

FIGURE 1

Since ϕ fixes each Re_i, it fixes each $\mathfrak{D}[jk]$ [3, Lemma 3.2]. Define $\sigma \in \operatorname{End}_R(\mathfrak{D})$ by $a^\sigma[31] = \phi(a[31])$. $\sigma^2 = 1$, since $\phi^2 = 1$. Write $\mathfrak{D} = \mathfrak{D}_1 \oplus \mathfrak{D}_{-1}$, where $\mathfrak{D}_i$ is the i-eigenspace of σ. $P\phi$ interchanges $a_* = (z_1 \times z_3)_*$ with $P\phi a_* = (z_1 \times z_4)_*$, where $z_1 \times z_3$ and $z_1 \times z_4$ both have the form $\gamma_2\gamma_3 e_1 + \gamma_1\gamma_3 e_2 + \gamma_1\gamma_2 e_3 + \cdots$. It follows that ϕ fixes e_2 and e_3 as well as e_1. Then ϕ fixes $1 \in J$, so $\phi \in S$. The equation $N(e_2 + a[31]) = -\gamma_1\gamma_3 n(a)$ implies that $n(a^\sigma) = n(a)$ for $a \in \mathfrak{D}$. Thus the restrictions of n to $\mathfrak{D}_1$ and $\mathfrak{D}_{-1}$ are nondegenerate. If $\mathfrak{D} = \mathfrak{D}_{-1}$, $P\phi$ and $P\zeta_{e_{3*},e_3^*}$ agree on e_{2*}, y_*, and all points on e_2^*. It follows that $P\phi$ equals $P\zeta_{e_{3*},e_3^*}$ [3, Lemma 8.3] and $P\phi$ is an involution of the first kind. If $\mathfrak{D} \neq \mathfrak{D}_{-1}$, there is $x \in \mathfrak{D}_1$ such that $n(x)$ is a unit. Then $(e_{2*}, e_{3*}, y_*, (e_1 + x[31] + \gamma_1\gamma_3 n(x) e_3)_*)$ is a four-point fixed by $P\phi$, and $P\phi$ is of the second kind. □

4. Group-theoretic classification of involutions. We prove now that involutions of the first kind can be distinguished from those of the second kind by their group-theoretic properties within the little projective group. Together with Theorem

3.4, this implies that an isomorphism of little projective groups preserves each kind of involution.

For $\eta \in S$, let $C(\eta)$ be the centralizer of η in S and let $C(P\eta)$ be the centralizer of $P\eta$ in PS. Let $P\zeta$ be an involution of the first or second kind and let $P\psi \in C(P\zeta)$. Then $\psi\zeta\psi^{-1} = \alpha\zeta$ for $\alpha \in R - m$. Since J decomposes into eigenspaces of distinct ranks for ζ, $\alpha = 1$ and $\psi \in C(\zeta)$. Hence $C(P\zeta) = PC(\zeta) \cong C(\zeta)/(R^{\times} \cap S)$, where $R^{\times}$ is the group of units of R.

LEMMA 4.1. *If $R = F$ is a field and $P\zeta$ is an involution of the first kind, then $C(P\zeta)$ has a normal series where all factor groups are abelian except for one which is simple.*

PROOF. We can assume that $\zeta = \zeta_{e_{1*}, e_1^*}$ [Proposition 2.1]. Since $C(P\zeta) \cong C(\zeta)/(F^{\times} \cap S)$, it suffices to prove that $C(\zeta)$ has such a normal series. If $\psi \in S$, $\psi \in C(\zeta)$ if and only if ψ preserves Fe_1 and $J_0(e_1)$, by the proof of Proposition 2.2. Let N be the kernel of the homomorphism from $C(\zeta)$ to $F^{\times}$ taking ψ to α such that $\psi e_1 = \alpha e_1$. $C(\zeta)/N$ is abelian. Let $\mathfrak{O}(J_0)$ be the orthogonal group of $J_0(e_1)$ relative to the quadratic form $x \to T(x^{\#}, e_1)$. Since elements of S preserve $T(x^{\#}, y)$ [**6**, p. 10], we can define a homomorphism $\lambda: N \to \mathfrak{O}(J_0)$ by restriction. The kernel of λ is $\{1, \zeta\}$ [**3**, Lemma 5.1]. The image of λ is the reduced orthogonal group $\mathfrak{O}'(J_0)$ [**6**, p. 31]. $\mathfrak{O}'(J_0)$ modulo its center is simple, since $J_0(e_1)$ is an isotropic space of dimension 10 [**1**, pp. 195 and 209]. $\square$

LEMMA 4.2. *If $R = F$ is a field with more than three elements and $P\zeta$ is an involution of the second kind, then $C(P\zeta)$ has a normal series with two nonsolvable factor groups.*

PROOF. We can assume that $\gamma = 1$ and that $P\zeta = P\zeta_Q$ for a quaternion subalgebra Q of $\mathfrak{O}$ [Theorem 1.3 and Proposition 2.3]. Since $C(P\zeta_Q) \cong C(\zeta_Q)/(F^{\times} \cap S)$, it suffices to prove that $C(\zeta_Q)$ has a normal series with two nonsolvable factor groups. $C(\zeta_Q)$ consists of the elements of S preserving the ± 1-eigenspaces of ζ_Q. Identify the 1-eigenspace of ζ_Q with $H(Q_3, \pi)$, where π is the standard involution conjugate transpose of Q_3. Let τ be the restriction homomorphism from $C(\zeta_Q)$ to $\mathrm{End}_F(H(Q_3, \pi))$. We claim that neither the kernel nor the image of τ is solvable. If Q is a division algebra, we are done by the proof of [**10**, pp. 333–334]. Assume that Q is split, so $Q \cong F_2$ [**8**, p. 169].

The proof of [**10**, p. 334] shows that the kernel of τ is isomorphic to the group of elements of Q of norm 1. This group is isomorphic to $\mathrm{SL}_2(F)$, which is not solvable if F has more than three elements [**1**, p. 169].

To see that the image of τ is not solvable, let $T = T_{x[ij], e_j}$ for $x \in Q$, $i \neq j$. $T \in S(J)$ [**6**, p. 18]. $T \in C(\zeta_Q)$, since ζ_Q is an automorphism fixing $x[ij]$ and e_j. Define a homomorphism λ from the group of invertible elements of Q_3 to the group of invertible linear transformations of $H(Q_3, \pi)$ by $\lambda(A) = AXA^{\pi}$, $A \in Q_3$, $X \in H(Q_3, \pi)$. $\tau(T) = \lambda(1 + xe_{ij})$, where the e_{ij} are matrix units decomposing Q_3 over Q. Let W be the multiplicative subgroup of Q_3 generated by $1 + xe_{ij}$, $x \in Q$, $i \neq j$. Identify $Q \cong F_2$ and let the matrix units of F_2 be written as f_{st}. If $\alpha \in F$, $s \neq t$ and $i \neq j$, W contains

$$(1 + e_{ij})(1 - f_{tt}e_{ji})(1 + \alpha f_{st}e_{ij})(1 + f_{tt}e_{ji})(1 - e_{ij}) = 1 + \alpha f_{st}e_{ii}.$$

Thus W contains $1 + \alpha f_{st} e_{ij}$, $\alpha \in F$, if either $s \neq t$ or $i \neq j$. Let η be the natural isomorphism of $(F_2)_3$ onto F_6 and let the matrix units of F_6 be written g_{uv}. Then $\eta(W)$ contains $1 + \alpha g_{uv}$ for all $\alpha \in F$ and $u \neq v$. These elements generate $\mathrm{SL}_6(F)$ [**1**, p. 156], so $W = \eta^{-1}(\mathrm{SL}_6(F))$. One verifies that the kernel of λ is ± 1. Since $\mathrm{PSL}_6(F)$ is simple [**1**, p. 169], $\lambda(W)$ is not solvable. Since $\lambda(1 + xe_{ij}) = \tau(T)$, $\lambda(W)$ is contained in $\tau(C(\zeta_Q))$ and $\tau(C(\zeta_Q))$ is not solvable. $\square$

Let $PH_i(J)$ be the set of involutions of the ith kind in $PS(J)$.

LEMMA 4.3. *Let θ be an isomorphism of $PS(J)$ onto $PS(J')$, where $R = F$ and $R' = F'$ are fields. Then $\theta(PH_i(J)) = PH_i(J')$.*

PROOF. Since $PH_1(J)$ is a conjugacy class of $PS(J)$, it suffices to prove that either $\theta(PH_1(J)) \subset PH_1(J')$ or $\theta^{-1}(PH_1(J')) \subset PH_1(J)$. We are done by Theorem 3.4 and Lemmas 4.1 and 4.2, unless both F and F' have three elements. Assume that is the case. If $P\zeta_1 \in PH_1(J)$, the proof of Lemma 4.1 shows that $C(\zeta_1)$ has a normal subgroup N of index at most two such that $N/\{1, \zeta_1\} \cong \mathfrak{O}'(J_0)$ and $\mathfrak{O}'(J_0)$ modulo its center is simple. The center of $\mathfrak{O}'(J_0)$ has order at most two [**1**, p. 133]. Let Q' be a quaternion subalgebra of $\mathfrak{O}'$. Since F' is finite, Q' is split [**1**, p. 144]. The second paragraph of the proof of Lemma 4.2 shows that $C(\zeta_{Q'})$ has a normal subgroup isomorphic to $\mathrm{SL}_2(F')$. $\mathrm{SL}_2(F')$ is solvable of order 24 [**1**, p. 170], so $C(\zeta_1)$ is not isomorphic to $C(\zeta_{Q'})$. $C(P\zeta_1) \cong C(\zeta_1)$ and $C(P\zeta_{Q'}) \cong C(\zeta_{Q'})$, since $F^\times \cap S = \{\alpha \in F | \alpha^3 = 1\} = \{1\}$. Thus $C(P\zeta_1)$ is not isomorphic to $C(P\zeta_{Q'})$. If $P\zeta_2 \in PH_2(J)$, there is $\psi \in G$ such that $P\psi\zeta_2\psi^{-1} = P\zeta_{Q'}$ for some quaternion subalgebra Q' of $\mathfrak{O}'$ [Proposition 2.3], so $C(P\zeta_2) \cong CP(\zeta_{Q'})$. Thus $C(P\zeta_1)$ and $C(P\zeta_2)$ are not isomorphic. We are done by Theorem 3.4. $\square$

THEOREM 4.4. *If θ is an isomorphism of $PS(J)$ onto $PS(J')$, then $\theta(PH_i(J)) = PH_i(J')$.*

PROOF. The canonical homomorphism from $PS(J)$ to $PS(J/mJ)$ is surjective [**3**, Corollary 6.5]. Let $PS_m(J)$ be its kernel, so $PS(J)/PS_m(J) \cong PS(J/mJ)$. $\theta(PS_m(J)) = PS_{m'}(J')$, since $PS_m(J)$ is the unique maximal normal subgroup of $PS(J)$ [**3**, Corollary 7.5]. Thus θ induces an isomorphism θ_m of $PS(J/mJ)$ onto $PS(J'/m'J')$. By Lemma 4.3, $\theta_m(PH_i(J/mJ)) = PH_i(J'/m'J')$. It follows from Theorem 3.4 that $\theta(PH_i(J)) = PH_i(J')$. $\square$

5. Commuting involutions of the first kind. We prove in this section $P\zeta_{a_*,b^*}$ and $P\zeta_{c_*,d^*}$ commute and their product is an involution of the first kind if and only if $a_* | d^*$ and $c_* | b^*$. The key step is to characterize the action of involutions of the first kind in terms of the harmonic properties of the plane.

LEMMA 5.1. *$P\zeta_{e_{1*},e_1^*}$ commutes with $P\zeta_{a_*,b^*}$ if and only if one of the following conditions holds*:

(1) $a_* = e_{1*}$ *and* $b^* = e_1^*$,

(2) $a_* | e_1^*$ *and* $e_{1*} | b^*$,

(3) $a, b \in J_{1/2}(e_1)$.

PROOF. Write e_{1*} as e_*. $P\zeta_{e_*,e^*}$ commutes with $P\zeta_{a_*,b^*}$ if and only if $P\zeta_{e_*,e^*}$ fixes a_* and b^* [Proposition 2.2]. This holds if and only if ζ_{e_*,e^*} fixes Ra and Rb, since one checks that $\zeta^{*-1}_{e_*,e^*} = \zeta_{e_*,e^*}$. This is equivalent to assuming that a and b are elements of $Re + J_0(e)$ or $J_{1/2}(e)$, the eigenspaces of ζ_{e_*,e^*}. This holds if and only if a and b are elements of Re, $J_0(e)$, and $J_{1/2}(e)$, by the proof of Proposition 2.2. The lemma follows, since $a_* \nsim b^*$ and the spaces Re, $J_0(e)$, and $J_{1/2}(e)$ are orthogonal with respect to $T(x, y)$. $\square$

LEMMA 5.2. *If $a_* \nsim b^*$, $c_* \nsim d^*$, $a_*|d^*$, and $c_*|\, b^*$, then*

$$P\zeta_{a_*,b^*}P\zeta_{c_*,d^*} = P\zeta_{(b\times d)_*,(a\times c)^*}$$

and $(a_*, c_*, (b \times d)_*)$ is a three-point.

PROOF. $a_* \nsim b^*$ implies that $b^* \nsim d^*$ and $a_* \nsim (b \times d)_*$, by [3, Lemma 8.2] and its dual. Then $(a \times (b \times d))^* = d^*$, so $c_* \nsim d^*$ implies that $(a_*, c_*, (b \times d)_*)$ is a three-point. We can assume that this three-point equals (e_{1*}, e_{2*}, e_{3*}) [3, Proposition 2.1]. Then $b^* = (c \times (b \times d))^* = e_1^*$ and $d^* = (a \times (b \times d))^* = e_2^*$. Since ζ_{e_{i*},e_i^*} is 1 on $Re_i + J_0(e_i)$ and -1 on $J_{1/2}(e_i)$, $P\zeta_{e_{1*},e_1^*}P\zeta_{e_{2*},e_2^*} = P\zeta_{e_{3*},e_3^*}$. The lemma follows, since $a_* = e_{1*}$, $b^* = e_1^*$, $c_* = e_{2*}$, and $d^* = e_2^*$. $\square$

When $R = F$ is a field, if $x_* \nsim y_*$ and $w_*|(x \times y)^*$, Faulkner has defined the harmonic conjugate of w_* with respect to x_* and y_* [6, p. 42]. We remark that his results and the following lemma extend directly to octonion planes over local rings, but we do not need this extension.

LEMMA 5.3. *If $R = F$ is a field, $a_* \nsim b^*$, and $a_*|d^*$, then $P\zeta_{a_*,b^*}$ takes every point on d^* to its harmonic conjugate with respect to a_* and $(b \times d)_*$.*

PROOF. $d^* \nsim b^*$ and $a_* \nsim (b \times d)_*$, since $a_* \nsim b^*$ [6, p. 36]. There is $f_*|b^*$ such that $f_* \nsim (b \times d)_*$ [6, p. 36]. $a_* \nsim b^* = ((b \times d) \times f)^*$, so $(a_*, (b \times d)_*, f_*)$ is a three-point, and we can assume that it equals (e_{1*}, e_{2*}, e_{3*}) [6, p. 33]. It follows that $b^* = e_1^*$ and $d^* = e_3^*$. The harmonic conjugate of $(\alpha_1 e_1 + s[12] + \alpha_2 e_2)_*$ with respect to e_{1*} and e_{2*} is $(\alpha_1 e_1 - s[12] + \alpha_2 e_2)_*$ [6, p. 42]. Then $P\zeta_{e_{1*},e_1^*}$ takes every point on e_3^* to its harmonic conjugate with respect to e_{1*} and e_{2*} as required. $\square$

LEMMA 5.4. *Assume that $R = F$ is a field, $a_* \nsim b^*$, and $a, b \in J_{1/2}(e_1)$. Then $P\zeta_{e_{1*},e_1^*}P\zeta_{a_*,b^*}$ is not an involution of the first kind.*

PROOF. Since F is a field, there is a line f^* on e_{1*} and a_* [6, p. 35]. $f^* \nsim e_1^*$, since $e_{1*}|f^*$ and $e_{1*} \nsim e_1^*$ [6, p. 36]. Take $g_*|e_1^*$ such that $g_* \nsim (f \times e_1)_*$. $e_{1*} \nsim e_1^* = ((f \times e_1) \times g)^*$, so there is $P\phi \in PS$ taking the three-point $(e_{1*}, (f \times e_1)_*, g_*)$ to (e_{1*}, e_{2*}, e_{3*}). Since $((f \times e_1) \times g)^* = e_1^*$, $P\phi$ fixes e_{1*} and e_1^*. Lemma 5.1 implies that $\phi a, \phi b \in J_{1/2}(e_1)$. Thus we can replace $P\zeta_{a_*,b^*}$ with its conjugate by $P\phi$. Then $f^* = (e_1 \times (f \times e_1))^* = e_3^*$, so $a_*|e_3^*$.

$e_3^* \nsim b^*$, since $a_*|e_3^*$ and $a_* \nsim b^*$. a and $e_3 \times b$ are in $J_{1/2}(e_1) \cap J_0(e_3)$, so $a_* = (s[12])_*$ and $(e_3 \times b)_* = (t[12])_*$ for $n(s) = 0 = n(t)$. $a_* \nsim b^*$ implies that $a_* \nsim (e_3 \times b)_*$, so $n(s, t)$ is a unit. Replace t by a scalar multiple to make $n(s, t) = -\gamma_1^{-1}\gamma_2^{-1}$.

Let W be the orthogonal complement of $Fs + Ft$ in $\mathfrak{D}$ with respect to $n(x, y)$. Define a linear transformation ψ on $J_0(e_3)$ to interchange e_1 and e_2 with $s[12]$ and $t[12]$ respectively and to be the identity on $W[12]$. $\psi^2 = 1$ and ψ belongs to the orthogonal group of the quadratic form $T(x^\#)$ on $J_0(e_3)$. Since its determinant is 1, ψ is the product of an even number of hyperplane reflections [**1**, p. 129]. Then ψ is induced by an element η of G preserving Fe_3 [**3**, Theorem 5.3]. $P\zeta_{a_*,b^*}$ takes every point on e_3^* to its harmonic conjugate with respect to a_* and $(e_3 \times b)^*$ [Lemma 5.3]. Likewise $P\zeta_{e_{1*},e_1^*}$ takes every point on e_3^* to its harmonic conjugate with respect to e_{1*} and e_{2*}. It follows that $P\zeta_{a_*,b^*}$ agrees with $P\eta\zeta_{e_{1*},e_1^*}\eta^{-1}$ on all points on e_3^*. Let

$$x_* = (\alpha e_1 + \beta e_2 + (\delta s + \lambda t + w)[12])_*,$$

where $\alpha, \beta, \delta, \lambda \in F$, $w \in W$, and $x^\# = 0$. Then

$$\begin{aligned} P\zeta_{a_*,b^*}(x_*) &= (\psi\zeta_{e_{1*},e_1^*}\psi x)_* \\ &= (-\alpha e_1 - \beta e_2 + (\delta s + \lambda t - w)[12])_*; \qquad (1) \end{aligned}$$

$$P\zeta_{e_{1*},e_1^*}P\zeta_{a_*,b^*}(x_*) = (-\alpha e_1 - \beta e_2 + (-\delta s - \lambda t + w)[12])_*. \qquad (2)$$

Take a preimage of $P\zeta_{e_{1*},e_1^*}P\zeta_{a_*,b^*}$ in G and let τ be its restriction to $J_0(e_3)$. (2) implies that τ is a scalar multiple of the map which is 1 on $W[12]$ and -1 on $W[12]^\perp$. Thus τ has eigenspaces of dimensions 4 and 6.

Assume that $P\zeta_{e_{1*},e_1^*}P\zeta_{a_*,b^*} = P\zeta_{c_*,d^*}$. It is not true that $c_*|e_1^*$ and $e_{1*}|d^*$, else Lemma 5.2 would imply that $a_*|e_1^*$ and $e_{1*}|b^*$. Since $P\zeta_{e_{1*},e_1^*}$ and $P\zeta_{c_*,d^*}$ commute, Lemma 5.1 implies that $c, d \in J_{1/2}(e_1)$. Take $v \in W$ such that $n(v) \neq 0$, and set

$$y_* = (e_1 + \gamma_1\gamma_2 n(v)e_2 + v[12])_*.$$

(2) shows that $(y \times \tau y)^* = e_3^*$. Since F is a field, there is a line on c_* and y_* [**6**, p. 35]. Then Lemma 5.3 and the equation $(y \times \tau y)^* = e_3^*$ imply that c_* is on e_3^*. Thus we can apply the analogue of equation (1) for $P\zeta_{c_*,d^*}$. Take a preimage of $P\zeta_{c_*,d^*}$ in G and let ξ be its restriction to $J_0(e_3)$. (1) implies that ξ has eigenspaces of dimensions 2 and 8, contradicting the fact that ξ is a scalar multiple of τ. □

THEOREM 5.5. *If $a_* \nsim b^*$ and $c_* \nsim d^*$, the following conditions are equivalent*:
(1) *$P\zeta_{a_*,b^*}$ and $P\zeta_{c_*,d^*}$ commute and their product is an involution of the first kind.*
(2) *$a_*|d^*$ and $c_*|b^*$.*

PROOF. (1) ⇒ (2). By conjugation, we can assume that $c_* = e_{1*}$ and $d^* = e_1^*$. By Lemma 5.1, either $a_*|e_1^*$ and $e_{1*}|b^*$ or $a, b \in J_{1/2}(e_1)$. Taking images in $PS(J/mJ)$ and applying Lemma 5.4 shows that the latter condition does not hold. (2) ⇒ (1), by Lemmas 5.1 and 5.2. □

We say that $P\zeta_{a_*,b^*}$ and $P\zeta_{c_*,d^*}$ *commute exactly* if the conditions of Theorem 5.5 are satisfied.

COROLLARY 5.6. *$a_* \nsim c_*$ if and only if there exist b^* and d^* such that $P\zeta_{a_*,b^*}$ and $P\zeta_{c_*,d^*}$ commute exactly.*

PROOF. If $P\zeta_{a_*,b^*}$ and $P\zeta_{c_*,d^*}$ commute exactly, then $a_* \nsim c_*$ [Lemma 5.2]. Conversely, if $a_* \nsim c_*$, we can assume that $a_* = e_{1*}$ and $c_* = e_{2*}$ [3, Proposition 2.1], so we can take $b^* = e_1^*$ and $d^* = e_2^*$. $\square$

COROLLARY 5.7. *$a_*|d^*$ if and only if there exist b^* and c_* such that $P\zeta_{a_*,b^*}$ and $P\zeta_{c_*,d^*}$ commute exactly.*

PROOF. If $a_*|d^*$, there is $g_*|d^*$ such that $a_* \nsim g_*$ [3, dual of Lemma 2.3]. We can assume that $a_* = e_{1*}$ and $g_* = e_{3*}$, so $d^* = e_2^*$. It suffices to take $b^* = e_1^*$ and $c_* = e_{2*}$. The converse is trivial. $\square$

6. Classification of isomorphisms of collineation groups. Let PH be a subgroup of $PG(J)$ containing $PS(J)$ and let PH' be a subgroup of $PG(J')$ containing $PS(J')$. We prove that every isomorphism of PH onto PH' has the form $P\phi \to P\tau P\phi P\tau^{-1}$ where $P\tau\colon PJ \to PJ'$ is a collineation or a correlation.

LEMMA 6.1. *Let $a_*, b^*, c^* \in PJ$ be such that $a_* \nsim b^*$, $a_* \nsim c^*$, and $b^* \sim c^*$. Then there is $d^* \in PJ$ such that $a_* \nsim d^*$, $b^* \nsim d^*$, and $c^* \nsim d^*$.*

PROOF. We can assume that $a_* = e_{1*}$ and $b^* = e_1^*$, as in §2. Since $b^* \sim c^*$, $c \equiv \alpha e_1 + x[12] + y[31] \pmod{mJ}$. Since $a_* \nsim c^*$, α is a unit. There is $z \in \mathfrak{O}$ such that $n(x, z) \in m$ and $n(z)$ is a unit. It suffices to set $d = e_1 + z[12] + \gamma_1\gamma_2 n(z)e_2$, since the coefficient of e_3 in $c \times d$ is a unit. $\square$

PROPOSITION 6.2. *Let θ be an isomorphism of $PS(J)$ onto $PS(J')$. Assume that there are $a_*, b^*, c^* \in PJ$ such that*

(1) $a_* \nsim b^*$, $a_* \nsim c^*$ and there is a point d_* on both b^* and c^*, and

(2) $\theta P\zeta_{a_*,b^*} = P\zeta_{s_*,t^*}$ and $\theta P\zeta_{a_*,c^*} = P\zeta_{u_*,v^*}$, where $t^* \nsim v^*$.

Then there is a collineation $P\tau\colon PJ \to PJ'$ such that $\theta P\zeta_{x_*,y^*} = P\tau P\zeta_{x_*,y^*}P\tau^{-1}$ for all $x_* \nsim y^*$.

PROOF. Let $f_*, g^*, h^* \in PJ$ satisfy $f_* \nsim g^*, f_* \nsim h^*$, and $g^* \nsim h^*$. Let $\theta P\zeta_{f_*,g^*} = P\zeta_{n_*,p^*}$ and $\theta P\zeta_{f_*,h^*} = P\zeta_{q_*,r^*}$ [Theorem 4.4]. We claim that $n_* = q_*$. $f_* \nsim (g \times h)_*$, since $f_* \nsim g^*$. $a_* \nsim d_*$, since $a_* \nsim b^*$. Thus there is $P\phi \in PS(J)$ such that $P\phi a_* = (g \times h)_*$ and $P\phi d_* = f_*$. $P\zeta_{f_*,g^*}$ and $P\zeta_{f_*,h^*}$ commute exactly with $P\zeta_{P\phi a_*,P\phi b^*}$ and $P\zeta_{P\phi a_*,P\phi c^*}$. Applying θ shows that $P\zeta_{n_*,p^*}$ and $P\zeta_{q_*,r^*}$ commute exactly with $\theta P\zeta_{P\phi a_*,P\phi b^*}$ and $\theta P\zeta_{P\phi a_*,P\phi c_*}$ [Theorems 4.4 and 5.5]. Setting $P\psi = \theta P\phi \in PS(J')$ gives

$$\theta P\zeta_{P\phi a_*,P\phi b^*} = \theta\left(P\phi P\zeta_{a_*,b^*}P\phi^{-1}\right) = P\psi P\zeta_{s_*,t^*}P\psi^{-1} = P\zeta_{P\psi s_*,P\psi t^*}$$

and likewise $\theta P\zeta_{P\phi a_*,P\phi c^*} = P\zeta_{P\psi u_*,P\psi v^*}$. Thus $P\zeta_{n_*,p^*}$ and $P\zeta_{q_*,r^*}$ commute exactly with $P\zeta_{P\psi s_*,P\psi t^*}$ and $P\zeta_{P\psi u_*,P\psi v^*}$, so n_* and q_* are on $P\psi t^*$ and $P\psi v^*$. By hypothesis, $t^* \nsim v^*$, so $n_* = (\psi t \times \psi v)_* = q_*$, as asserted.

We have shown that if $f_* \nsim g^*$, $f_* \nsim h^*$, and $g^* \nsim h^*$, then $\theta P\zeta_{f_*,g^*} = P\zeta_{n_*,p^*}$ and $\theta P\zeta_{f_*,h^*} = P\zeta_{n_*,r^*}$. Lemma 6.1 implies that this remains true without the hypothesis that $g^* \nsim h^*$. Thus there is a bijection $P\tau_1$ from the points of PJ onto the points of PJ' such that $\theta P\zeta_{x_*,y^*} = P\zeta_{P\tau_1 x_*,p^*}$ for all $x_* \nsim y^*$. $x_{1*} \nsim x_{2*}$ if and only if $P\tau_1 x_{1*} \nsim P\tau_1 x_{2*}$ [Corollary 5.6].

Take $a^*, b_*, c_* \in PJ$ such that $b_* \nsim a^*$, $c_* \nsim a^*$, and $b_* \nsim c_*$. $\theta P\zeta_{b_*,a^*} = P\zeta_{P\tau_1 b_*,s^*}$ and $\theta P\zeta_{c_*,a^*} = P\zeta_{P\tau_1 c_*,u^*}$, where $P\tau_1 b_* \nsim P\tau_1 c_*$. Thus θ satisfies the duals of conditions (1) and (2). By duality, there is a bijection $P\tau_2$ from the lines of PJ onto the lines of PJ' such that

$$\theta P\zeta_{x_*,y^*} = P\zeta_{P\tau_1 x_*, P\tau_2 y^*} \tag{3}$$

for all $x_* \nsim y^*$; also $y_1^* \nsim y_2^*$ if and only if $P\tau_2 y_1^* \nsim P\tau_2 y_2^*$. By Corollary 5.7, $x_* | y^*$ if and only if $P\tau_1 x_* | P\tau_2 y^*$. $x_* \nsim y^*$ if and only if $P\zeta_{x_*,y^*}$ is defined, so $x_* \nsim y^*$ if and only if $P\tau_1 x_* \nsim P\tau_2 y^*$. Thus $x_* \to P\tau_1 x_*$ and $y^* \to P\tau_2 y^*$ define a collineation $P\tau$. We are done by (3). $\square$

A correlation of two octonion planes consists of bijections between the points of each plane and the lines of the other preserving the relations "on" and "connected to". Let $P\psi$ be the canonical correlation of PJ' interchanging x_* and x^*. If $P\phi \in P\Gamma(J')$, $P\psi P\phi P\psi = P\phi^{*-1}$. If $x_* \nsim y^*$ in PJ', $P\psi P\zeta_{x_*,y^*} P\psi = P\zeta_{y_*,x^*}$. Every correlation of PJ onto PJ' has the form $P\psi P\eta$ where η is a norm semisimilarity of J onto J' [3, Theorem 8.4].

PROPOSITION 6.3. *Let θ be an isomorphism of $PS(J)$ onto $PS(J')$. Assume that there are $a_*, b^*, c^* \in PJ$ such that*

(1) $a_* \nsim b^*$, $a_* \nsim c^*$, and there is a point d_* on both b^* and c^*, and

(2) $\theta P\zeta_{a_*,b^*} = P\zeta_{s_*,t^*}$ and $\theta P\zeta_{a_*,c^*} = P\zeta_{u_*,v^*}$, where $s_* \nsim u_*$.

Then there is a correlation $P\tau$: $PJ \to PJ'$ such that $\theta P\zeta_{x_*,y^*} = P\tau P\zeta_{x_*,y^*} P\tau^{-1}$ for all $x_* \nsim y^*$.

PROOF. Conjugation by the canonical correlation $P\psi$ of PJ' induces an automorphism Ψ of $PS(J')$. ($\phi \in S$ implies $\phi^{*-1} \in S$, by the proof of [3, Lemma 1.7].) $\Psi\theta$ satisfies the hypotheses of Proposition 6.2, so there is a collineation $P\eta$: $PJ \to PJ'$ such that $\Psi\phi(P\zeta_{x_*,y^*}) = P\eta P\zeta_{x_*,y^*} P\eta^{-1}$ for all $x_* \nsim y^*$. Set $P\tau = P\psi P\eta$. $\square$

PROPOSITION 6.4. *Let θ be an isomorphism of $PS(J)$ onto $PS(J')$, where $R = F$ and $R' = F'$ are fields. Then there is a collineation or a correlation $P\tau$: $PJ \to PJ'$ such that $\theta P\zeta_{x_*,y^*} = P\tau P\zeta_{x_*,y^*} P\tau^{-1}$ for all $x_* \nsim y^*$.*

PROOF. Take $a_*, b^* \in PJ$ such that $\theta P\zeta_{a_*,b^*} = P\xi_{e_{1*},e_1^*}$ [Theorem 4.4]. There is $c^* \in PJ$ such that $a_* \nsim c^*$ and $b^* \nsim c^*$ (since $PS(J)$ is transitive on pairs $a_* \nsim b^*$). Let $\theta P\zeta_{a_*,c^*} = P\zeta_{f_*,g^*}$. If $f_* \nsim e_{1*}$, we are done by Proposition 6.3. If $f_* = e_{1*}$, we are done by applying Proposition 6.2 to θ^{-1}. (The hypothesis of Proposition 6.2 that there is a point on both e_1^* and g^* is satisfied, since F' is a field [6, p. 35].) Thus we can assume that $f_* \sim e_{1*}$ and $f_* \neq e_{1*}$. Let $f = \alpha e_1 + s[12] + r[31]$, where $n'(s) = 0 = n'(r)$ and at least one of s or r is nonzero. By symmetry, assume that $s \neq 0$. Take $t \in \mathfrak{O}'$ such that $n'(s, t) \neq 0$ and $n'(t) = 0$.

Consider the nondegenerate quadratic form $Q(x) = T'(x^\#)$ on $J_0'(e_3)$. If $z \in J_0'(e_3)$ and $Q(z) \neq 0$, let $S_z \in \mathfrak{O}(J_0'(e_3))$ be the hyperplane reflection $x \to x - Q(z)^{-1}Q(x, z)z$. Let W be the orthogonal complement of the span of e_1, e_2, $s[12]$, and $t[12]$. Since $n(s) = 0$ and $s \neq 0$, $\mathfrak{O}'$ is split and $J_0'(e_3)$ has Witt index five [8, p. 169]. Thus there is $w \in W$ such that $Q(w) = Q((s + t)[12])^{-1}$. $S_w S_{(s+t)[12]}$ belongs to the reduced orthogonal group $\mathfrak{O}'(J_0'(e_3))$, so it extends to $\eta \in S(J')$ fixing e_3 [**6**, p. 31]. η fixes e_1 and e_2 and takes $s[12]$ to $-t[12]$. Since η fixes each e_i, it preserves each $\mathfrak{O}'[jk]$ [**3**, Lemma 3.2]. Since η preserves e_1 and $J_0'(e_1)$, $P\eta$ fixes e_{1*} and e_1^* and thus commutes with $P\zeta_{e_{1*},e_1^*}$ [Proposition 2.2]. Applying θ^{-1} shows that $\theta^{-1}P\eta$ commutes with $P\zeta_{a_*,b^*}$. Then $\theta^{-1}P\eta$ fixes a_*, so

$$P\zeta_{a_*,(\theta^{-1}P\eta)c^*} = (\theta^{-1}P\eta)P\zeta_{a_*,c^*}(\theta^{-1}P\eta^{-1}).$$

Applying θ shows that

$$\theta P\zeta_{a_*,(\theta^{-1}P\eta)c^*} = P\eta\big(\theta P\zeta_{a_*,c^*}\big)P\eta^{-1} = P\eta P\zeta_{f_*,g^*}P\eta^{-1} = P\zeta_{P\eta f_*,P\eta g^*}. \tag{4}$$

Since η preserves each $\mathfrak{O}'[jk]$, $\eta f = \alpha e_1 - t[12] + u[31]$ for $u \in \mathfrak{O}'$. The coefficient of e_3 in $f \times \eta f$ is nonzero, so $f_* \not\sim P\eta f_*$. There is a point on both c^* and $(\theta^{-1}P\eta)c^*$, since F' is a field [**6**, p. 35]. We are done by applying Proposition 6.3 with equations (4) and $\theta P\zeta_{a_*,c^*} = P\zeta_{f_*,g^*}$. □

THEOREM 6.5. *Let PH be a subgroup of $P\Gamma(J)$ containing $PS(J)$ and let PH' be a subgroup of $P\Gamma(J')$ containing $PS(J')$. Let θ be an isomorphism of PH onto PH' such that $\theta PS(J) = PS(J')$. Then there is a collineation or a correlation $P\tau$: $PJ \to PJ'$ such that $\theta P\phi = P\tau P\phi P\tau^{-1}$ for all $P\phi \in PH$.*

PROOF. Let $PS_m(J)$ be the kernel of the canonical homomorphism from $PS(J)$ to $PS(J/mJ)$. $PS_m(J)$ is the unique largest subgroup of $PS(J)$, and $PS(J)/PS_m(J) \cong PS(J/mJ)$ [**3**, Corollaries 6.5 and 7.5]. Thus θ induces an isomorphism θ_m of $PS(J/mJ)$ onto $PS(J'/m'J')$. Take $a_*, b^*, c^* \in PJ$ such that $a_* \not\sim b^*$, $a_* \not\sim c^*$, and $b^* \not\sim c^*$. Let $\theta P\zeta_{a_*,b^*} = P\zeta_{s_*,t^*}$ and $\theta P\zeta_{a_*,c^*} = P\zeta_{u_*,v^*}$ [Theorem 4.4]. If p: $PJ \to P(J/mJ)$ and p': $PJ' \to P(J'/m'J')$ are the canonical maps,

$$\theta_m P\zeta_{pa_*,pb^*} = P\zeta_{p's_*,p't^*}, \qquad \theta_m P\zeta_{pa_*,pc^*} = P\zeta_{p'u_*,p'v^*}.$$

Since $pb^* \not\sim pc^*$, Proposition 6.4 implies that either $p's_* \not\sim p'u_*$ or $p't^* \not\sim p'v^*$, so either $s_* \not\sim u_*$ or $t^* \not\sim v^*$. Thus Propositions 6.2 and 6.3 imply that there is a collineation or a correlation $P\tau$: $PJ \to PJ'$ such that $\theta P\zeta_{x_*,y^*} = P\tau P\zeta_{x_*,y^*}P\tau^{-1}$ for all $x_* \not\sim y^*$. If $P\phi \in PH$,

$$\begin{aligned}(\theta P\phi)P\tau P\zeta_{x_*,y^*}P\tau^{-1}(\theta P\phi^{-1}) &= (\theta P\phi)\theta P\zeta_{x_*,y^*}(\theta P\phi^{-1}) = \theta\big(P\phi P\zeta_{x_*,y^*}P\phi^{-1}\big)\\ &= \theta P\zeta_{P\phi x_*,P\phi y^*} = P\tau P\zeta_{P\phi x_*,P\phi y^*}P\tau^{-1}\\ &= P\tau P\phi P\zeta_{x_*,y^*}P\phi^{-1}P\tau^{-1}.\end{aligned}$$

Proposition 2.2 implies that $(\theta P\phi)P\tau = P\tau P\phi$, as required. □

COROLLARY 6.6. *Let H be a subgroup of $\Gamma(J)$ containing $S(J)$ and let H' be a subgroup of $\Gamma(J')$ containing $S(J')$. Let θ be an isomorphism of H onto H' such that $\theta S(J) = S(J')$. Then there is a norm semisimilarity τ: $J \to J'$ and a map χ: $H \to R' - m'$ such that either $\theta\phi = (\chi\phi)\tau\phi\tau^{-1}$ or $\theta\phi = (\chi\phi)\tau\phi^{*-1}\tau^{-1}$ for all $\phi \in H$. If $\phi_1, \phi_2 \in H$,*

$$\chi(\phi_1\phi_2) = (\chi\phi_1)\big(\sigma\sigma_1\sigma^{-1}(\chi\phi_2)\big) \tag{5}$$

where τ is σ-semilinear and ϕ_1 is σ_1-semilinear.

PROOF. Proposition 2.2 implies that the centralizer of $PS(J)$ in $P\Gamma(J)$ is trivial, so $R - m$ is the centralizer of $S(J)$ in $\Gamma(J)$. Then θ maps $H \cap (R - m)$ onto $H' \cap (R' - m')$ and θ induces an isomorphism $P\theta$ of PH onto PH'. By Theorem 6.5, there is a norm semisimilarity τ of J onto J' such that either $P\theta P\phi = P\tau P\phi P\tau^{-1}$ or $P\theta P\phi = P\tau P\phi^{*-1}P\tau^{-1}$ for all $P\phi \in PH$. Thus there is a map $\chi\colon H \to R' - m'$ such that either $\theta\phi = (\chi\phi)\tau\phi\tau^{-1}$ or $\theta\phi = (\chi\phi)\tau\phi^{*-1}\tau^{-1}$ for $\phi \in H$. (5) is equivalent to the condition that θ is a homomorphism. $\square$

COROLLARY 6.7. (1) *The hypothesis that* $\theta PS(J) = PS(J')$ *can be deleted from Theorem* 6.5 *if* $PH \subset PG(J)$ *and* $PH' \subset PG(J')$.

(2) *The hypothesis that* $\theta S(J) = S(J')$ *can be deleted from Corollary* 6.6 *if* $H \subset G(J)$ *and* $H' \subset G(J')$. *In this case,* χ *is a homomorphism whose kernel contains* $S(J)$.

PROOF. (1) If K is a group, let $[K, K]$ denote the subgroup generated by the commutators of elements of K. $[S(J), S(J)] = S(J)$ [**3**, equation (v) and Corollary 6.5]. It is immediate that $[G(J), G(J)] \subset S(J)$, so $[PH, PH] = PS(J)$ and $[PH', PH'] = PS(J')$. Hence $\theta PS(J) = PS(J')$. (2) As above, $[H, H] = S(J)$ and $[H', H'] = S(J')$, so $\theta S(J) = S(J')$. (5) and the assumption that $H \subset G(J)$ imply that χ is a homomorphism. $\chi(S(J)) = 1$, since $S(J) = [H, H]$ and $R' - m'$ is abelian. $\square$

COROLLARY 6.8. (1) *The hypothesis that* $\theta PS(J) = PS(J')$ *can be deleted from Theorem* 6.5 *if* $R = F$ *and* $R' = F'$ *are fields.*

(2) *The hypothesis that* $\theta S(J) = S(J')$ *can be deleted from Corollary* 6.6 *if* $R = F$ *and* $R' = F'$ *are fields.*

PROOF. (1) Since $PS(J)$ is a normal subgroup of $P\Gamma(J)$ and $\theta PH = PH' \supset PS(J')$, $\theta PS(J)$ is normalized by $PS(J')$. Since F' is a field, any proper subgroup of $P\Gamma(J')$ normalized by $PS(J')$ contains $PS(J')$ [**3**, Corollary 7.2]. Thus $\theta PS(J)$ contains $PS(J')$, and replacing θ by θ^{-1} gives the reverse containment. (2) Since $S(J)$ is a normal subgroup of $\Gamma(J)$ and $\theta H = H' \supset S(J')$, $\theta S(J)$ is normalized by $S(J')$. Since F' is a field, any subgroup of $\Gamma(J')$ normalized by $S(J')$ is either contained in $F' - 0$ or contains $S(J')$ [**3**, Theorem 7.1]. Since $S(J)$ is nonabelian, $\theta S(J)$ contains $S(J')$. The reverse containment holds by symmetry. $\square$

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